Interpolating between Temperley-Lieb and Brauer representation theory: generalised roots of unity

(based on arXiv:2512.24535 joint with P. P. Martin)

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Overview

1. Kadar-Martin-Yu Categories/Algebras
2. Representation Theory by "Tower of Recollment"
3. (Non)Semisimplicity Criteria: Generalised Roots of Unity
   - Maps by Reflection Geometry

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Kadar-Martin-Yu Categories: Context

For $\alpha \in \mathbb{K}$, we are going to define a discrete family of nested $\mathbb{K}$-linear subcategories of the diagrammatic Brauer category $Br(\alpha)$:

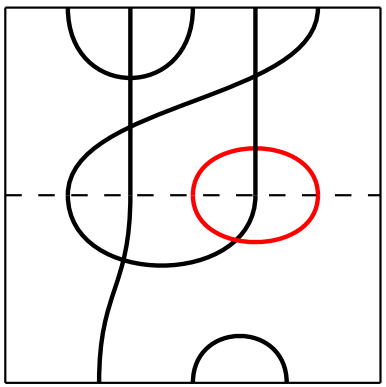
$$TL(\alpha) = J_{-1}(\alpha) \subset J_0(\alpha) \subset \ldots \subset J_l(\alpha) \subset \ldots \subset Br(\alpha) = J_\infty(\alpha).$$

The Brauer category $Br(\alpha)$ has:

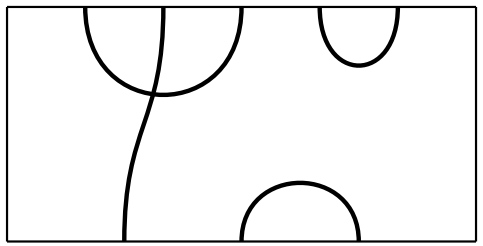
- $\mathbb{N}$ as objects.
- $Br(n, m)$ spanned by type (n, m) "diagrams" (pairings) with a diagrammatic composition up to loop replacement:

$Br(5,5) \ni$

$Br(5,3) \ni$



$= \alpha$

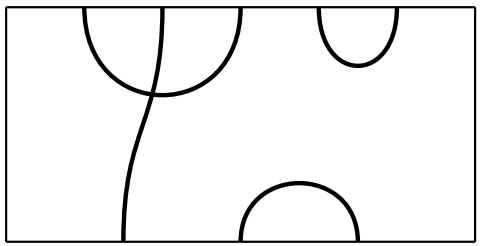


$\in Br(5,3)$

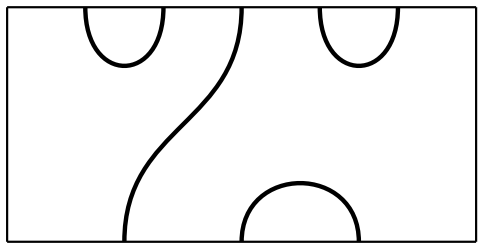
- $\otimes : Br \times Br \to Br$ given by horizontal juxtaposition of diagrams.

Kadar-Martin-Yu Categories: Context (continued)

The Temperley-Lieb (TL), $TL(\alpha)$ is the wide (monoidal, $\mathbb{K}$-linear) subcategory where we restrict to "crossingless diagrams", *e.g.*:



$\notin TL(5,3)$,



$\in TL(5,3)$

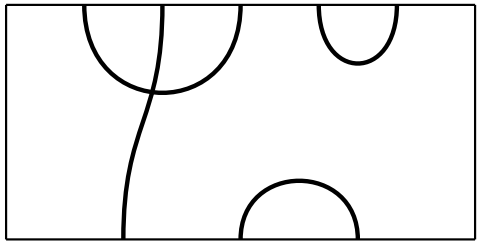
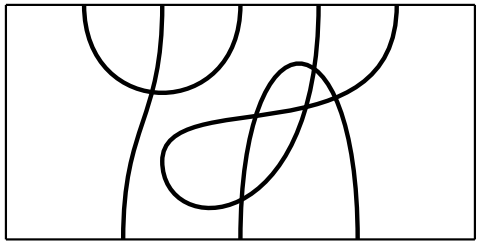
Note that TL does not see much of Brauer, e.g. for $TL_n := TL(n,n)$ and likewise $Br_n := Br(n,n)$:

$$\frac{\dim(TL_n)}{\dim(Br_n)} = \frac{C_n}{(2n-1)!!} = \frac{2^n}{(n+1)!} \to 0, \qquad \text{as} \qquad n \to \infty.$$

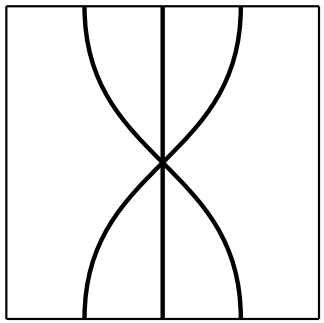
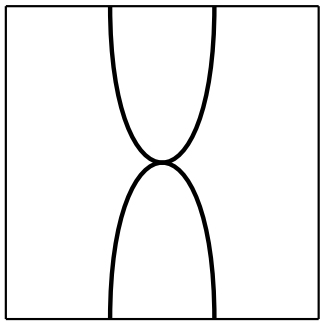
Context summary: We have $TL(\alpha) \subset Br(\alpha)$ diagrammatic categories and their endomorphism algebras $TL_n(\alpha) \subset Br_n(\alpha)$. The TL "crossingless" condition kills almost everything...

Kadar-Martin-Yu Categories: Context (warning)

ACHTUNG: Diagrams represent pair partitions... pair partitions have **many** representatives and some are better than others, *e.g.*:

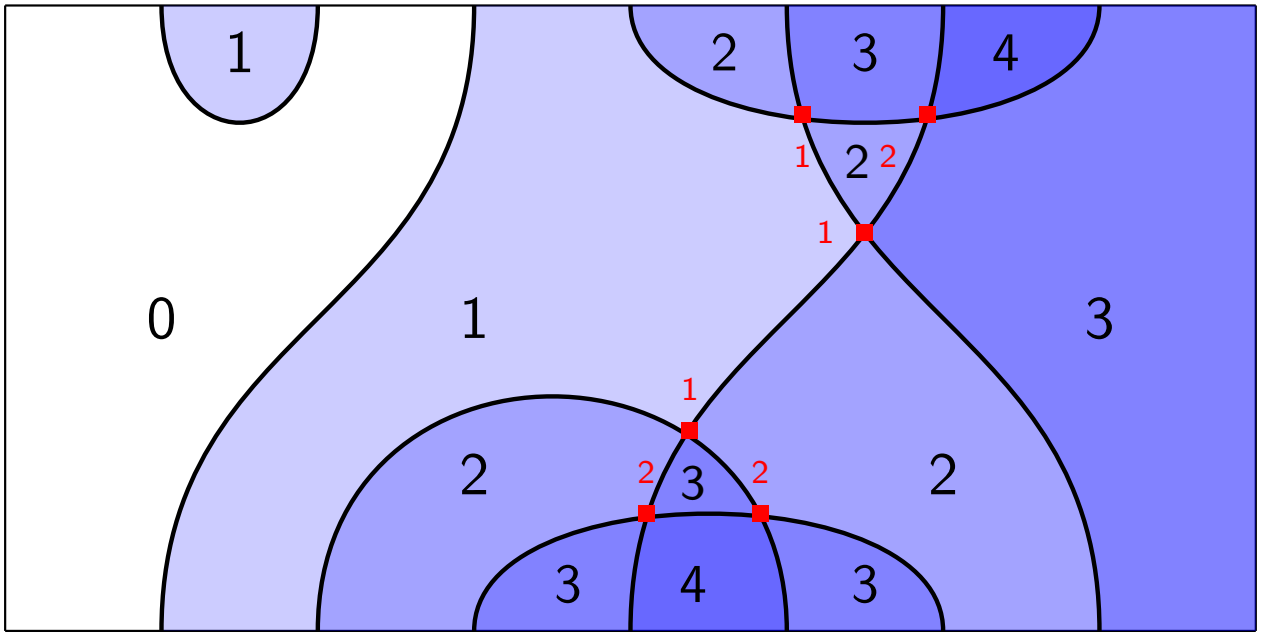



Diagrams are **regular**, *i.e.* no instances of:

We will pretty much always draw diagrams with **minimal crossings**.

Kadar-Martin-Yu Categories: The Height Condition



The height of a **diagram**: $ht(\text{diag.}) = \max_{\text{k} \in \text{diag.}} k$, *e.g.* above $ht = 2$.

(it is defined to be -1 if no crossings)

The height of a **pair partition**:

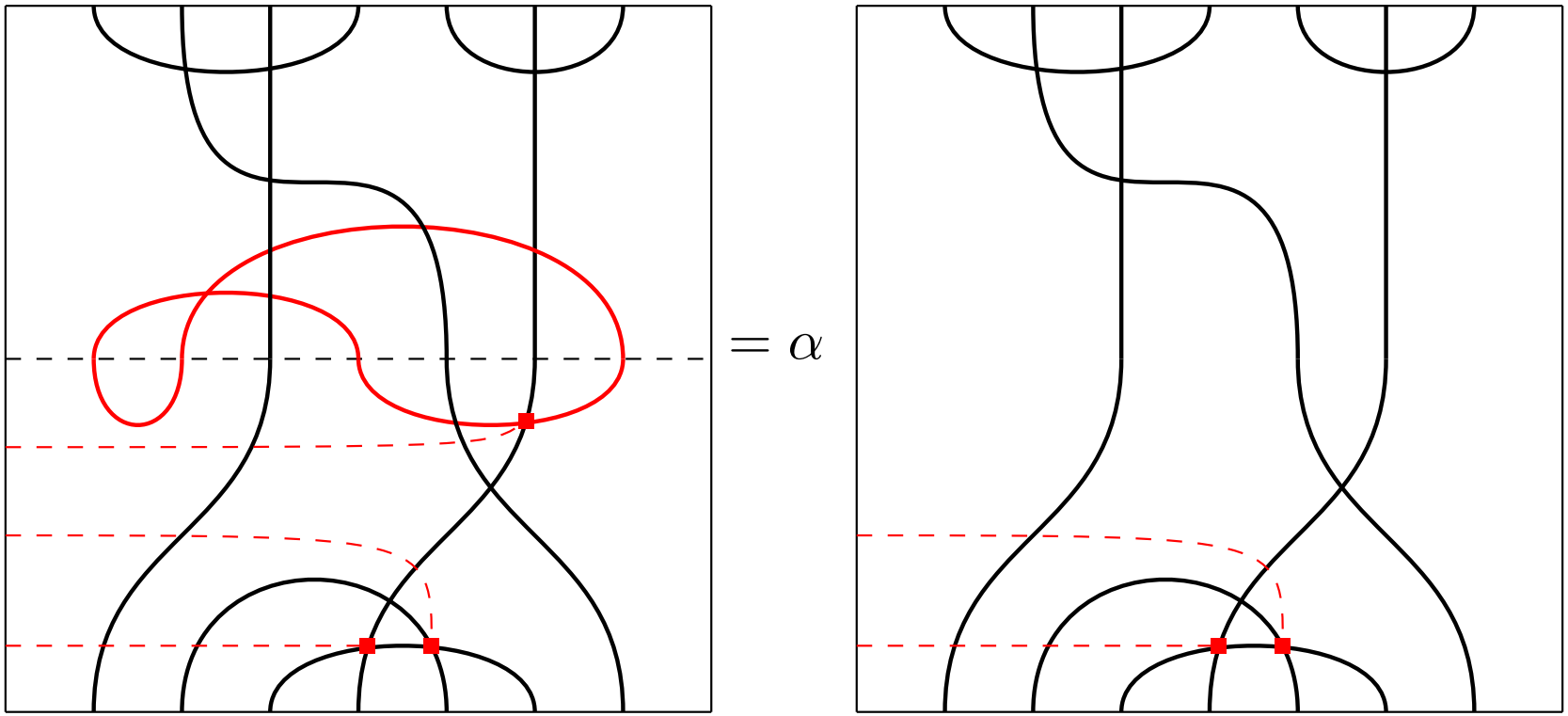
$$ht(p) = \min_{\text{diag. of } p} ht(\text{diag.}) \quad \text{e.g. above} \quad ht \leq 2$$

Kadar-Martin-Yu Categories: The Height Condition (ctd)

Lemma (Height is non-increasing)

$$ht(p'\#p) \leq \max\{ht(p'), ht(p)\}$$

Proof: $(p' \circ p = \alpha^{\#\text{loops}} p'\#p)$



Kadar-Martin-Yu Categories

Definition (Kadar, Martin, and Yu, 2014 ; KMY Category)

For $l = -1, 0, 1, \ldots, \infty$ define $J_l(\alpha) \subset Br(\alpha)$ to be the wide, $\mathbb{K}$-linear subcategory where now, $J_l(n, m)$ is spanned by pairings with height $\leq l$.

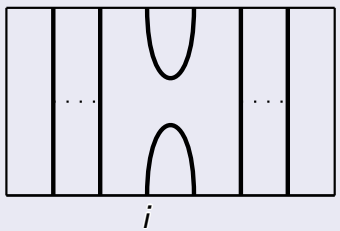
Note: J_l is not monoidal unless $l = -1$ or $l = \infty$.

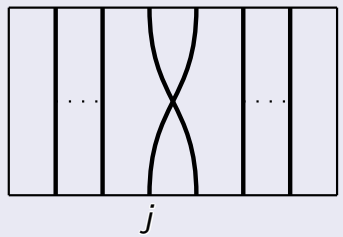
Today: we study $J_{l,n}(\alpha) = J_{l,n} := J_l(n, n)$. Observe

$$TL_n = J_{-1,n} \subset J_{0,n} \subset J_{1,n} \subset \ldots \subset J_{n-2,n} = J_{\infty,n} = Br_n$$

Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}(\alpha)$ is generated by:

$e_i =$ 

, $s_j =$ 

,

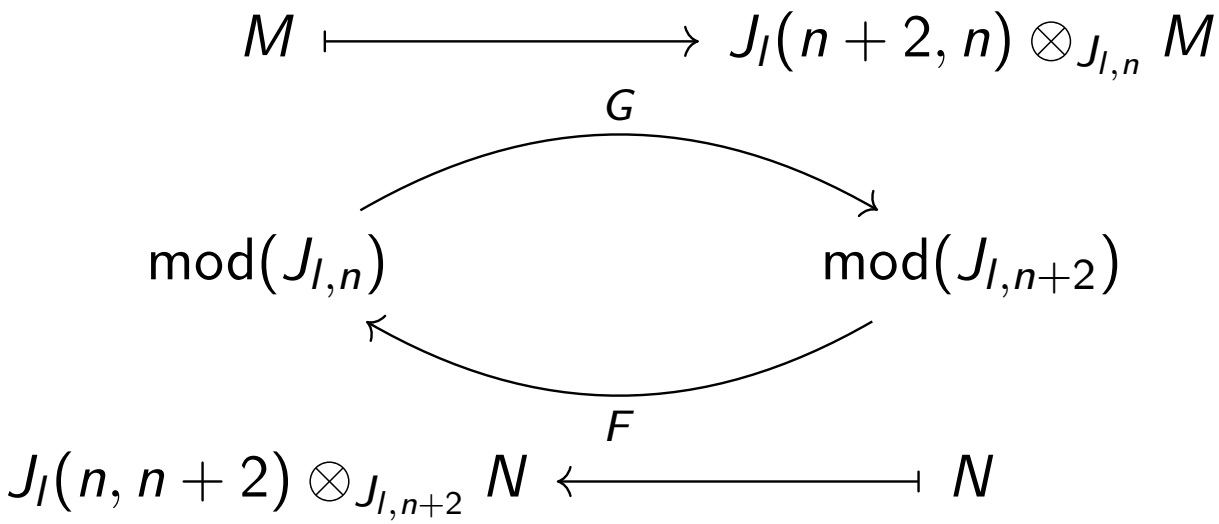
for $i = 1, \ldots, n-1$, and $j = 1, \ldots, \min(l+1, n-1)$. *i.e.* $\mathbb{K}S_n \subset Br_n$ whereas $\mathbb{K}S_{\min(l+2,n)} \subset J_{l,n}$.

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Representation Theory: Globalisation/Localisation

Use hom-spaces as bimodules:



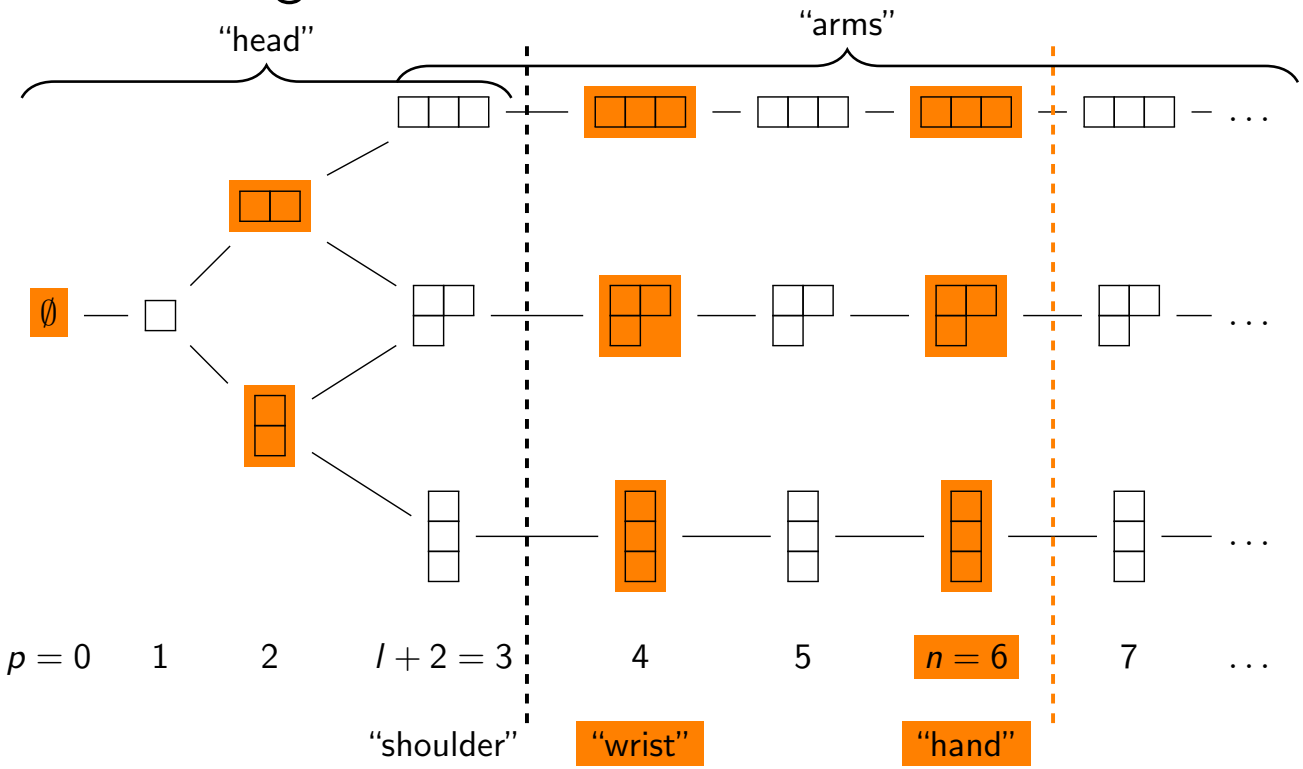
Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$. Use to label simples inductively:

$$\begin{aligned} \Lambda_{l,n} &= \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_{l,n}^{\leq n-2}\right) \\ &= \Lambda_{l,n-2} \sqcup \Lambda(\mathbb{K}\, S_{\min(n,l+2)}) = \Lambda_{l,n-2} \sqcup \{\lambda \vdash \min(n, l+2)\}, \end{aligned}$$

$$\Rightarrow \qquad \Lambda_{l,n} = \{(p,\lambda) \mid p = n, n-2, \ldots, 0/1, \quad \lambda \vdash \min(p, l+2)\}$$

Representation Theory: Anatomy of $\mathrm{mod}(J_{l,-})$

e.g. Rollet/Bratelli diagram for $l = 1$



For each label have a **standard module**, constructed inductively:

$$\Delta^n_{(n,\lambda)} \simeq \mathcal{S}_\lambda, \qquad \Delta^n_{(n-2k,\lambda)} = G(\Delta^{n-2}_{(n-2k,\lambda)}).$$

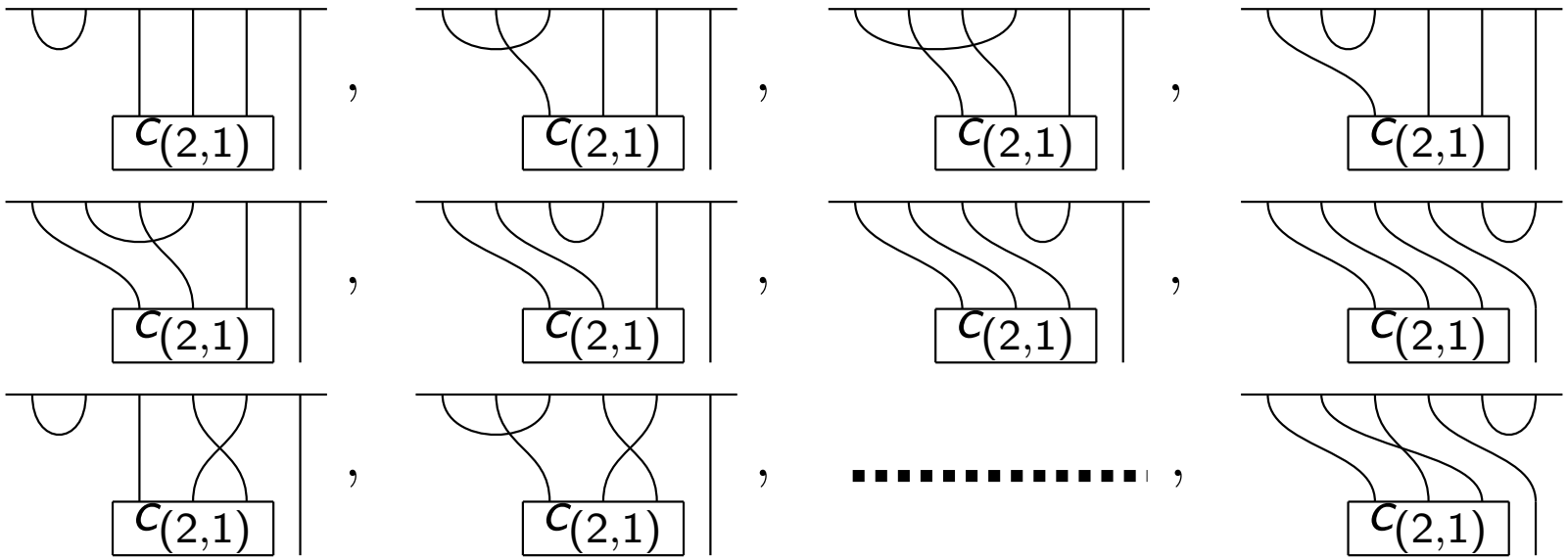
Encode induction/restriction rules on **Rollet diagram**, *e.g.* above.

Representation Theory: Standard Modules

Standard modules have simple heads $L_{(p,\lambda)}$ and determine $[L_{(p,\lambda)} : P_{(p,\lambda)}]$.
Example: $n = 6$ and label $(4,(2,1))$. Write

$$\mathcal{S}_{(2,1)} = \mathbb{C}S_3 \cdot c_{(2,1)} = \mathbb{C}\{c_{(2,1)}, (2\,3)c_{(2,1)}\},$$

c_λ is a young symmetriser s.t.: $c_\lambda^2 = c_\lambda,\ c_\lambda^* = c_\lambda,\ c_\lambda \cdot \mathbb{C}S_{|\lambda|} \cdot c_\lambda = \mathbb{C}\, c_\lambda$.
A basis for $\Delta^6_{(4,\lambda)}$ is:



Overview

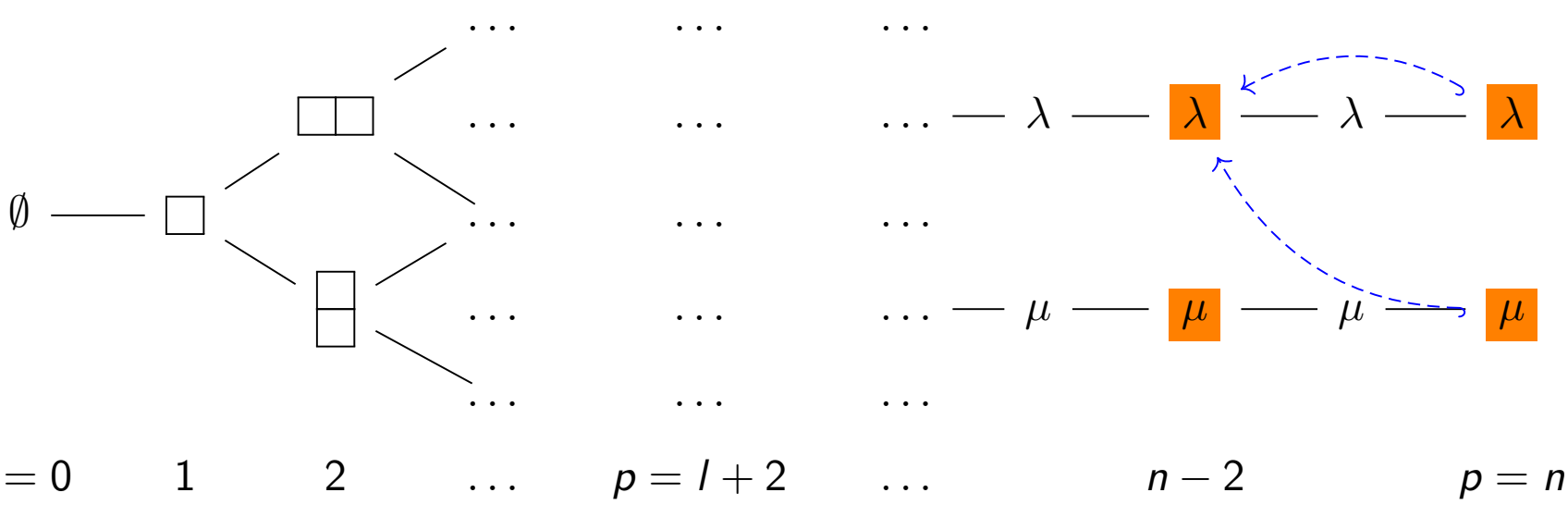
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Representation Theory: Semisimplicity

Q: When is $J_{l,n}$ semisimple?
A: When standards = simples. By tower of recollment frame work [Cox et al., 2004], enough to have no maps from "hand" to "wrist":



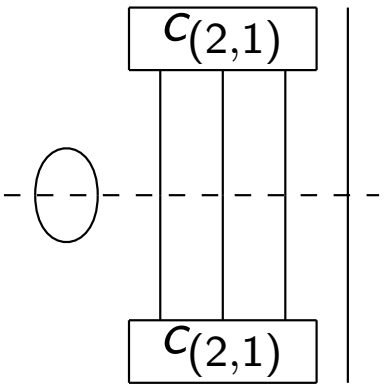
Reduce to looking for (Specht) submodules in $\Delta^n_{(n-2,\lambda)}$.
In turn this is reduced to studying non-degeneracy of a bilinear form

$$\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha].$$

Representation Theory: Bilinear Form

Q: When is $J_{l,n}$ semisimple?

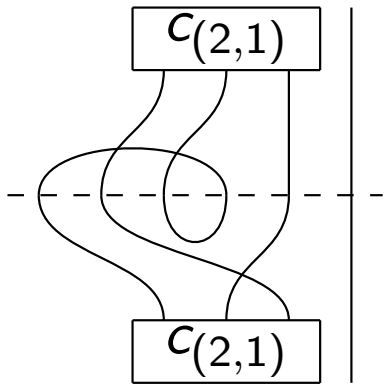
A: ... study non-degeneracy of $\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha]$.



$$= \alpha\, c^{(4)}_{(2,1)}\, id_4\, c^{(4)}_{(2,1)}$$

$$= \alpha\, c^{(4)}_{(2,1)}$$

$$\langle _, _ \rangle = \alpha,$$



$$= c^{(4)}_{(2,1)}\, (1\ 2\ 3)\, c^{(4)}_{(2,1)}$$

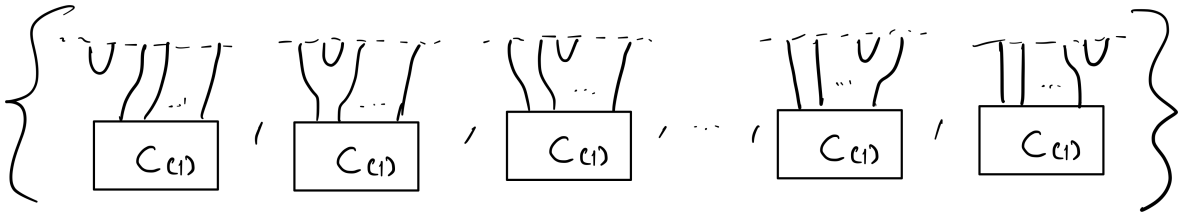
$$= -\frac{1}{2}\, c^{(4)}_{(2,1)}$$

$$\langle _, _ \rangle = -\frac{1}{2}.$$

Form is symmetric and contravariant with respect to $J_{l,n}$-action, *i.e.*

$$\langle \xi, y\zeta \rangle = \langle y^* \xi, \zeta \rangle.$$

Representation Theory: Bilinear Form - TL Case

$\mathcal{B} =$ 

Gram matrix of form:

$$\Gamma(\Delta_{(n-2,(1))}) = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow \begin{aligned} P_n^{(1)}(\alpha) &:= |\Gamma(\Delta_{(n-2,(1))})| \\ &= U_{n-1}(\alpha/2) \end{aligned}$$

Chebyshev recursion: $P_{n+1}^{(1)}(\alpha) = \alpha P_n^{(1)}(\alpha) - P_{n-1}^{(1)}(\alpha)$,

$$P_3^{(1)} = \alpha^2 - 1, \qquad P_4^{(1)} = \alpha(\alpha^2 - 2), \qquad P_4^{(1)} = \left(\alpha^2 - \alpha - 1\right)\left(\alpha^2 + \alpha - 1\right).$$

Roots: $P_n^{(1)}(\alpha) = 0$ at $\alpha = q + q^{-1}$ and $q^{2n} = 1$ (or $\alpha = 2\cos(k\pi/n)$).

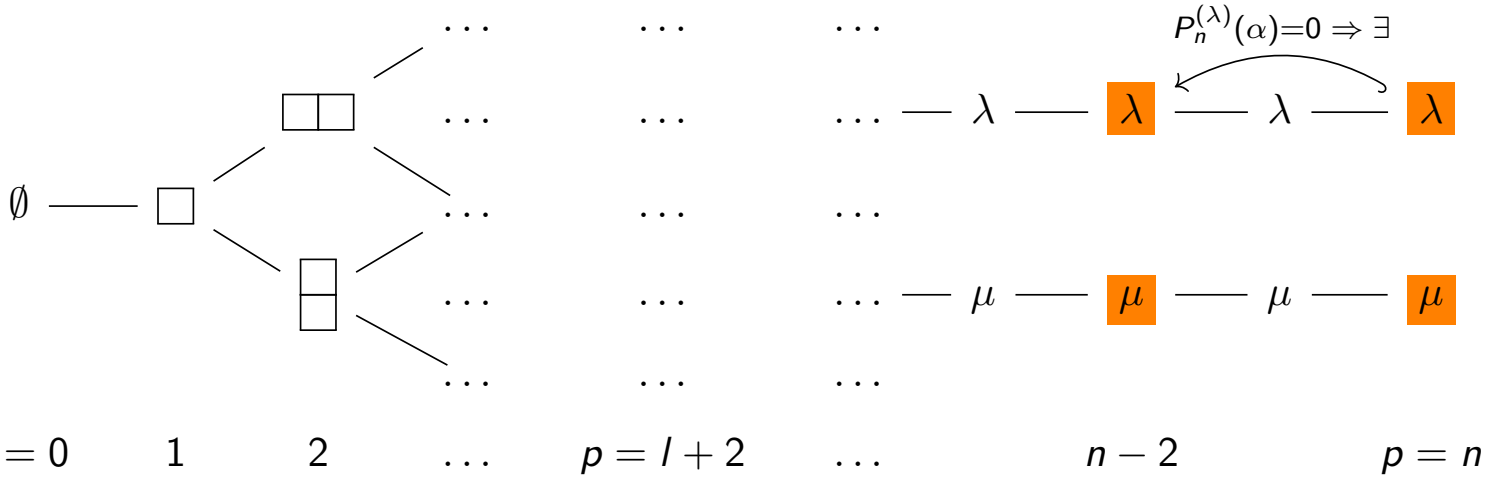
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Morris and Martin, 2025] - Generalised Roots of Unity

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

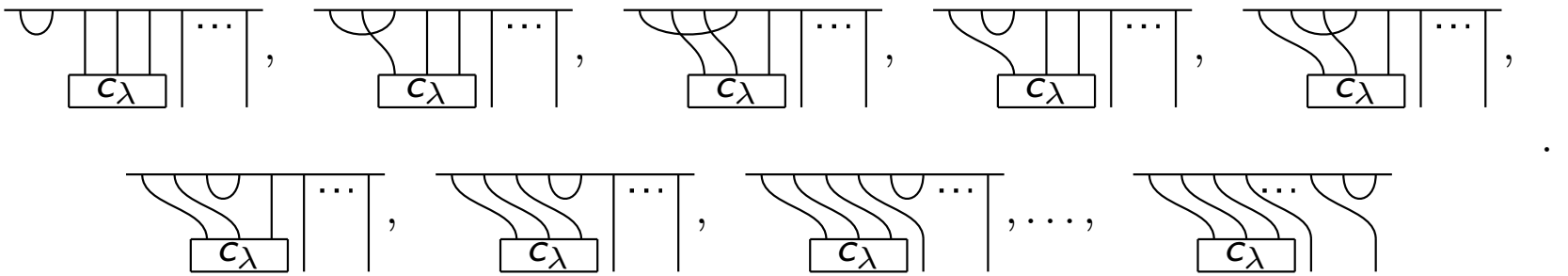
$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**. Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.



Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, *e.g.* $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



Need to compute determinant of:

$$\left(\begin{array}{ccccccc|cccccccc}
\alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
u_{12}b \\ u_{13}b \\ u_{14}b \\ u_{23}b \\ u_{24}b \\ u_{34}b \\ u_{45}b \\
u_{56}b \\ u_{67}b \\ u_{78}b \\ u_{89}b \\ u_{9,10}b \\ u_{10,11}b \\ u_{11,12}b \\ u_{12,13}b
\end{array}
\begin{array}{l}
\} \text{ head} \\
\\
\\
\\
\\
\\
\\
\} \text{ tail}
\end{array},$$

By expanding along last row we see the determinant satisfies Chebyshev recursion in n. We therefore, obtain the desired form: $\det = C^{(\lambda)}(\alpha)P_n^{(\lambda)}(\alpha)$.

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)$$

k_1 (rows 8–10) k_2 (rows 19–21)

This has two Chebyshev expansions: $\det = C^{(\lambda)}(\alpha)\, P^{(\lambda)}_{n_1}(\alpha) = D^{(\lambda)}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$.
Need to

1. Show $P^{(\lambda)}_n(\alpha) = Q^{(\lambda)}_n(\alpha)$. (Interesting)
2. Show $P^{(\lambda)}_{n_1}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$ divides determinant. (Boring)

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

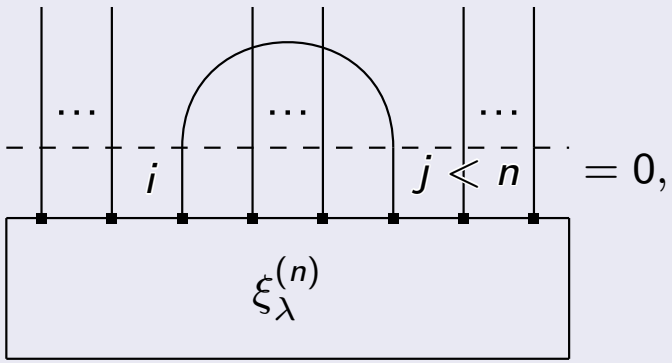
1. Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
2. Show $P_{n_1}^{(\lambda)}(\alpha)\, Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

To prove 1. we show both series have the same initial conditions by a row reduction procedure... uses a nice family of elements $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$.

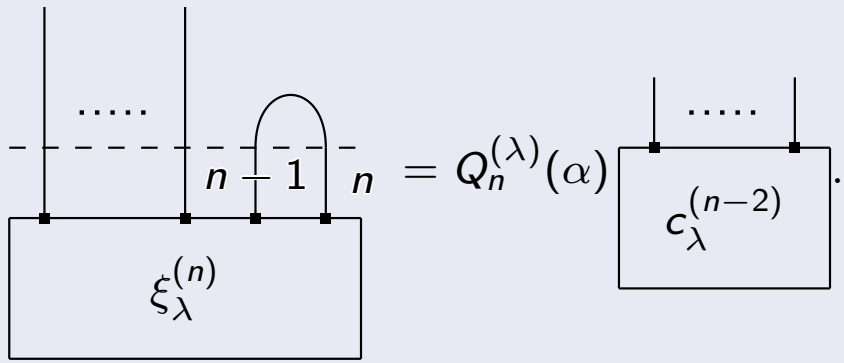
Lemma

For each $n \geq l+4$ there exists a unique $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$ (polynomial coefficients) such that:

1. *$\xi_\lambda^{(n)}$ is non-zero for all specialisations $\alpha \in \mathbb{C}$,*
2. *We have (in $\Delta^{n-2}_{(n-2,\lambda)}$)*



$= 0,$



$= Q_n^{(\lambda)}(\alpha)$

.

When $\alpha \in \mathbb{C}$ s.t. $Q_n^{(\lambda)}(\alpha) = 0$, $\xi_\lambda^{(n)}$ generates a submodule $J_{l,n}\xi_\lambda^{(n)} \simeq \mathcal{S}_\lambda \subset \Delta^n_{(n-2,\lambda)}$. □

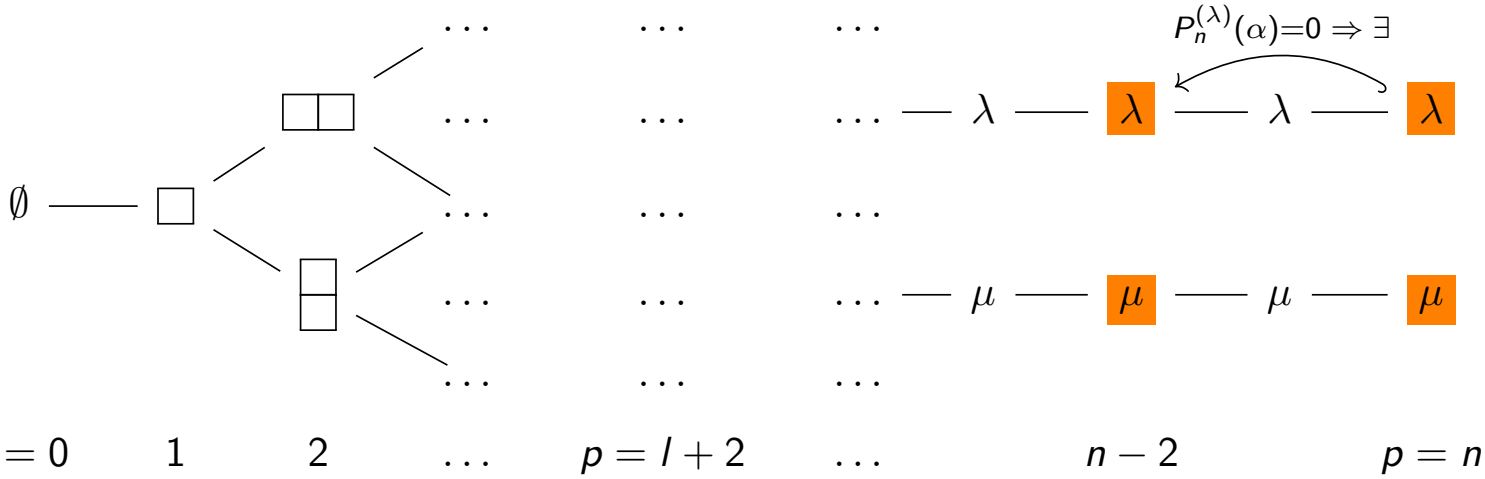
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**. Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.



Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



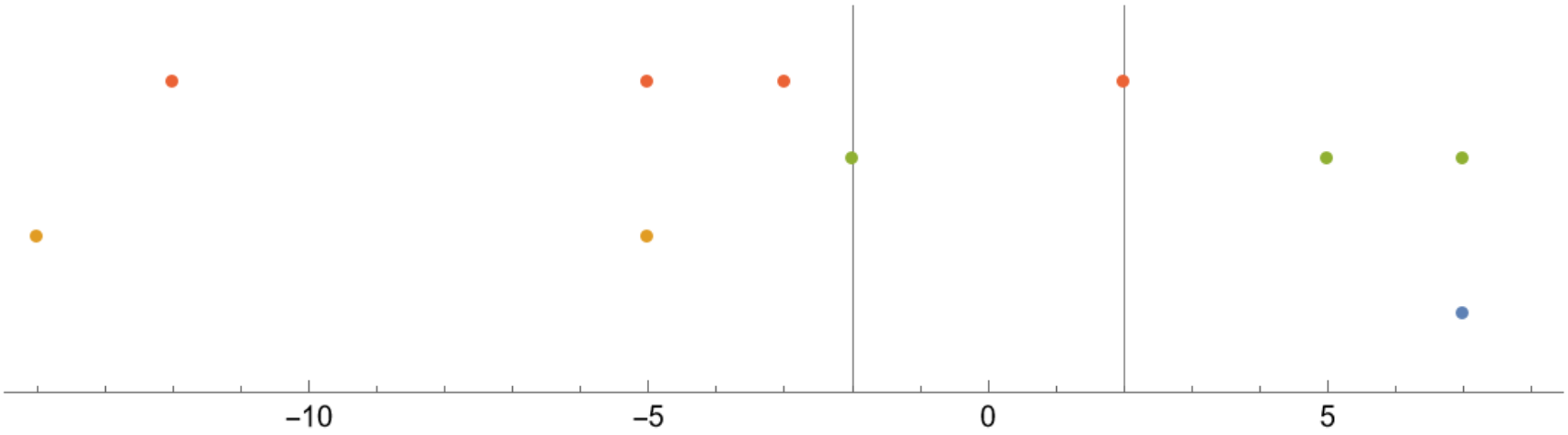


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = -1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



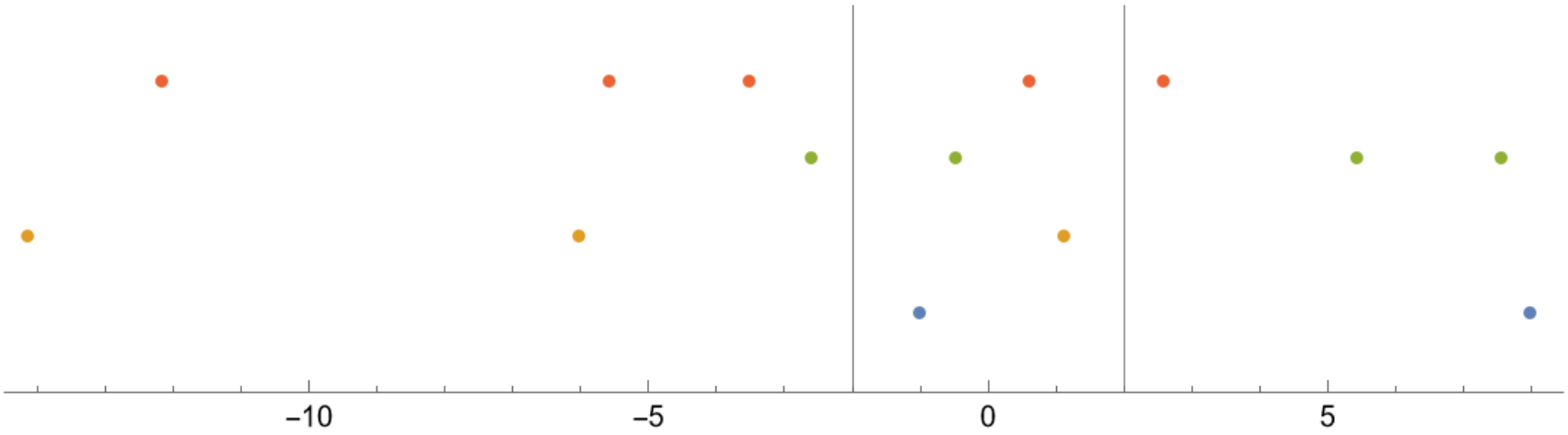


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 0$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



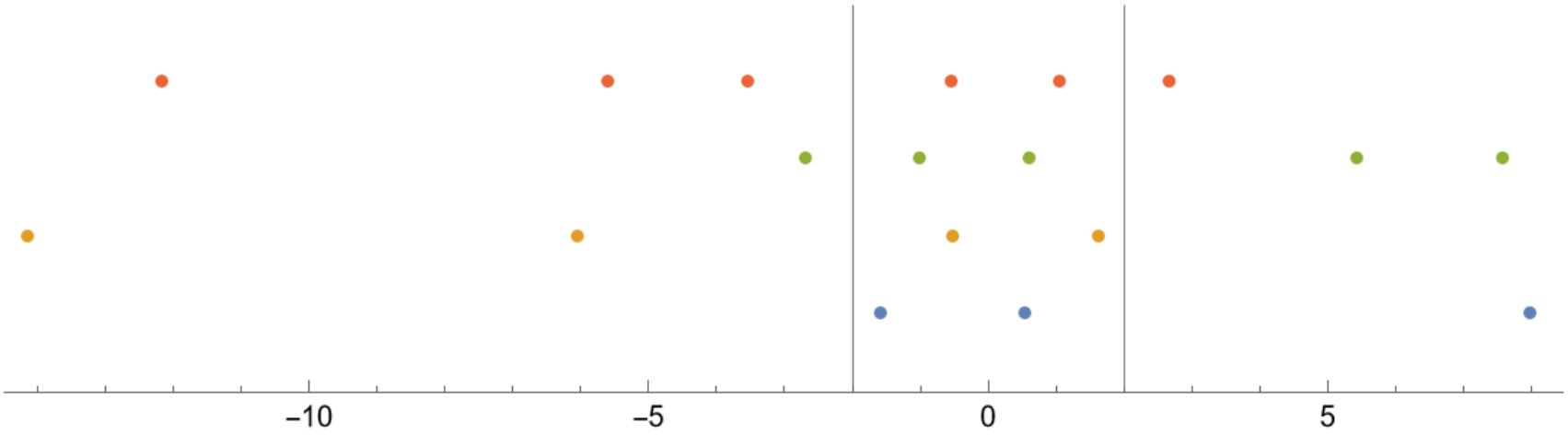


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



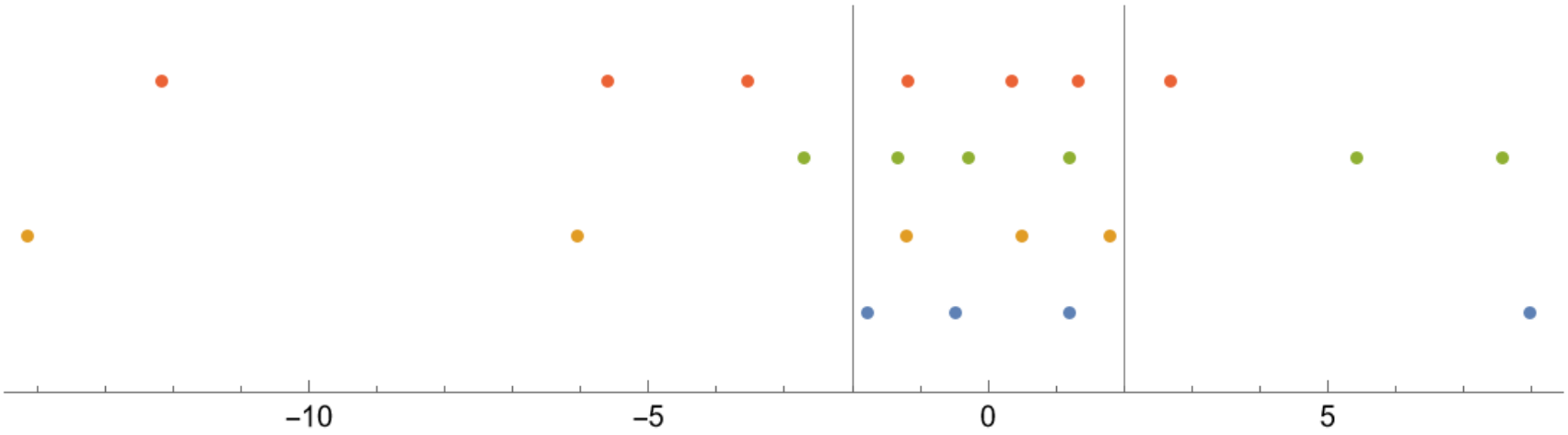


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 2$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



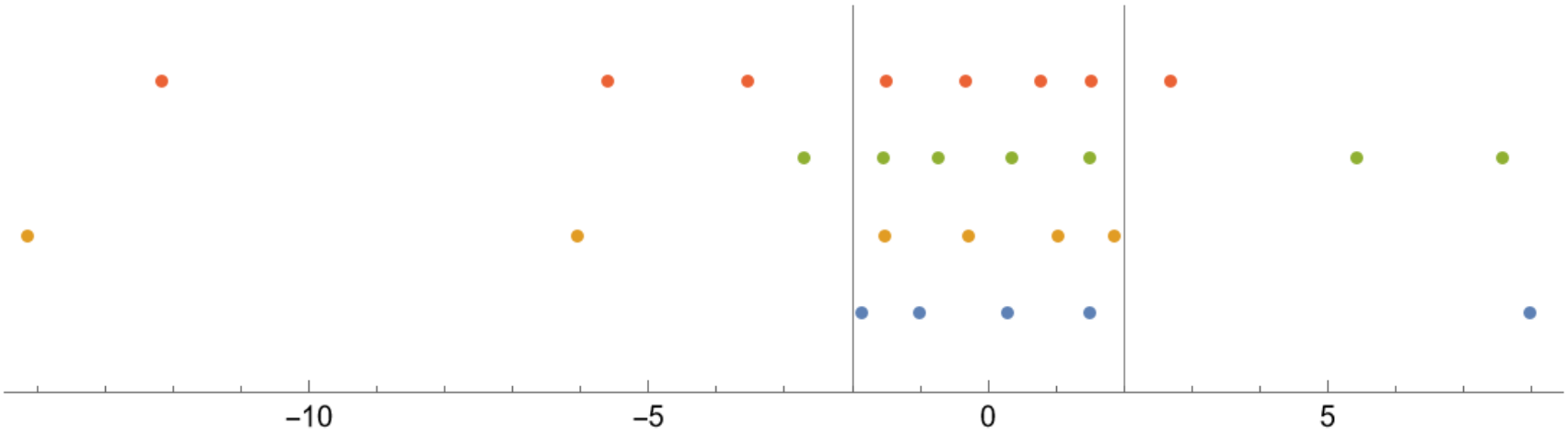


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 3$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



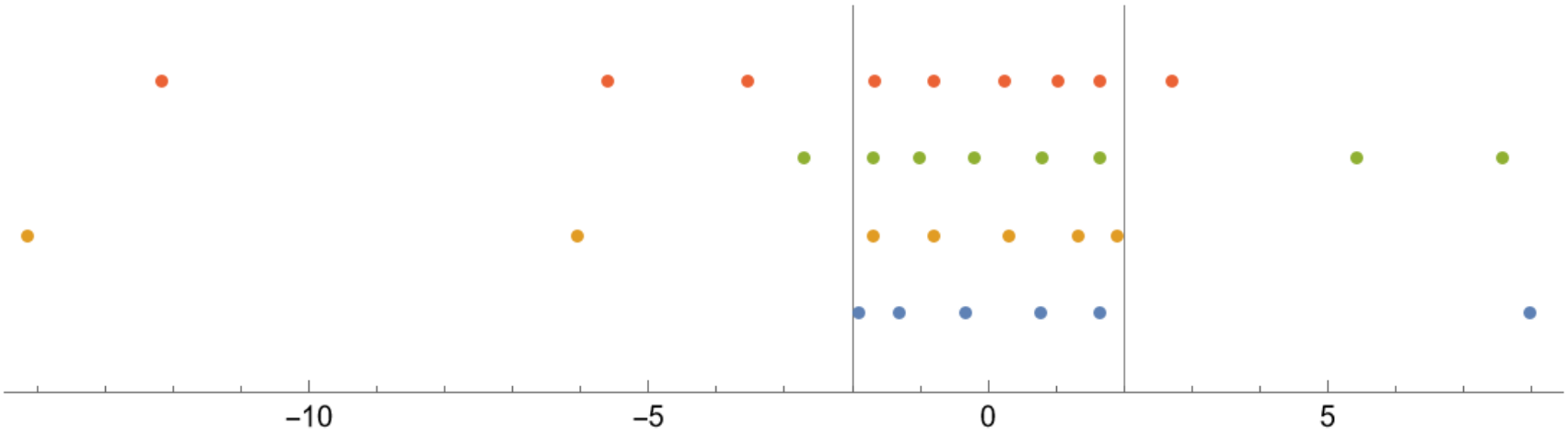


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 4$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



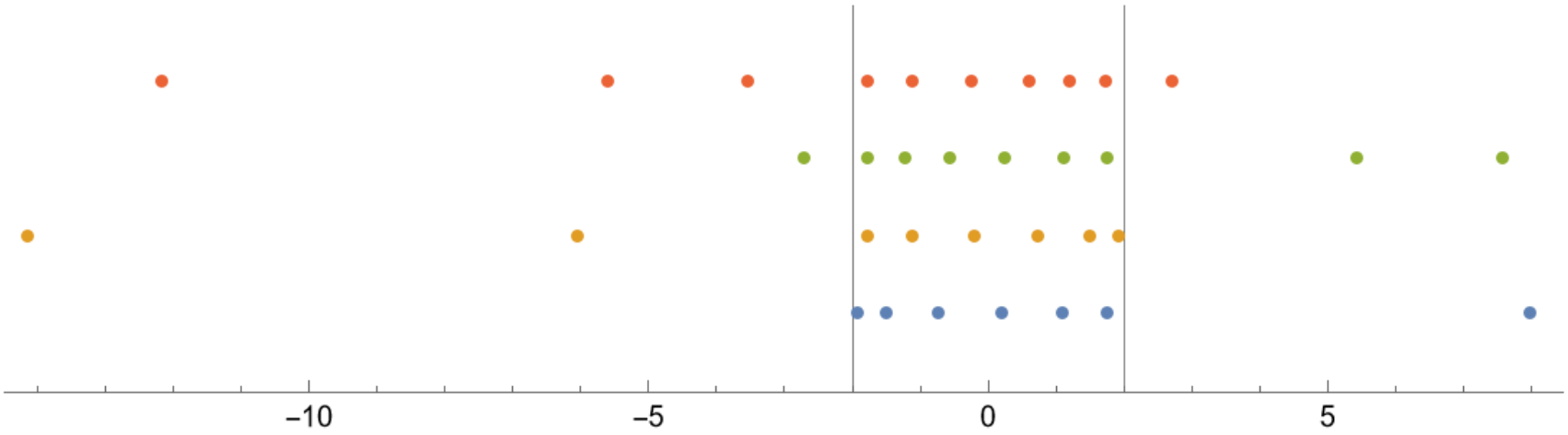


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 5$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



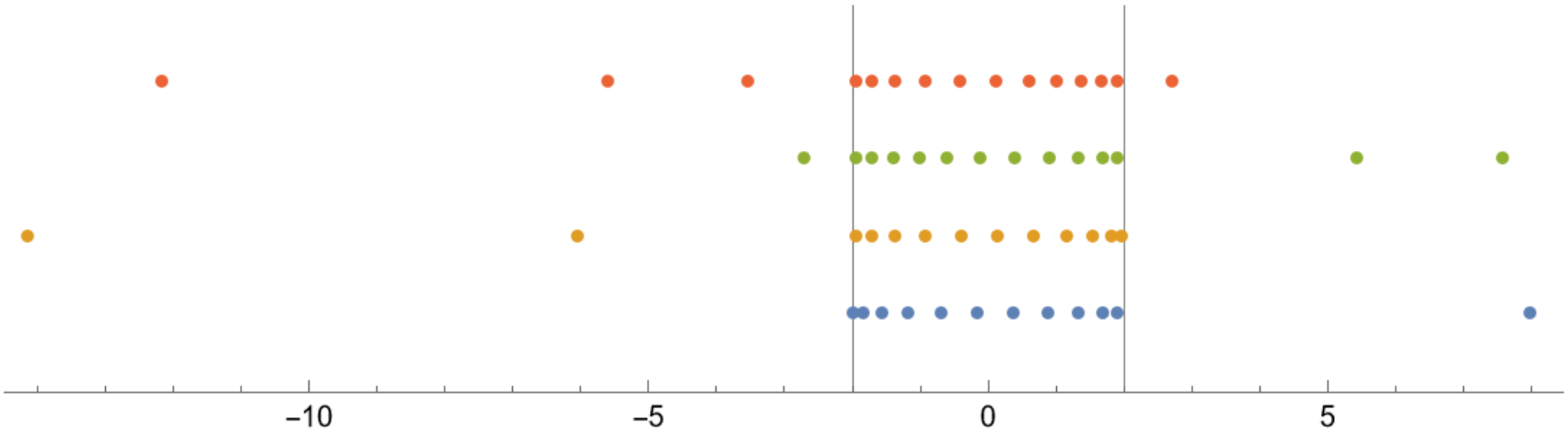


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 10$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



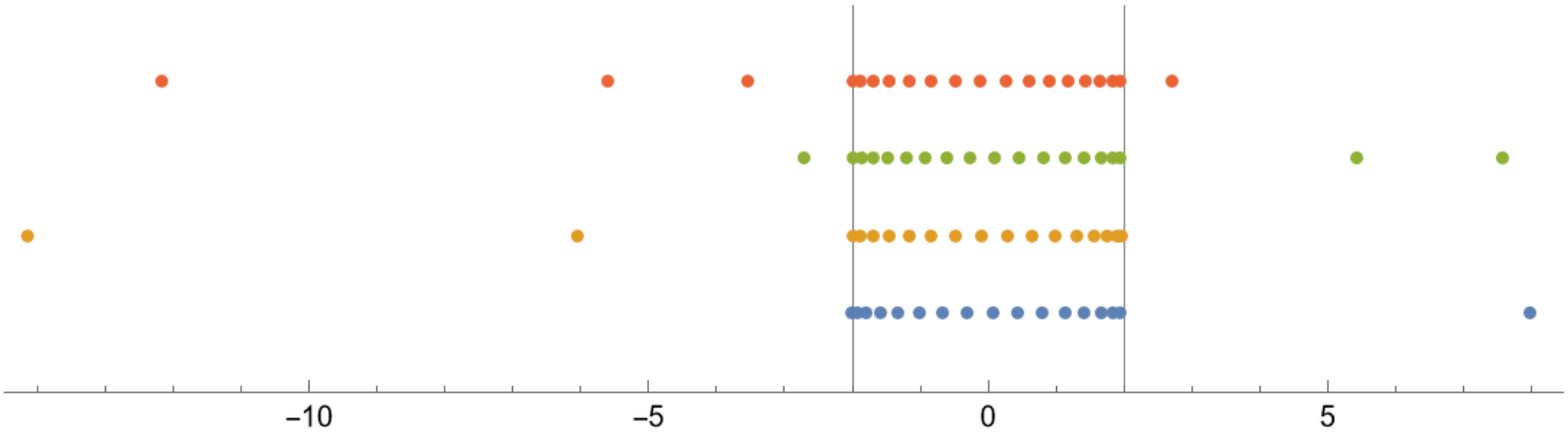


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 15$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



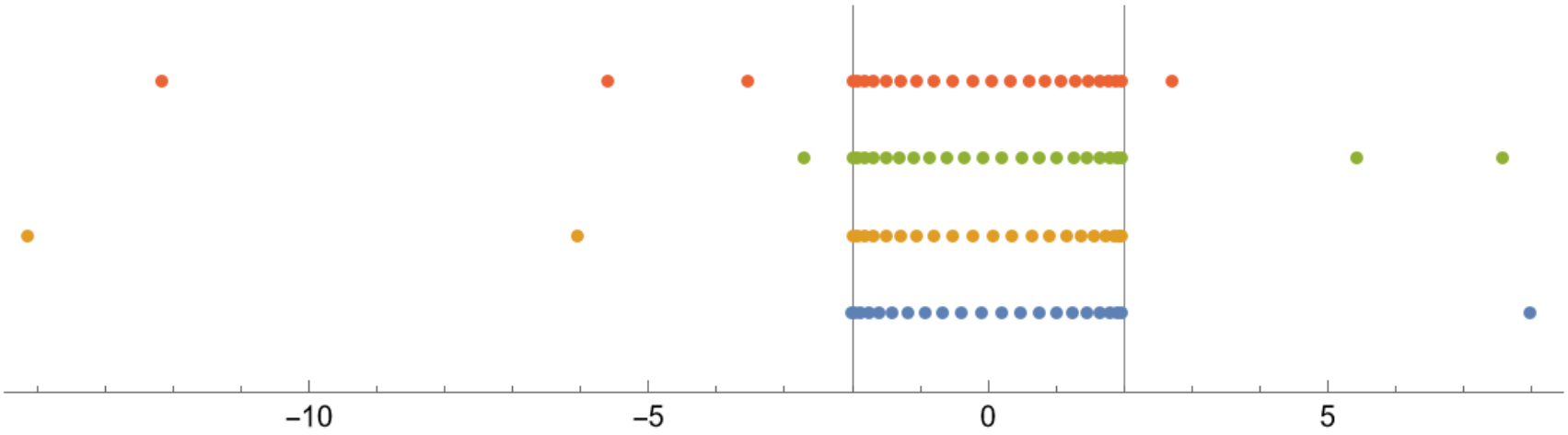


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 20$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



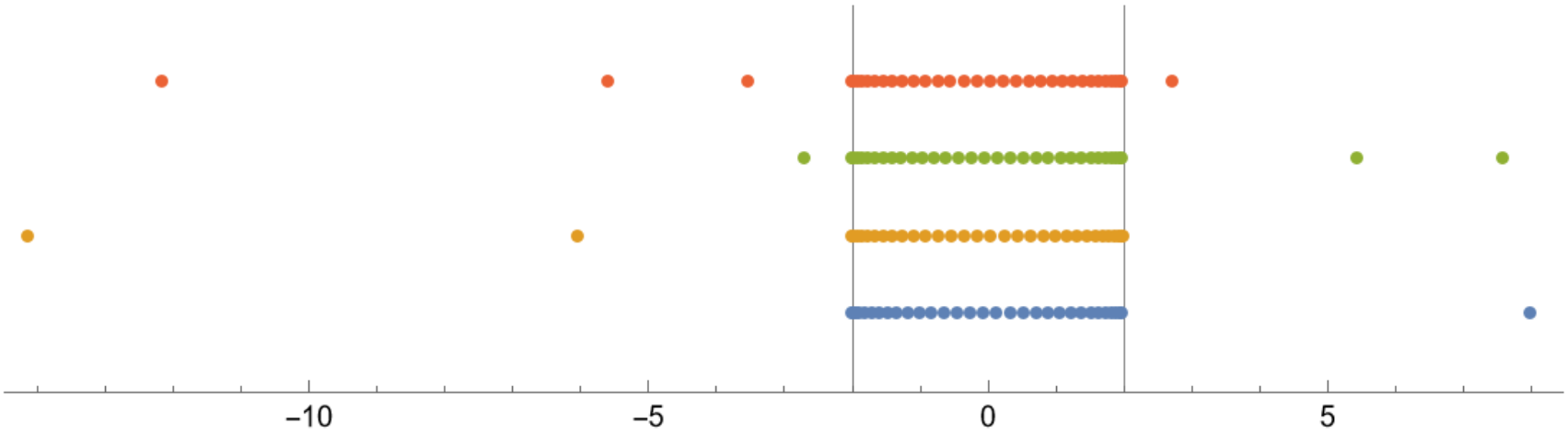


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 30$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



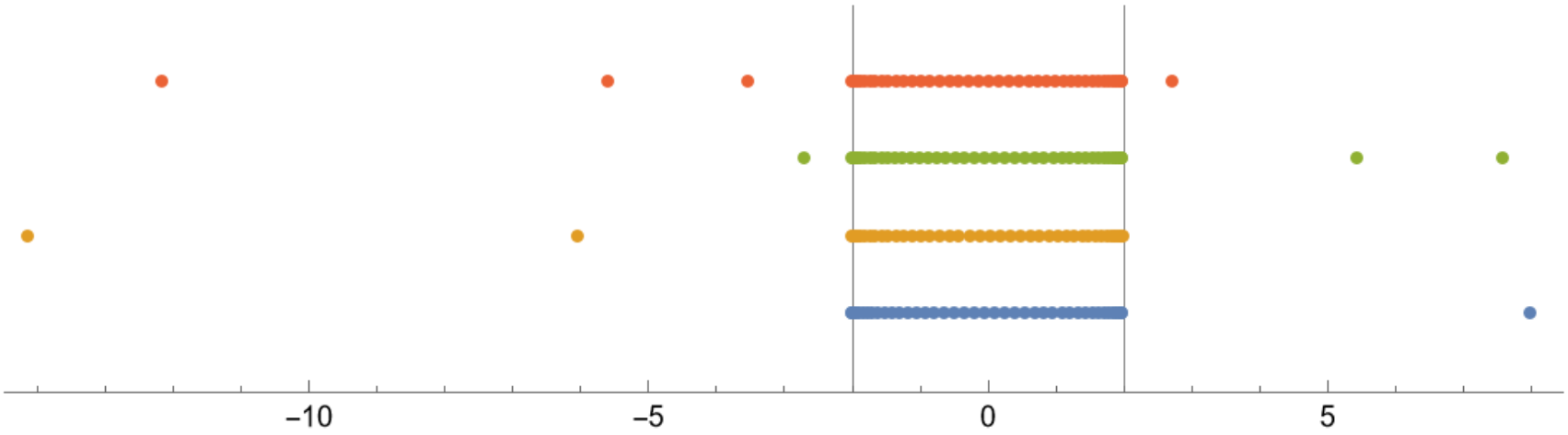


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 40$

Rep. Theory - Generalised Roots of Unity

Lets take a look at a series and run it backwards...

k	$P^{(5,1)}_{l+4+k}$
$\vdots$	$\vdots$
-3	$5\alpha^4+32\alpha^3+\alpha^2-50\alpha+48$
-2	$6(\alpha+2)\left(\alpha^2+5\alpha-8\right)$
-1	$(\alpha-2)(\alpha+1)(\alpha+3)(\alpha+8)^*$
0	$\alpha^5+10\alpha^4+5\alpha^3-88\alpha^2-60\alpha+96$
1	$\alpha^6+10\alpha^5+4\alpha^4-98\alpha^3-71\alpha^2+142\alpha+48$
2	$(\alpha+1)\left(\alpha^6+9\alpha^5-6\alpha^4-102\alpha^3+26\alpha^2+204\alpha-96\right)$
$\vdots$	$\vdots$

*In all cases computed, divides irreducible dimension polynomials for orthogonal groups.

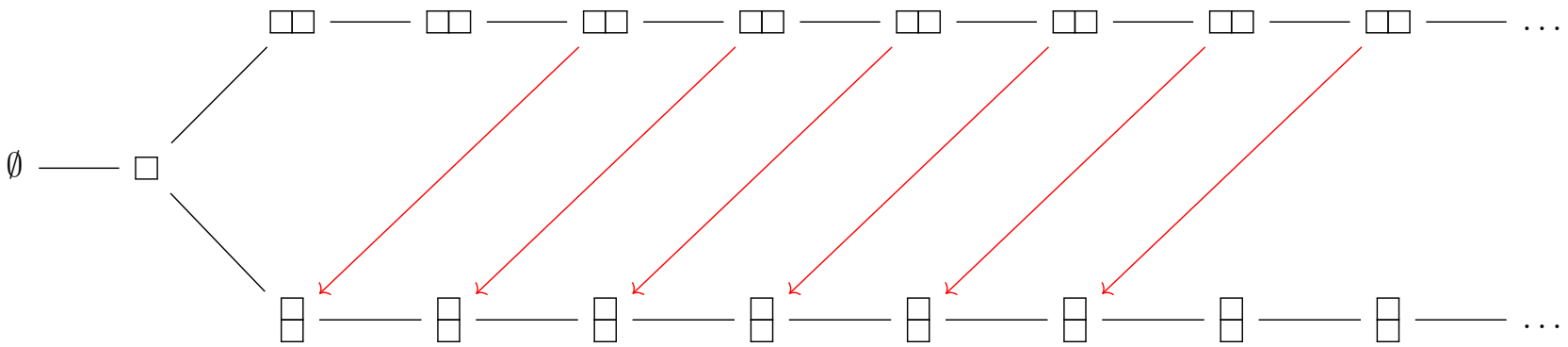
Coincidences? All data contained in the elements $\xi^{(n)}_{\lambda}$...

Representation Theory: Bilinear Form - Det. Factorisation

Computed for all $\lambda \vdash l + 2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).
Q: What about the constant piece of the factorisation?
Example: For $l = 0$ we have $\lambda = (2), (1^2)$, $d_\lambda = 1$

$$C^{(2)}(\alpha) = (\alpha - 1), \qquad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

Zeros correspond to maps "across arms" i.e. at $\alpha = -2$ we have $\Delta^n_{(n,(2))} \hookrightarrow \Delta^n_{(n-2,(1^2))}$.



Representation Theory: Bilinear Form - Det. Factorisation

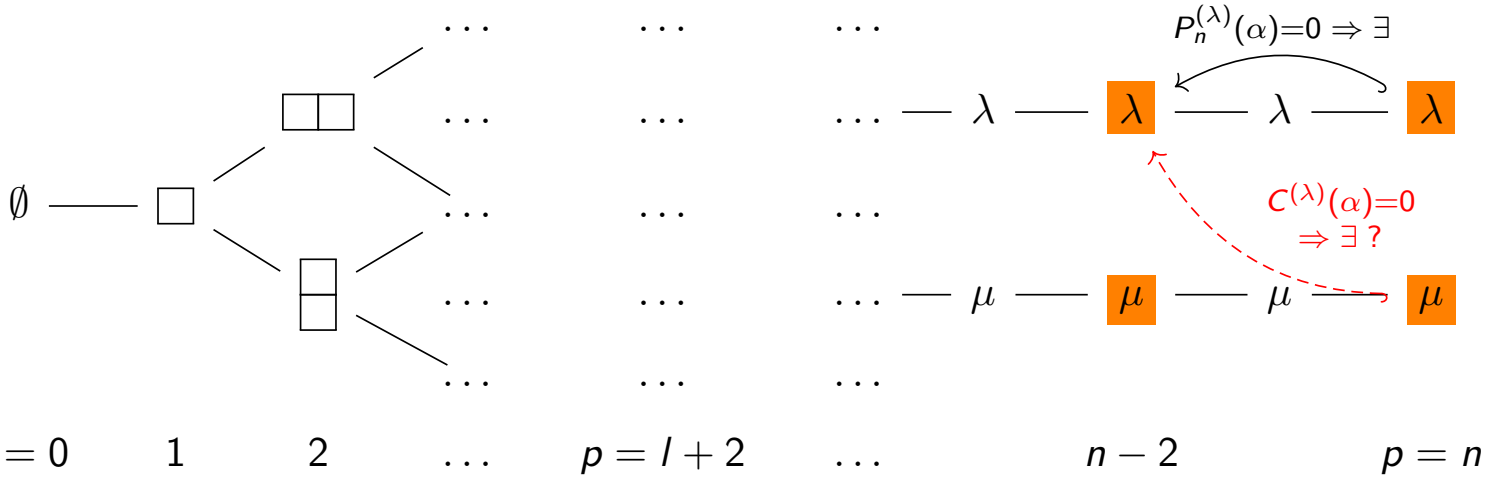
Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda} \dots$$

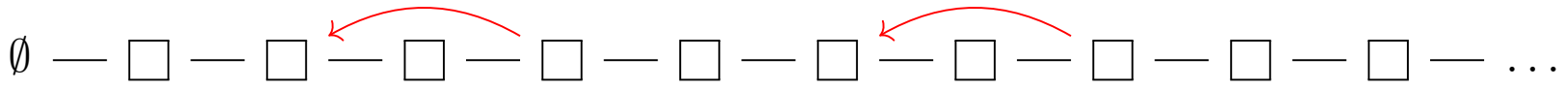
Conjecture

The $C^{(\lambda)}(\alpha)$ have integer roots which correspond to "non-TL behaviour".

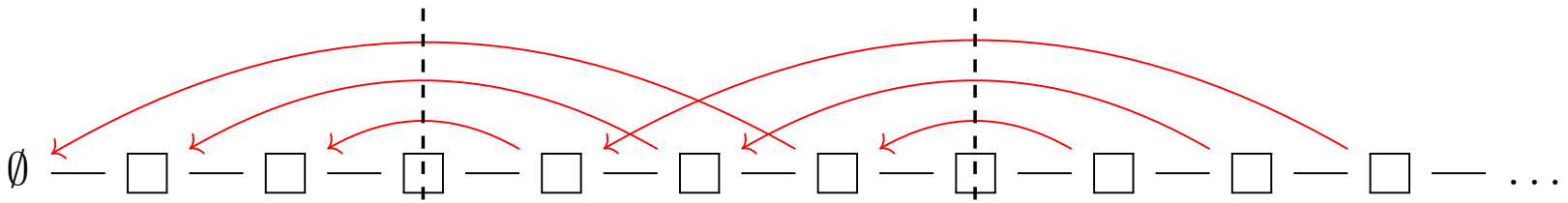


Representation Theory: Reflection Geometry

In the TL and Brauer cases maps between standard modules can be described by reflection geometries.
For example, consider TL at $\alpha = [2]_q$ where $[4]_q = 0$. First put "hand" to "wrist" maps on the Rollet:



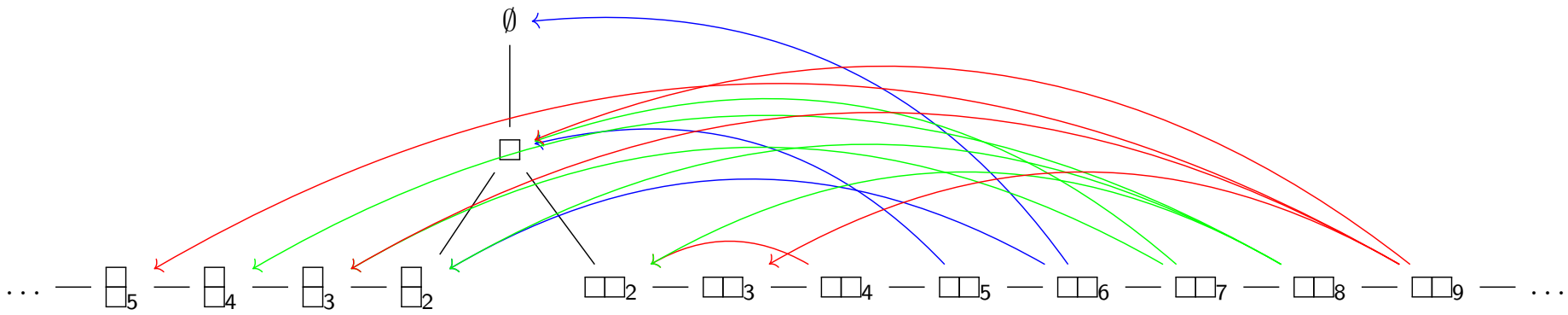
The remaining maps can be determined by frobenius reciprocity:



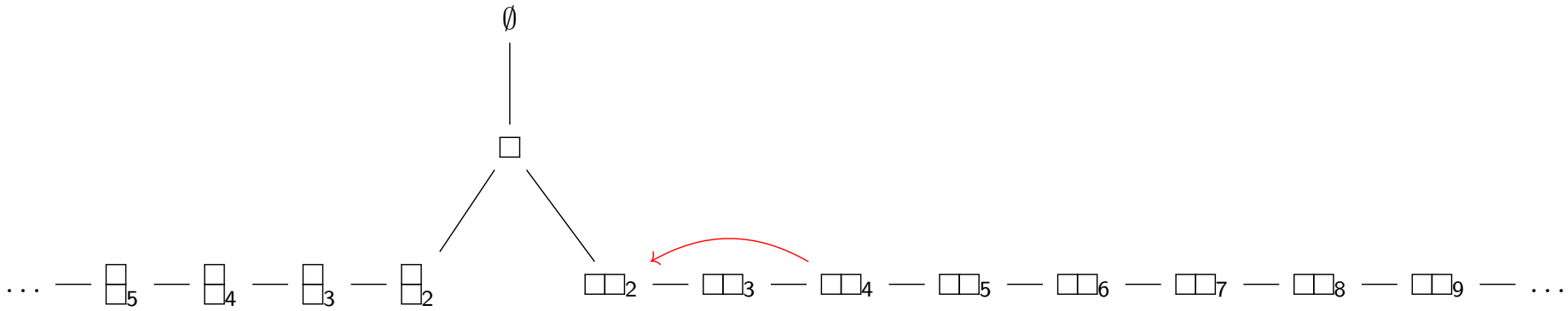
The Block structure can be realised by the action of affine reflections on the Rollet graph! (In this case it is 1D)

Rep. Theory: Reflection Geometry beyond TL?

Let's consider $l = 0$. We have $P_4^{(2)} = \alpha(\alpha^2 + \alpha - 4) = 0$ - we take α such that $(\alpha^2 + \alpha - 4) = 0$:

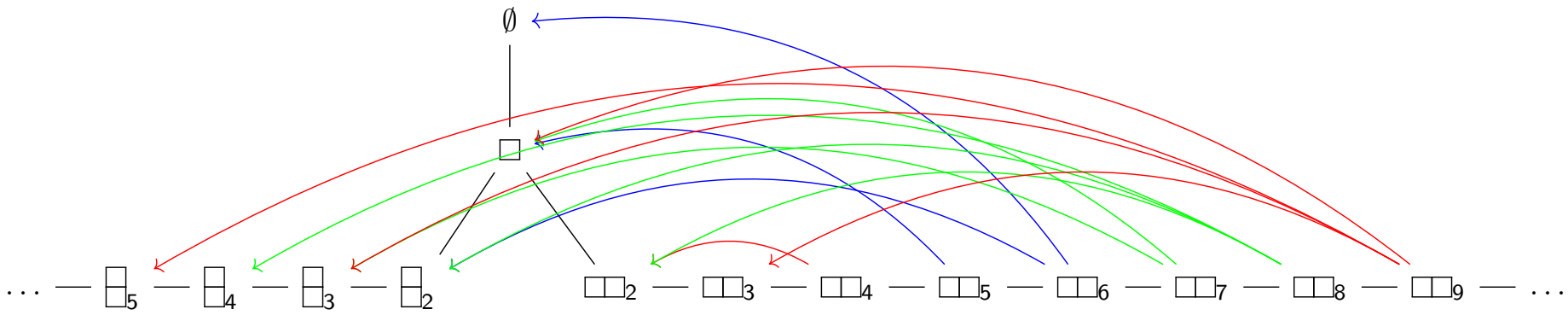


Let's slow it down...

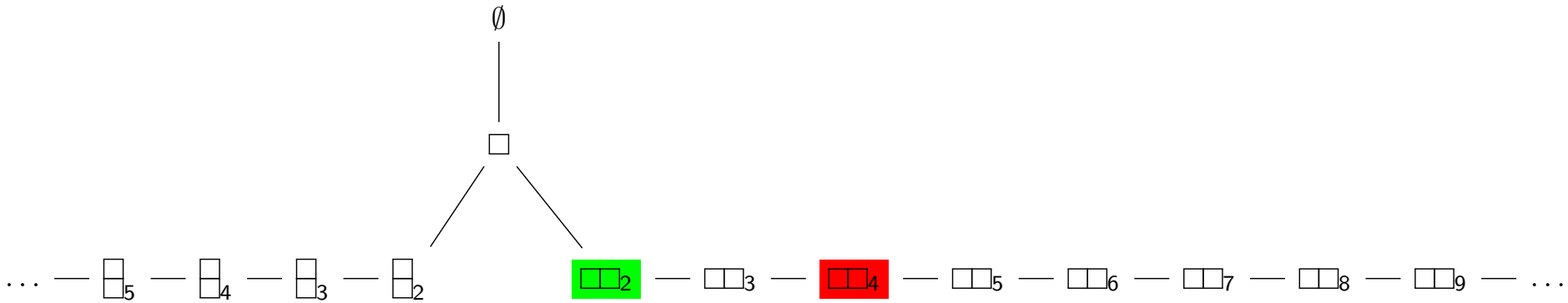


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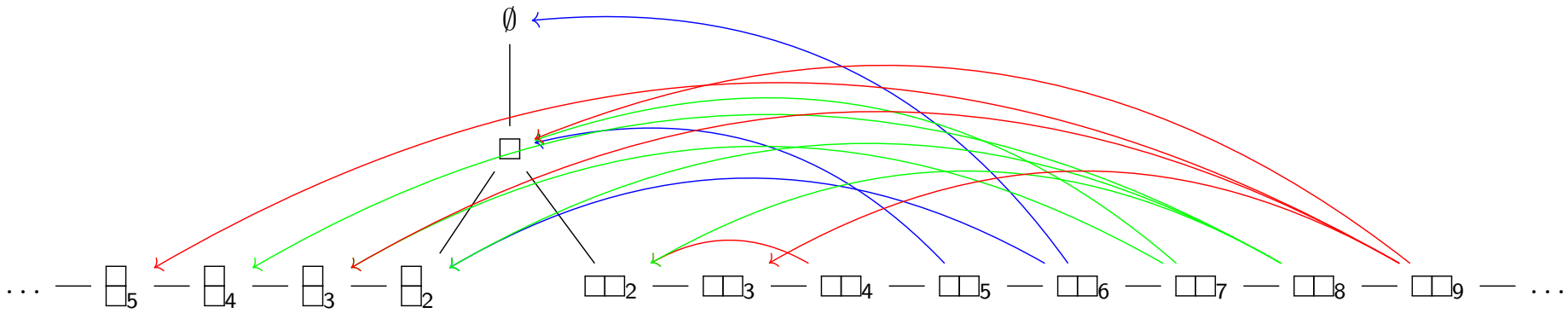


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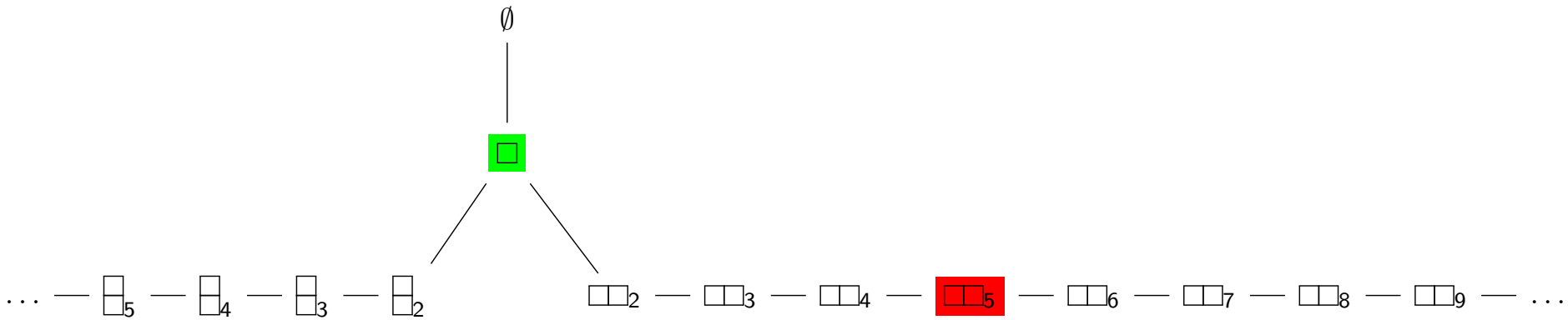


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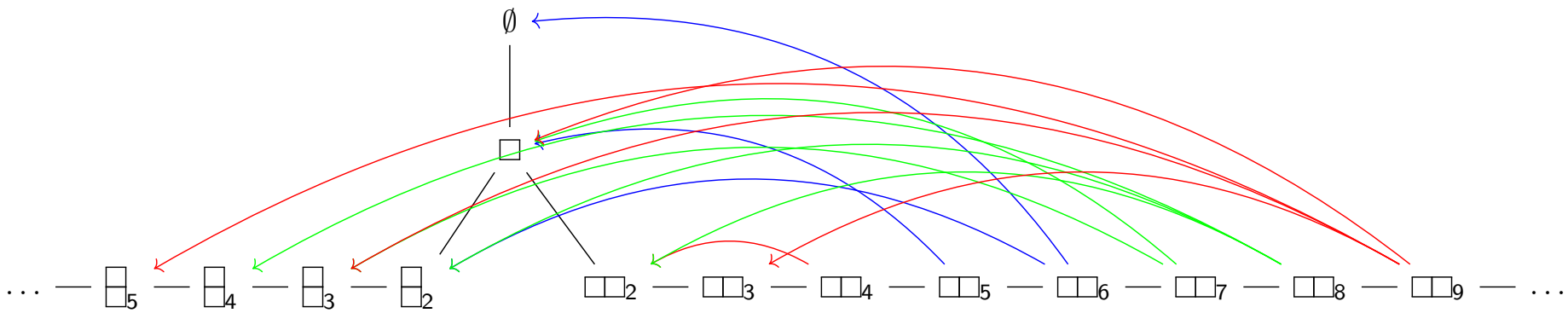


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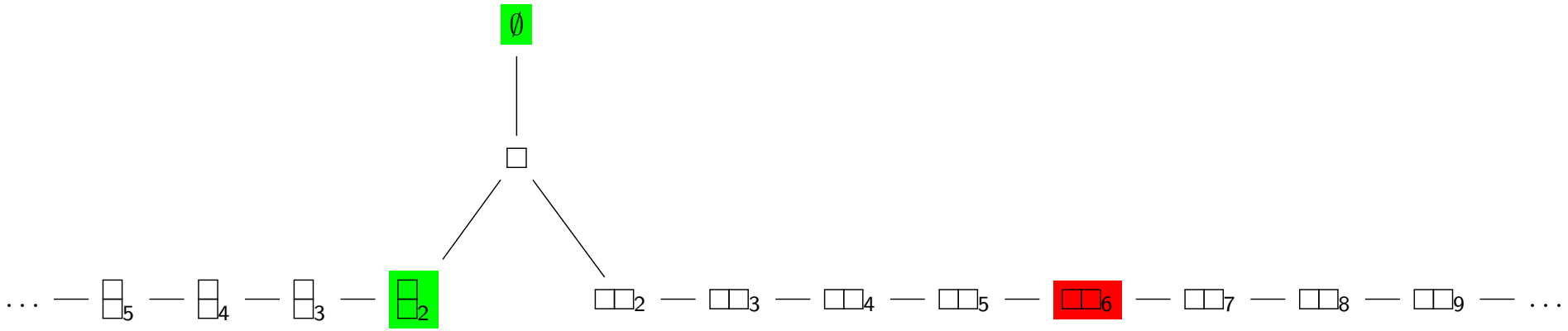


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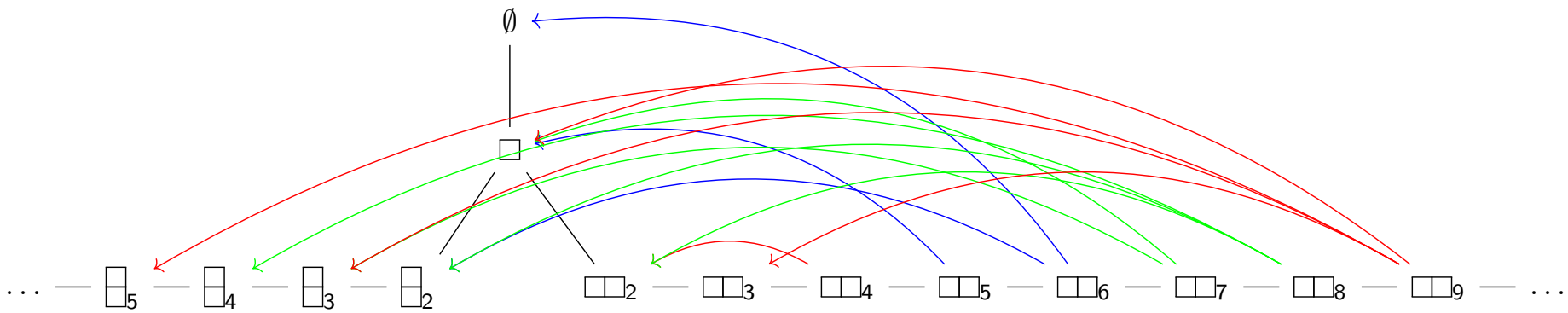


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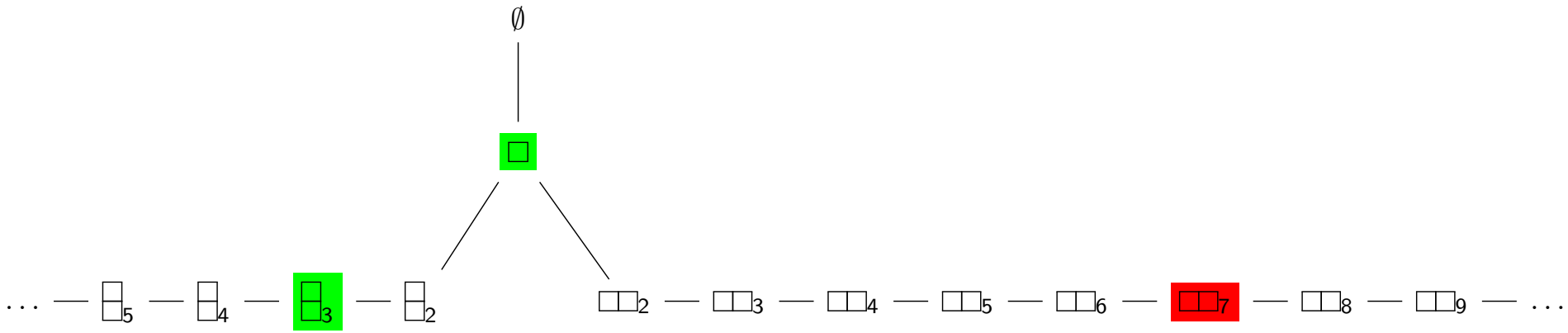


Rep. Theory: Reflection Geometry beyond TL?

Let's consider $l = 0$. We have $P_4^{(2)} = \alpha(\alpha^2 + \alpha - 4) = 0$ - we take α such that $(\alpha^2 + \alpha - 4) = 0$:

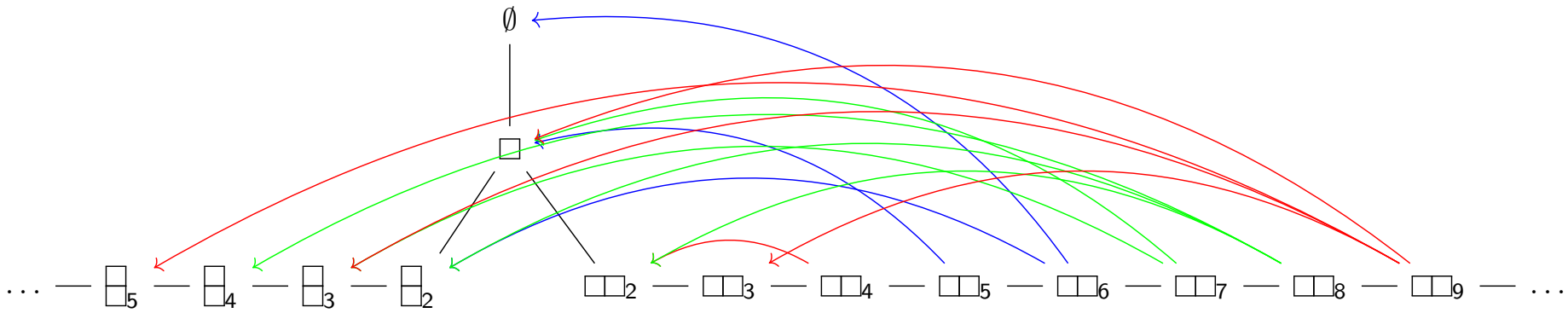


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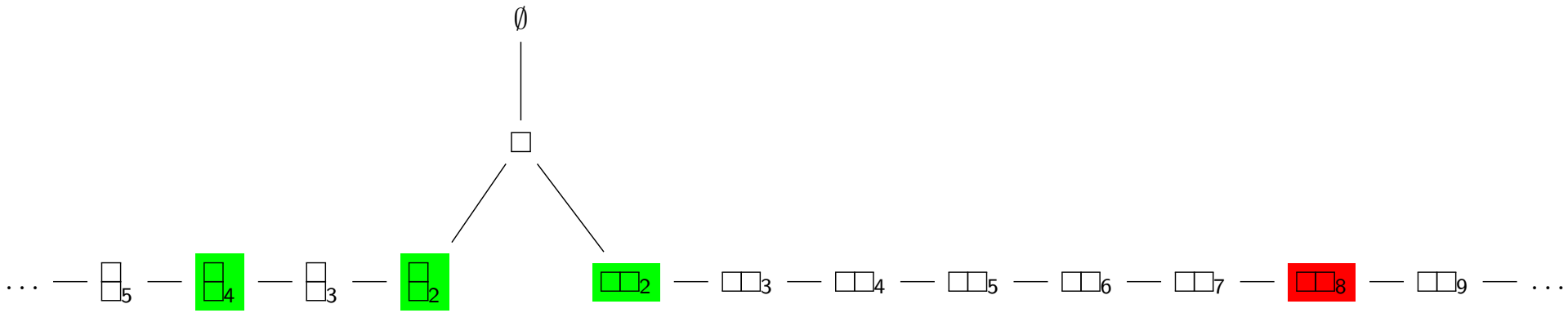


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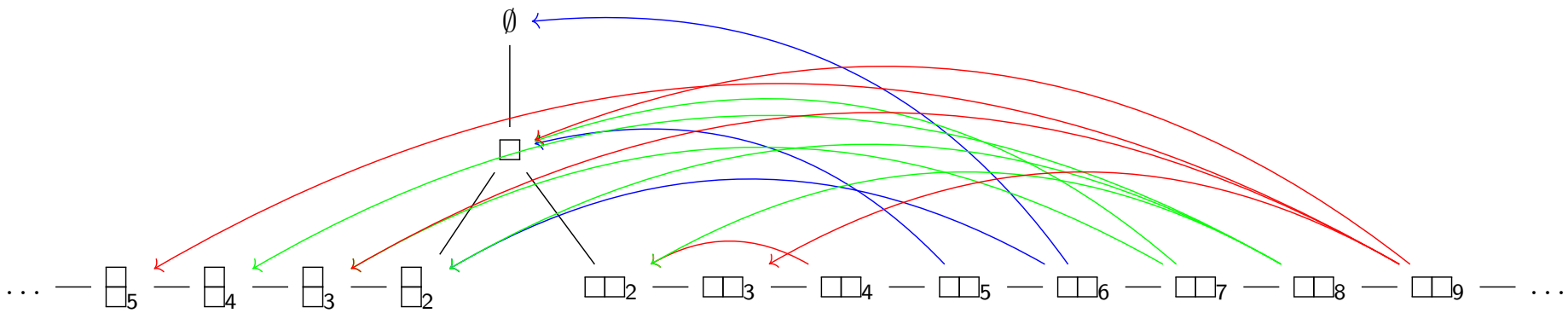


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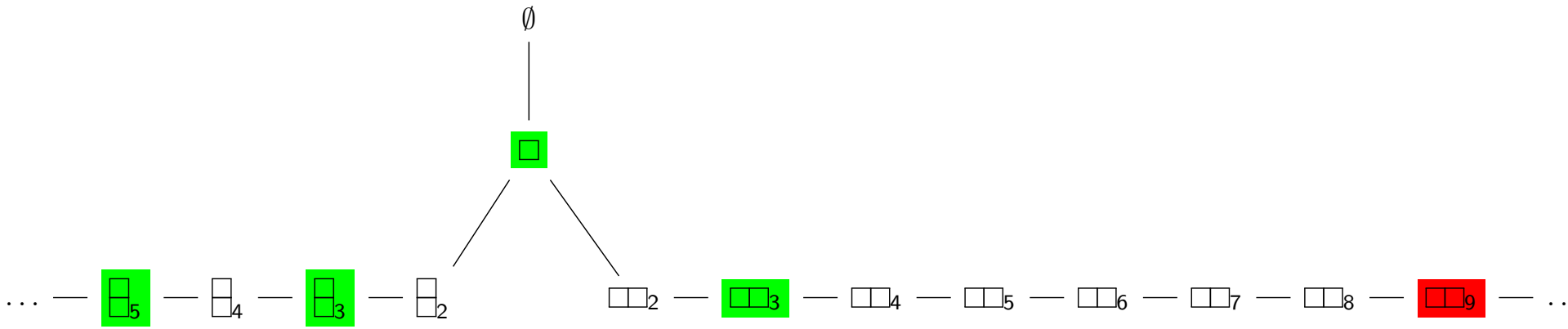


Rep. Theory: Reflection Geometry beyond TL?

Let's consider $l = 0$. We have $P_4^{(2)} = \alpha(\alpha^2 + \alpha - 4) = 0$ - we take α such that $(\alpha^2 + \alpha - 4) = 0$:



Let's slow it down...



Thank You!

Questions?

References:

- Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv 2407.07028. arXiv: 2407.07028 [math.RT]. URL: https://arxiv.org/abs/2407.07028.
- Cox, Anton et al. (2004). *Representation theory of towers of recollement: theory, notes, and examples*. arXiv 0411395. arXiv: math/0411395 [math.RT].
- Kadar, Zoltan, Paul P. Martin, and Shona Yu (2014). *On geometrically defined extensions of the Temperley-Lieb category in the Brauer category*. arXiv 1401.1774. DOI: 10.48550/ARXIV.1401.1774. URL: https://arxiv.org/abs/1401.1774.
- Morris, Benjamin and Paul P. Martin (2025). *On semisimplicity criteria and non-semisimple representation theory for the Kadar-Yu algebras*. arXiv: 2512.24535 [math.RT]. URL: https://arxiv.org/abs/2512.24535.