Towards a factorised solution of the Yang-Baxter equation with $U_q(\mathfrak{sl}_n)$ symmetry

Yang-Baxter equations and parameter permutations

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Outline

1. Lie algebra $\mathfrak{sl}_n$
   - Differential realisation of $\mathfrak{sl}_n$
   - Factorised L-operator for $\mathfrak{sl}_n$
2. Quantum Group $U_q(\mathfrak{sl}_n)$
   - q-difference realisation and L-operator
   - $n = 4$ case
3. Yang-Baxter equation and Parameter Permutations

Lie algebra $\mathfrak{sl}_n$

The Lie algebra $\mathfrak{sl}_n$

For $n > 1$, $\mathfrak{sl}_n$ is the complex Lie algebra generated by elements e_i, f_i, h_i for $i = 1, \ldots, n-1$ subject to the Chevalley-Serre relations:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_i,$$
$$(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) = 0, \quad \text{for } i \neq j,$$

where a_{ij} are components of the A_n-type Cartan matrix.

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where a_{ij} are components of the A_n-type Cartan matrix.

- Cartan-Weyl basis: A basis of root vectors for $\mathfrak{sl}_n$, $\{E_{ij} \mid 1 \leq i, j \leq n\}$ defined iteratively by

$$h_i = E_{ii} - E_{i+1,i+1}, \quad \sum_{i=1}^{n} E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i,$$
$$E_{ij} = [E_{i,j-1}, E_{j-1,j}], \quad \text{for } j > i+1,$$
$$E_{ij} = [E_{i,i-1}, E_{i-1,j}], \quad \text{for } j < i-1.$$

Differential realisation of $\mathfrak{sl}_n$

The algebra $\mathfrak{sl}_n$ has a representation on $\mathcal{V}^{(n)}$, the space of polynomials in $n(n-1)/2$ variables (x_{ij} for $1 \leq j < i \leq n$)[1]

[1] S. E. Derkachov and A. N. Manashov. "*R*-Matrix and Baxter *Q*-Operators for the Noncompact $SL(N,C)$ Invariant Spin Chain". In: *Symmetry, Integrability and Geometry: Methods and Applications* (Dec. 2006).

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$$E_{ji} = (Z\, D(\boldsymbol{\rho})\, Z^{-1})_{ij},$$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad \tilde{D}(\boldsymbol{\rho}) = \begin{pmatrix} -\rho_n & \tilde{D}_{12} & \tilde{D}_{13} & \dots & \tilde{D}_{1n} \\ & -\rho_{n-1} & \tilde{D}_{23} & \dots & \tilde{D}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & -\rho_2 & \tilde{D}_{n-1,n} \\ & & & & -\rho_1 \end{pmatrix},$$

and $\tilde{D}_{ij} := -\partial_{ji} - \sum_{k=j+1}^{n} x_{kj}\partial_{ki}$.

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$$e = x(m + N_x), \quad f = -\partial_x, \quad h = m + 2N_x, \quad (N_x = x\partial_x).$$

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- The vector $1 \in \mathcal{V}_{\boldsymbol{\rho}}^{(n)}$ satisfies $E_{ij}.1 = 0$ for lower triangular elements ($i > j$), and $h_i.1 = m_i.1$ where

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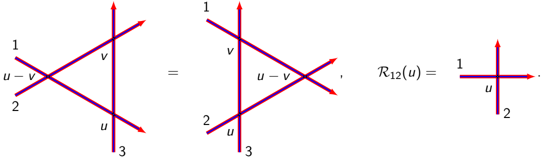
- If $m_i = -n_i \in \mathbb{Z}_{\leq 0}$ for all i, then $\mathcal{V}_{\boldsymbol{\rho}}^{(n)}$ contains a finite-dimensional irreducible submodule which is described by the Young tableau corresponding to the partition $(\ell_1, \ell_2, \dots, \ell_{n-1})$ where $\ell_j = \sum_{k=j}^{n-1} n_k$.

Differential realisation of $\mathfrak{sl}_n$

Goal is to solve YBE with an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}}^{(n)} \otimes \mathcal{V}_{\boldsymbol{\tau}}^{(n)})$, which we write graphically as



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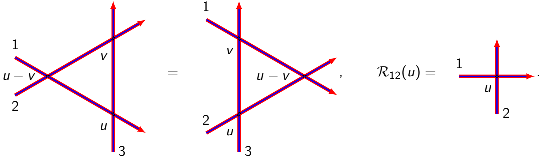
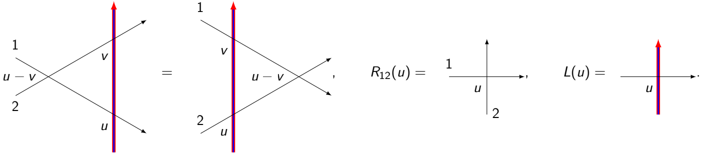


First step is to solve YBE in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}}^{(n)})$



Factorised L-operator for $\mathfrak{sl}_n$

Universal $\mathfrak{sl}_n$ L-operator $\tilde{L}(u) = uI + e_{ij} \otimes E_{ji} \in \mathrm{End}(\mathbb{C}^n) \otimes U(\mathfrak{sl}_n)$.
Satisfies the abstract "RLL"-relation

$$R_{12}(u-v)\tilde{L}_1(u)\tilde{L}_2(v) = \tilde{L}_2(v)\tilde{L}_1(u)R_{12}(u-v) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes U(\mathfrak{sl}_n)$$

with the defining $\mathfrak{sl}_n$ R-matrix $R = uI + P \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$.

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If we evaluate the second factor of $\tilde{L}(u)$ in the differential representation $\mathcal{V}_{\boldsymbol{\rho}}^{(n)}$ we obtain a factorised L-operator

$$L(u;\boldsymbol{\rho}) = Z(uI + \tilde{D}(\boldsymbol{\rho}))Z^{-1}.$$

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Spectral parameter u and representation parameters $\boldsymbol{\rho}$ are absorbed into the combinations $u_i := u - \rho_i$ on diagonals of the central factor $uI + \tilde{D}(\boldsymbol{\rho})$.
This is a useful parameterisation so write $L(\boldsymbol{u})$ and $uI + \tilde{D}(\boldsymbol{\rho}) := \tilde{D}(\boldsymbol{u})$.

The (Jimbo-Drinfeld) Quantum Group $U_q(\mathfrak{sl}_n)$

For a parameter $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$, $U_q(\mathfrak{sl}_n)$ $(n > 1)$ is the complex algebra generated by elements e_i, f_i, and the invertible $k_i = q^{h_i}$ for $i = 1, \dots, n-1$ subject to the relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm1} - (q + q^{-1}) g_i g_{i\pm1} g_i + g_{i\pm1} g_i^2 = 0, \quad \text{for } g_i = e_i, f_i.$$

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where a_{ij} are components of the A_n-type Cartan matrix.

We will work with elements e_i, f_i, h_i.

(We are using q-number convention $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$.)

q-difference realisation of $U_q(\mathfrak{sl}_n)$

How does the story carry over?

[2] V. K. Dobrev, P. Truini, and L. C. Biedenharn. "Representation theory approach to the polynomial solutions of q-difference equations: $U_q(\mathfrak{sl}(3))$ and beyond". In: *Journal of Mathematical Physics* 35.11 (1994), pp. 6058–6075; S. E. Derkachov et al. "Iterative Construction of $U_q(\mathfrak{sl}(n+1))$ Representations and Lax Matrix Factorisation". In: *Letters in Mathematical Physics* 85.2-3 (Sept. 2008), pp. 221–234.

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- It is known[2] that $U_q(\mathfrak{sl}_n)$ has a similar representation on $\mathcal{V}^{(n)}$ which can be described by the same parameters $\boldsymbol{\rho} \in \mathbb{C}^n$. Generators of $U_q(\mathfrak{sl}_n)$ are now realised as q-difference operators. The key ingredients are the shift operators, and q-derivative

$$q^{\alpha N_{ij}}(f(\dots, x_{ij}, \dots)) = f(\dots, q^{\alpha} x_{ij}, \dots), \quad (N_{ij} = x_{ij}\partial_{ij})$$

$$D_{ij} := \frac{1}{x_{ij}} \frac{q^{N_{ij}} - q^{-N_{ij}}}{q - q^{-1}} = \frac{1}{x_{ij}}[N_{ij}]_q, \quad (\lim_{q \to 1} D_{ij} = \partial_{ij}).$$

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- There is not a closed form expression for $U_q(\mathfrak{sl}_n)$ generators evaluated in the representation $\mathcal{V}_{\boldsymbol{\rho}}^{(n)}$.

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q-difference realisation of $U_q(\mathfrak{sl}_n)$: n=2

<u>$n = 2$ case:</u> $U_q(\mathfrak{sl}_2)$ has a similar on $\mathcal{V}^{(2)}$ (polynomials in the single variable x), given by

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- Non-uniqueness: e.g. $\tilde{\rho}(\_) := q^{(N_x-1)N_x/2}\rho(\_)q^{-(N_x-1)N_x/2}$

$$\tilde{\rho}(e) = xq^{N_x}[N_x + m]_q, \quad \tilde{\rho}(f) = -D_x q^{-N_x+1}, \quad \tilde{\rho}(h) = m + 2N_x.$$

Factorised L-operator: $n = 2$

The algebra $U_q(\mathfrak{sl}_n)$ has a universal L-operator (Jimbo) given by

$$\tilde{L}(u) = \sum_{i,j} e_{ij} \otimes \hat{E}_{ji}(u), \quad \hat{E}_{ij}(u) = \begin{cases} q^{u+(E_{ii}+E_{jj}-1)/2} E_{ij}, & j > i, \\ (q^{-1})^{u+(E_{ii}+E_{jj}-1)/2} E_{ij}, & i > j, \\ [u + E_{ii}]_q, & i = j, \end{cases}$$

which has the defining RLL-relation with the defining $U_q(\mathfrak{sl}_n)$ R-matrix. Here $E_{ij} \in U_q(\mathfrak{sl}_n)$ refer to q-deformed Cartan-Weyl elements.

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Here $E_{ij} \in U_q(\mathfrak{sl}_n)$ refer to q-deformed Cartan-Weyl elements.
In $n = 2$ case, evaluating the L-operator in the previous representation we obtain ($u_1 = u - m/2, u_2 = u - 1 + m/2$)

$$L(u_1, u_2) = \begin{pmatrix} [u_2 + 1 + N_x]_q & -D_x \\ x[m + N_x]_q & [u_1 - N_x]_q \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ q^{u_1} x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x q^{u_2} & 1 \end{pmatrix}$$

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This factorisation generalises the rational $\mathfrak{sl}_2$ factorisation in a straightforward way[3]

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In fact this suggests a more general phenomena ...
Recall in the rational $\mathfrak{sl}_n$ case, the factorisation $L(\boldsymbol{u}) = Z\,\tilde{D}(\boldsymbol{u})\,Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad \tilde{D}(\boldsymbol{u}) = \begin{pmatrix} u_n & \tilde{D}_{12} & \tilde{D}_{13} & \dots & \tilde{D}_{1n} \\ & u_{n-1} & \tilde{D}_{23} & \dots & \tilde{D}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & \tilde{D}_{n-1,n} \\ & & & & u_1 \end{pmatrix}$$

[3] Derkachov, Karakhanyan, and Kirschner, "Yang–Baxter-operators and parameter permutations".

Factorised L-operator

Can we obtain a factorised $U_q(\mathfrak{sl}_n)$ L-operator from the $\mathfrak{sl}_n$ expression in a similar manner?

Factorised L-operator

Can we obtain a factorised $U_q(\mathfrak{sl}_n)$ L-operator from the $\mathfrak{sl}_n$ expression in a similar manner? I.e. looking for a factorised L-operator like

$$L(\boldsymbol{u}) = Z_1\,\tilde{D}(\boldsymbol{u})\,Z_2^{-1}, \quad Z_i = \begin{pmatrix} 1 & & & \\ x_{21}q^{(a_i)_{21}} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{n1}q^{(a_i)_{n1}} & \dots & x_{n,n-1}q^{(a_i)_{n,n-1}} & 1 \end{pmatrix},$$

$$\tilde{D}(\boldsymbol{u}) = \begin{pmatrix} [u_n]_q q^{b_{11}} & \tilde{D}_{12}q^{b_{21}} & \dots & \tilde{D}_{1n}q^{b_{n1}} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1,n-1}} & \tilde{D}_{n-1,n}q^{b_{n,n-1}} \\ & & & [u_1]_q q^{b_{nn}} \end{pmatrix},$$

where now $\tilde{D}_{ji}q^{b_{ij}} = -D_{ij}q^{b_{ij}} - \sum_{k=i+1}^{n} x_{ki}D_{kj}q^{b_{ijk}}$.

Factorised L-operator: $n = 3$ case

$U_q(\mathfrak{sl}_3)$ has a factorised L-operator $L(u_1, u_2, u_3) = Z_1 \tilde{D} Z_2^{-1}$ of this form[4]

$$\tilde{D} = \begin{pmatrix} q^{N_{31}-N_{21}}[u_3]_q & \begin{matrix} -q^{(u_3+u_2+2+N_{31}+N_{32})/2} \\ \times\left(D_{21}+q^{N_{31}-N_{32}}x_{32}D_{31}\right) \end{matrix} & -q^{(u_3+u_1-1-N_{21}-N_{32})/2}D_{31} \\ 0 & q^{N_{21}-N_{32}-1}[u_2]_q & -q^{(u_1+u_2+N_{21}-3N_{31})/2}D_{32} \\ 0 & 0 & q^{-N_{31}+N_{32}}[u_1]_q \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ q^{(u_2-u_3+N_{32}-N_{31})/2}x_{21} & 1 & 0 \\ q^{(N_{21}+3N_{32}-u_3-3u_1+1)/2}x_{31} & q^{(u_1-u_2+N_{31}-N_{21})/2}x_{32} & 1 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ q^{(N_{31}-N_{32}+u_3-u_2)/2}x_{21} & 1 & 0 \\ q^{-u_3+(N_{32}-N_{21}-u_1-u_3-1)/2}x_{31} & q^{(u_2-u_1+N_{21}+3N_{31})/2}x_{32} & 1 \end{pmatrix}.$$

[4] P. Valinevich et al. "Factorization of the $\mathcal{R}$-matrix for the quantum algebra $U_q(s\ell_3)$". In: *Journal of Mathematical Sciences* 151 (2008), pp. 2848–2858.

Factorised L-operator: $n = 4$ case

$n = 4$ case: We are interested in the $n = 4$ case because of the Lie algebra isomorphism $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$. Use the previous factorisation as an ansatz for the $n = 4$ case.

[5] Dobrev, Truini, and Biedenharn, "Representation theory approach to the polynomial solutions of q-difference equations: $U_q(\mathfrak{sl}(3))$ and beyond".

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To build a $U_q(\mathfrak{sl}_4)$ L-operator it is enough to extract a general form for generating elements from the matrix product and then fix q-shift factors by imposing the $U_q(\mathfrak{sl}_4)$ relations. This only requires calculating near diagonal entries $(L(\boldsymbol{u}))_{ij}$ for $|i - j| \leq 1$.

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Diagonal Cartan-Weyl elements $E_{ii} \in U_q(\mathfrak{sl}_n)$ are obtained directly from entries $L(\boldsymbol{u})_{ii}$:

$$E_{ii} = -\rho_{5-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{4} (N_{ji} + 1), \quad h_i = E_{ii} - E_{i+1,i+1},$$

$(\rho_i = u - u_i)$. This agrees with known result[5].

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The remaining generating elements $E_{i+1,i} = f_i$, $E_{i,i+1} = e_i$ have the form

$$f_1 = -D_{21} q^{c_{21}}, \quad f_2 = -D_{32} q^{c_{32}} - x_{21} D_{31} q^{c_{321}},$$
$$f_3 = -D_{43} q^{c_{43}} - x_{32} D_{42} q^{c_{432}} - x_{31} D_{41} q^{c_{431}},$$
$$e_1 = x_{21} \left[m_1 + \sum_{l=2}^{4} N_{l1} - \sum_{l=3}^{4} N_{l2} \right]_q q^{d_{21}} + x_{31} D_{32} q^{d_{321}} + x_{41} D_{42} q^{d_{421}},$$
$$e_2 = x_{32}[m_2 + N_{32} + N_{42} - N_{43}]_q q^{d_{32}} + x_{42} D_{43} q^{d_{432}} - x_{31} D_{21} q^{c_{231}},$$
$$e_3 = x_{43}[m_3 + N_{43}]_q q^{d_{43}} - x_{41} D_{31} q^{c_{341}} + x_{42} D_{32} q^{c_{342}}.$$

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- $[e_i, f_j] = \delta_{ij} [h_i]_q$, $[e_1, e_3] = [f_1, f_3] = 0$ for $|i - j| > 1$, and cubic Serre-relations between $e_2, e_{2\pm1}$ $(f_2, f_{2\pm1})$. 15 relations.

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$$E_{42} = [f_3, f_2]_{q^{-1}} = -D_{42}q^{-1+N_{21}-N_{32}-N_{41}} - x_{21}D_{41}q^{-(1+N_{31})} + (q-q^{-1})x_{31}D_{41}D_{32}q^{N_{21}-N_{31}-1}.$$

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This affects the factorisation $L(\boldsymbol{u}) = Z_1\,\tilde{D}(\boldsymbol{u})\,Z_2^{-1}$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21}q^{a_{21}} & 1 & 0 & 0 \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & 0 \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21}q^{a_{21}} - (q-q^{-1})x_{31}D_{32}q^{a_{321}} & 1 & 0 & 0 \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & 0 \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix}$$

with a similar modification of the $3,2$ entry of Z_2.

YBE and Parameter permutations

So far we have solved a YBE of the form

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}}^{(n)})$$

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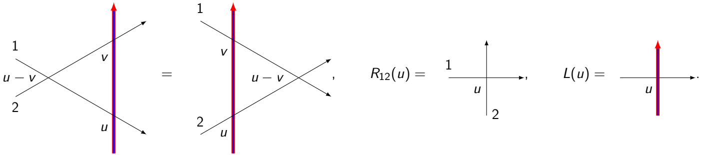

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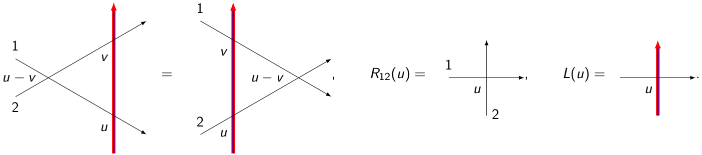
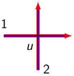


Goal: Find R-matrix $\mathcal{R}_{12}(u) =$ 

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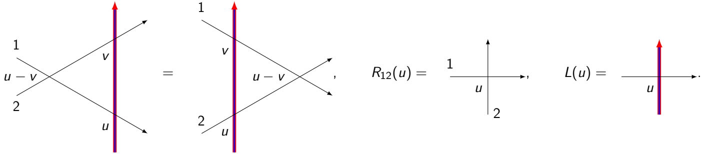
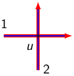
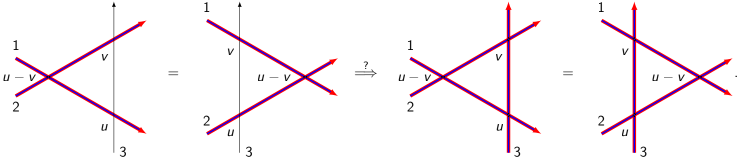


Goal: Find R-matrix $\mathcal{R}_{12}(u) =$  $\in \text{End}(\mathcal{V}_{\boldsymbol{\rho}}^{(n)} \otimes \mathcal{V}_{\boldsymbol{\tau}}^{(n)})$



YBE and Parameter permutations

In the "check" formalism $\hat{\mathcal{R}}_{12}(u) = \mathcal{P} \circ \mathcal{R}_{12}(u)$ should solve the defining relation:

$$\hat{\mathcal{R}}_{12}(u-v)L_1(\boldsymbol{u})L_2(\boldsymbol{v}) = L_1(\boldsymbol{v})L_2(\boldsymbol{u})\hat{\mathcal{R}}_{12}(u-v), \qquad (\star)$$

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Therefore, we can find a factorised solution of $(\star)$ by finding $2n-1$ "transpostion operators" $\mathcal{S}_i \in \mathrm{End}(V_{\boldsymbol{\rho}}^{(n)} \otimes V_{\boldsymbol{\tau}}^{(n)})$ which solve the simpler relations:

$$\mathcal{S}_i(\boldsymbol{u})L_{12}(\boldsymbol{u}, \boldsymbol{v}) = L_{12}(s_i(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_i(\boldsymbol{u}),$$

where $s_i = (i, i+1) \in S_{2n}$ for $1 \leq i \leq 2n-1$.

In fact it can be simplified further:

[6] Derkachov, Karakhanyan, and Kirschner, "Yang–Baxter-operators and parameter permutations"; Derkachov and Manashov, "R-Matrix and Baxter Q-Operators for the Noncompact $SL(N, C)$ Invariant Spin Chain"; Valinevich et al., "Factorization of the $\mathcal{R}$-matrix for the quantum algebra $U_q(s\ell_3)$".

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In fact it can be simplified further:

- $n-1$ "intertwining" operators $\mathcal{T}_j \in \mathrm{End}(\mathcal{V}_\rho)$ $j = 1, \ldots, n-1$

$$\mathcal{T}_j(u_j - u_{j+1})L(u_1, \ldots, u_n) = L(u_1, \ldots, u_j, u_{j-1}, \ldots, u_n)\mathcal{T}_j(u_j - u_{j+1}),$$

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- a single "exchange" operator $\mathcal{S}_n(u_n - v_1) \in \text{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\tau}})$

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This has been solved in the $\mathfrak{sl}_n$ case (fractional calculus), and $U_q(\mathfrak{sl}_m)$ (basic hypergeometric series) for $m = 2, 3$.[6]

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$U_q(\mathfrak{sl}_4)$ Intertwiners

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The $U_q(\mathfrak{sl}_4)$ intertwiners $\mathcal{T}_i(u_i - u_{i+1})L(\boldsymbol{u}) = L(s_i\boldsymbol{u})\mathcal{T}(u_i - u_{i+1})$ for $i = 1, 2, 3$ are given by

$$\mathcal{T}_{n-i}(\alpha_{n-i}) = (\Lambda_i)^{\alpha_{n-i}}\Phi_{(\alpha_{n-i})}(q^{2N_{i+1,i}}\boldsymbol{X}_i), \qquad (\alpha_i = u_i - u_{i+1})$$

$$\Phi_{(\alpha)}(\boldsymbol{Z}) = \frac{(q^{(1-\alpha)}\boldsymbol{Z}; q^2)}{(q^2\boldsymbol{Z}; q^2)} = \sum_{n=0}^{\infty} \frac{(q^{-2\alpha}; q^2)_n}{(q^2; q^2)_n}(q^2\boldsymbol{Z})^n, \qquad (|q| < 1)$$

where $\Lambda_i = x_{i+1,i}^{-1}q^{\beta_i}$, and $\boldsymbol{X}_i = 1 + x_{i+1,i}\sum_{k=i+2}^{4}\frac{x_{k,i+1}}{x_{k,i}}(q^{N_{ki}} - q^{-N_{ki}})q^{\gamma_i}$.
These satisfy $[\Lambda_i, \boldsymbol{X}_i] = 0$.

[7] Valinevich et al., "Factorization of the $\mathcal{R}$-matrix for the quantum algebra $U_q(s\ell_3)$".

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where $\Lambda_i = x_{i+1,i}^{-1} q^{\beta_i}$, and $\boldsymbol{X}_i = 1 + x_{i+1,i}\sum_{k=i+2}^{4} \frac{x_{k,i+1}}{x_{k,i}}(q^{N_{ki}} - q^{-N_{ki}})q^{\gamma_i}$. These satisfy $[\Lambda_i, \boldsymbol{X}_i] = 0$.

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Exchange Operators

Exchange operator swaps parameters u_n and v_1 between two different L-operators.

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Heavily reliant on factorisation of the L-operator e.g. in the $U_q(\mathfrak{sl}_2)$ case[8]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1}x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x-1} & -D_x \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -xq^{u_2} & 1 \end{pmatrix}.$$

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Used to simplify defining relations for exchange operator: e.g. assuming $\mathcal{S}_2$ is a multiplication operator

$$(\tilde{D}_1)^{-1}\mathcal{S}_2(u_2 - v_1)\tilde{D}_1 N_1(u_2)M_2(v_1) = N_1(v_1)M_2(u_2)\tilde{D}_2\mathcal{S}_2(u_2 - v_1)(\tilde{D}_2)^{-1}$$

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In general we can isolate dependence of u_n and u_1 in the rightmost and leftmost factors respectively. Not yet obtained for $U_q(\mathfrak{sl}_4)$.

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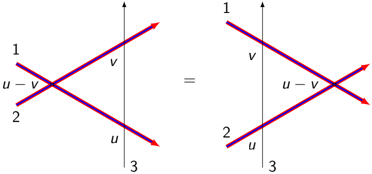
Yang-Baxter equation

So far we have solved (at least in some cases)



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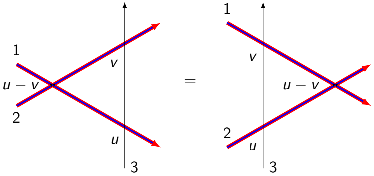

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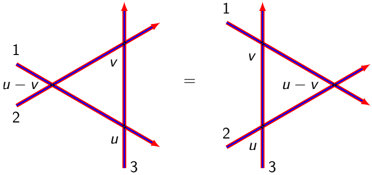

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For $\boldsymbol{A} = LHS, RHS$.

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$$s_{i_N} \dots s_{i_2} s_{i_1} \mapsto \mathcal{S}_{i_N}(s_{i_{N-1}} \dots s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \dots \mathcal{S}_{i_2}(s_{i_1}(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_{i_1}(\boldsymbol{u}, \boldsymbol{v}), \text{ does.}$$

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- $(s_i s_{i+1})^3 = \text{id} \Rightarrow \mathcal{T}_i(\beta)\mathcal{T}_{i+1}(\alpha+\beta)\mathcal{T}_i(\alpha) = \mathcal{T}_{i+1}(\alpha)\mathcal{T}_i(\alpha+\beta)\mathcal{T}_{i+1}(\beta)$ non-trivial

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E.g. The relation $\mathcal{T}_2(\alpha)\mathcal{T}_1(\alpha+\beta)\mathcal{T}_2(\beta) = \mathcal{T}_1(\beta)\mathcal{T}_2(\alpha+\beta)\mathcal{T}_1(\alpha)$ reduces to the (terminating) q-series result $\Theta_{i,j,k} = \Omega_{i,j,k}$,

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$$\Theta_{i,j,k} := \frac{(q^{-2(\alpha+\beta)};q^2)_{j+k}}{(q^2;q^2)_k(q^2;q^2)_j} q^{k(k-1+2(\alpha+i))-2i\alpha} \times \left(\sum_{n=0}^{i} \frac{(q^{-2\beta};q^2)_{i-n}}{(q^2;q^2)_{i-n}} \frac{(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n} q^{2n(\alpha-k)} \right) \propto {}_2\phi_1(\dots),$$

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- Coxeter relations for the transposition operators guarantee the general YBE, and are interesting identities in their own right.

References

Derkachov, S., D. Karakhanyan, and R. Kirschner. "Yang–Baxter-operators and parameter permutations". In: *Nuclear Physics B* 785.3 (Dec. 2007), pp. 263–285.

Derkachov, S. E. and A. N. Manashov. "R-Matrix and Baxter Q-Operators for the Noncompact $SL(N, C)$ Invariant Spin Chain". In: *Symmetry, Integrability and Geometry: Methods and Applications* (Dec. 2006).

Derkachov, S. E. et al. "Iterative Construction of $U_q(\mathfrak{sl}(n+1))$ Representations and Lax Matrix Factorisation". In: *Letters in Mathematical Physics* 85.2-3 (Sept. 2008), pp. 221–234.

Dobrev, V. K., P. Truini, and L. C. Biedenharn. "Representation theory approach to the polynomial solutions of q-difference equations: $U_q(\mathfrak{sl}(3))$ and beyond". In: *Journal of Mathematical Physics* 35.11 (1994), pp. 6058–6075.

Valinevich, P. et al. "Factorization of the $\mathcal{R}$-matrix for the quantum algebra $U_q(s\ell_3)$". In: *Journal of Mathematical Sciences* 151 (2008), pp. 2848–2858.

Thank you!

$\mathfrak{sl}_n$ case

In the rational $\mathfrak{sl}_n$ case, we have $L(\boldsymbol{u}) = Z\,\tilde{D}(\boldsymbol{u})\,Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad \tilde{D}(\boldsymbol{u}) = \begin{pmatrix} u_n & \tilde{D}_{12} & \tilde{D}_{13} & \dots & \tilde{D}_{1n} \\ & u_{n-1} & \tilde{D}_{23} & \dots & \tilde{D}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & \tilde{D}_{n-1,n} \\ & & & & u_1 \end{pmatrix}$$

- Intertwiners $\mathcal{T}_i(u_i - u_{i+1})$ are given by

$$\mathcal{T}_i(u_i - u_{i+1}) = (-\tilde{D}_{n-i,n+1-i})^{u_i - u_{i+1}},$$

so e.g. $\mathcal{T}_1(u_1 - u_2) = (\partial_{n-1,n})^{u_1 - u_2}$.

- Exchange operator $S_n(u_n - v_1)$ is given by

$$\mathcal{S}_n(u_n - v_1) = (((Z^{(y)})^{-1} Z^{(x)})_{N1})^{u_n - v_1}$$

(y_{ij} are variables for 2nd rep.). E.g. for $n = 2, 3$ we have

$\mathcal{S}_2(u_2 - v_1) = (x - y)^{u_2 - v_1}$, $\mathcal{S}_3(u_3 - v_1) = (x_{31} - y_{31} - y_{32}(x_{21} - y_{21}))^{u_3 - v_1}$.

q-deformed case

- In general we hope to be able to use the following ansatz for intertwiners,

$$\mathcal{T}_{n-i}(\alpha_{n-i}) = (\tilde{D}_i)^{\alpha_{n-i}}, \quad \tilde{D}_i = D_{i+1,i} q^{b_{i+1,i}} + \sum_{k=i+2}^{n} x_{k,i+1} D_{k,i} q^{b_{ki}}.$$

  For $\alpha_{n-i} \in \mathbb{N}$ we obtain a finite product which generalises to the ratio of q-Pochhammers.

- For exchange operator we have e.g. $n = 2$

$$\mathcal{S}_2(u_2 - v_1) = (x - y)^{u_2 - v_1}, \qquad \text{(Rational)}$$

$$\mathcal{S}_2(u_2 - v_1) = x^{u_2 - v_1} \frac{\left(\frac{y}{x} q^{1-u_2+v_1}; q^2\right)}{\left(\frac{y}{x} q^{1+u_2-v_1}; q^2\right)}. \qquad (q\text{-deformed})$$