

Interpolating between Temperley-Lieb and Brauer representation theory: generalised roots of unity

(based on arXiv:2512.24535 joint with P. P. Martin)

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QTCat Seminar, Universität Hamburg,
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Universität Hamburg

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Overview

- 1 Motivation
- 2 Kadar-Martin-Yu Categories/Algebras
- 3 Representation Theory by “Tower of Recollment”
 - Simple/standard modules
 - Semisimplicity Criteria: Generalised Roots of Unity
 - Maps by Reflection Geometry
- 4 Higher Dimensional “Height” Conditions?

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Representation Theory: We define a family of algebras

$$TL_n = J_{-1,n} \subset J_{0,n} \subset J_{1,n} \subset \dots \subset J_{n-2,n} = J_{\infty,n} = Br_n.$$

Temperley-Lieb and Brauer algebras are well understood and have a rich representation theory, which are significantly different. The algebras $J_{l,n}$ give the opportunity to “interpolate” between them in a discrete way.

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Kadar-Martin-Yu Categories: Context

For $\alpha \in \mathbb{K}$, we are going to define a discrete family of nested $\mathbb{K}$ -linear subcategories of the diagrammatic Brauer category $Br(\alpha)$:

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The Brauer category $Br(\alpha)$ has:

- $\mathbb{N}$ as objects.
- $Br(n, m)$ spanned by type (n, m) “diagrams” (pairings) with a diagrammatic composition up to loop replacement:

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$$\begin{array}{l} Br(5, 5) \ni \\ Br(5, 3) \ni \end{array} \begin{array}{c} \text{Diagram 1: A box with 5 vertical lines. The top 4 lines are paired into 2 loops. The bottom 3 lines are paired into 1 loop. A red circle highlights a crossing between the 2nd and 3rd lines from the left, which is part of a larger loop structure. A dashed horizontal line is drawn across the box at the level of the crossing.} \\ \text{Diagram 2: A box with 5 vertical lines. The top 4 lines are paired into 2 loops. The bottom 3 lines are paired into 1 loop. A red circle highlights a crossing between the 2nd and 3rd lines from the left, which is part of a larger loop structure. A dashed horizontal line is drawn across the box at the level of the crossing.} \end{array} = \alpha \begin{array}{c} \text{Diagram 3: A box with 5 vertical lines. The top 4 lines are paired into 2 loops. The bottom 3 lines are paired into 1 loop. A red circle highlights a crossing between the 2nd and 3rd lines from the left, which is part of a larger loop structure. A dashed horizontal line is drawn across the box at the level of the crossing.} \\ \text{Diagram 4: A box with 5 vertical lines. The top 4 lines are paired into 2 loops. The bottom 3 lines are paired into 1 loop. A red circle highlights a crossing between the 2nd and 3rd lines from the left, which is part of a larger loop structure. A dashed horizontal line is drawn across the box at the level of the crossing.} \end{array} \in Br(5, 3)$$

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$$\begin{array}{l} Br(5, 5) \ni \\ Br(5, 3) \ni \end{array} \begin{array}{c} \text{Diagram 1: A box with 5 top and 5 bottom strands. A red circle highlights a loop on the right side, which is part of a larger crossing structure.} \\ \text{Diagram 2: A box with 5 top and 3 bottom strands. It shows a different pairing of the strands.} \end{array} = \alpha \in Br(5, 3)$$

- $\otimes : Br \times Br \rightarrow Br$ given by horizontal juxtaposition of diagrams.

Kadar-Martin-Yu Categories: Context (continued)

The Temperley-Lieb (TL), $TL(\alpha)$ is the wide (monoidal, $\mathbb{K}$ -linear) subcategory where we restrict to “crossingless diagrams”, e.g.:



$\notin TL(5, 3),$



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Note that TL does not see much of Brauer, e.g. for $TL_n := TL(n, n)$ and likewise $Br_n := Br(n, n)$:

$$\frac{\dim(TL_n)}{\dim(Br_n)} = \frac{C_n}{(2n-1)!!} = \frac{2^n}{(n+1)!} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

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Context summary: We have $TL(\alpha) \subset Br(\alpha)$ diagrammatic categories and their endomorphism algebras $TL_n(\alpha) \subset Br_n(\alpha)$.

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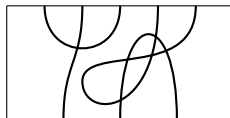
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Context summary: We have $TL(\alpha) \subset Br(\alpha)$ diagrammatic categories and their endomorphism algebras $TL_n(\alpha) \subset Br_n(\alpha)$. The TL “crossingless” condition kills almost everything...

ACHTUNG: Diagrams represent pair partitions...

Kadar-Martin-Yu Categories: Context (warning)

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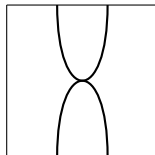
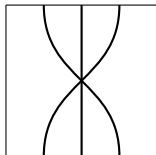


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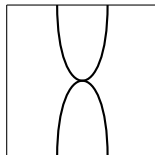
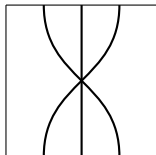


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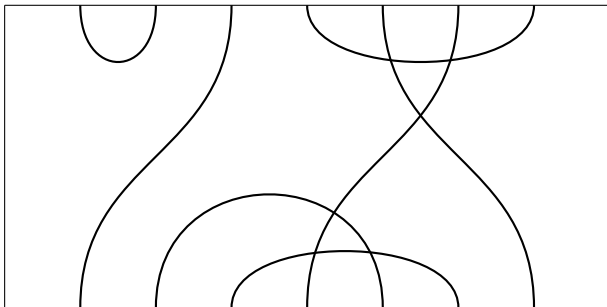


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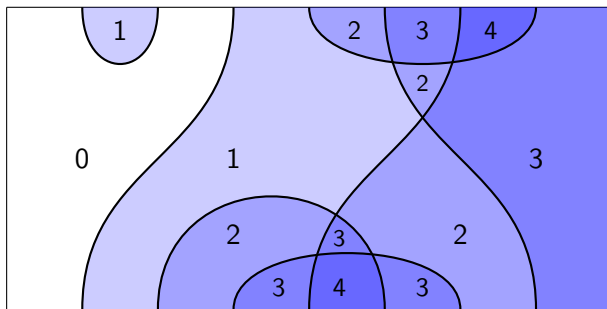


We will pretty much always draw diagrams with **minimal crossings**.

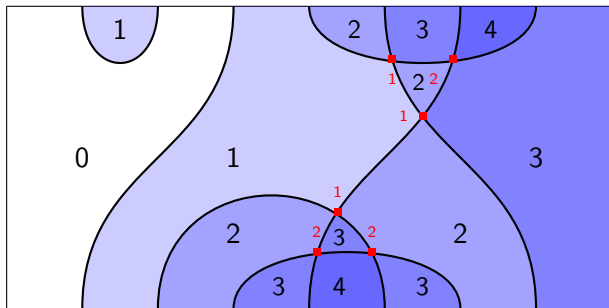
Kadar-Martin-Yu Categories: The Height Condition



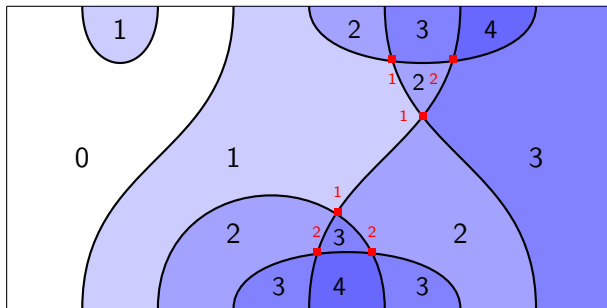
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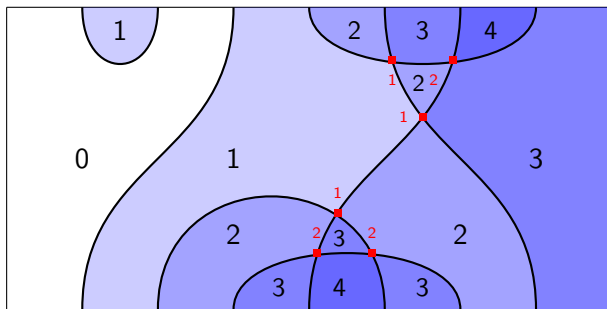


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The height of a **diagram**: $ht(\text{diag.}) = \max_{\text{red } \times_k \in \text{diag.}} k$, e.g. above $ht = 2$.
 (it is defined to be -1 if no crossings)

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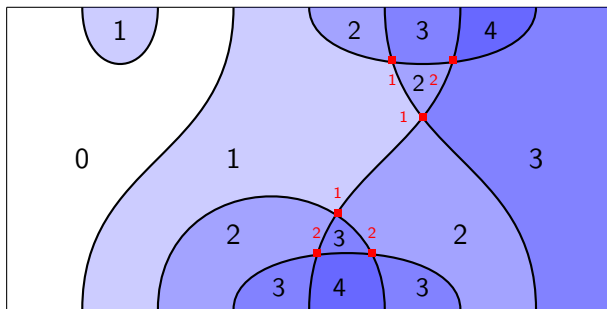
The height of a **diagram**: $ht(\text{diag.}) = \max_{\text{crossing } k} k$, e.g. above $ht = 2$.

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The height of a **pair partition**:

$$ht(p) = \min_{\text{diag. of } p} ht(\text{diag.}) \quad \text{e.g. above} \quad ht \leq 2$$

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Gadget*: Minimal crossing number diagrams are minimal height.

Kadar-Martin-Yu Categories: The Height Condition (ctd)

Lemma (Height is non-increasing)

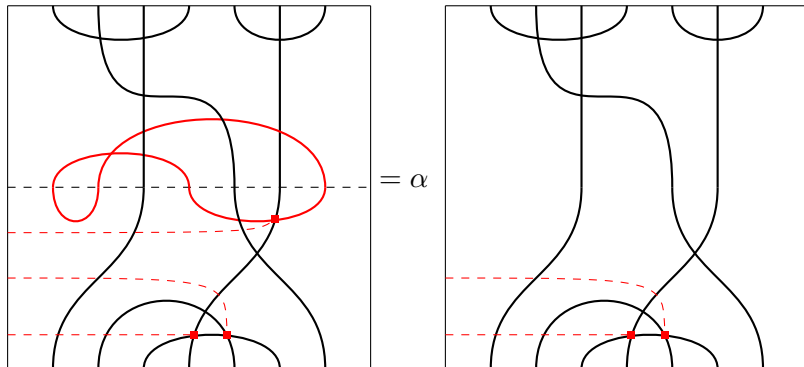
$$ht(p' \# p) \leq \max\{ht(p'), ht(p)\}$$

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Lemma (Height is non-increasing)

$$ht(p' \# p) \leq \max\{ht(p'), ht(p)\}$$

Proof: $(p' \circ p = \alpha^{\# \text{loops}} p' \# p)$



Kadar-Martin-Yu Categories

Definition (Kadar, Martin, and Yu, 2014 ; KMY Category)

For $l = -1, 0, 1, \dots, \infty$ define $J_l(\alpha) \subset Br(\alpha)$ to be the wide, $\mathbb{K}$ -linear subcategory where now, $J_l(n, m)$ is spanned by pairings with height $\leq l$.

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Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}(\alpha)$ is generated by:

$e_i =$  ,

The diagram shows a rectangular surface S_j with a handle i . The handle is represented by a vertical strip in the center where two sheets of the surface cross each other. Ellipses on either side of the handle indicate that the surface has multiple other handles.

for $i = 1, \dots, n-1$, and $j = 1, \dots, \min(l+1, n-1)$. i.e. $\mathbb{K}S_n \subset Br_n$
 whereas $\mathbb{K}S_{\min(l+2, n)} \subset J_{l, n}$.

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Representation Theory: Globalisation/Localisation

Use hom-spaces as bimodules:

$$\begin{array}{ccc}
 M & \xrightarrow{\quad} & J_I(n+2, n) \otimes_{J_{I,n}} M \\
 & \searrow \scriptstyle G & \nearrow \\
 \text{mod}(J_{I,n}) & & \text{mod}(J_{I,n+2}) \\
 & \nwarrow \scriptstyle F & \nearrow \\
 J_I(n, n+2) \otimes_{J_{I,n+2}} N & \xleftarrow{\quad} & N
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Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$.

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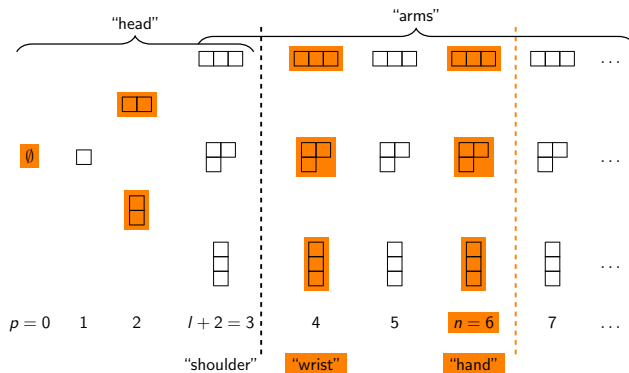
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Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$. Use to label simples inductively:

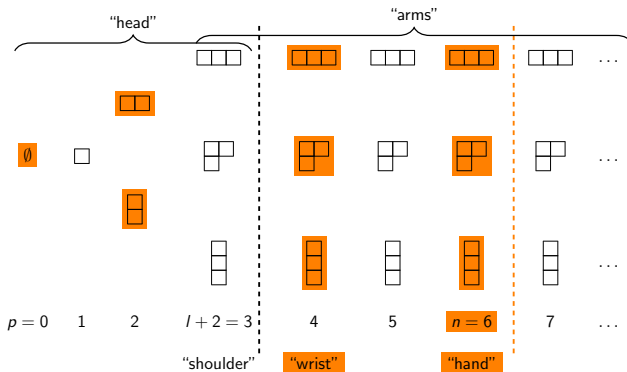
$$\begin{aligned}
 \Lambda_{l,n} &= \Lambda_{l,n-2} \sqcup \Lambda \left(J_{l,n} / J_{l,n}^{\leq n-2} \right) \\
 &= \Lambda_{l,n-2} \sqcup \Lambda(\mathbb{K} \textcolor{red}{S}_{\min(n, l+2)}) = \Lambda_{l,n-2} \sqcup \{ \lambda \vdash \textcolor{red}{\min}(n, l+2) \},
 \end{aligned}$$

$$\Rightarrow \quad \Lambda_{l,n} = \{ (p, \lambda) \mid p = n, n-2, \dots, 0/1, \quad \lambda \vdash \min(p, l+2) \}$$

Representation Theory: Anatomy of $\text{mod}(J_{l,-})$



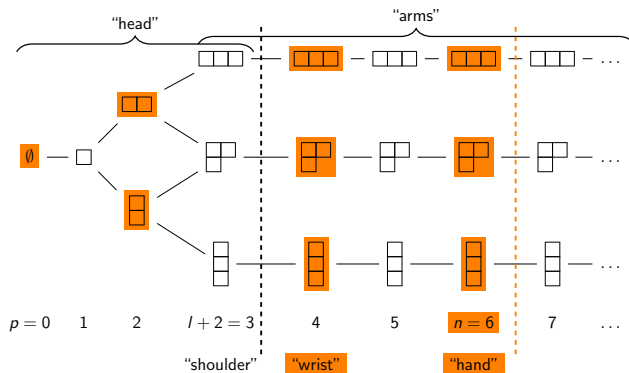
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For each label have a **standard module**, constructed inductively:

$$\Delta_{(n,\lambda)}^n \simeq \mathcal{S}_\lambda, \quad \Delta_{(n-2k,\lambda)}^n = G(\Delta_{(n-2k,\lambda)}^{n-2}).$$

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Encode induction/restriction rules on **Rollet diagram**, e.g. above.

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Standard modules have simple heads $L_{(p,\lambda)}$ and determine $[L_{(p,\lambda)} : P_{(p,\lambda)}]$.

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Example: $n = 6$ and label $(4, (2, 1))$. Write

$$\mathcal{S}_{(2,1)} = \mathbb{C}S_3 \cdot c_{(2,1)} = \mathbb{C}\{c_{(2,1)}, (2\,3)c_{(2,1)}\},$$

c_λ is a young symmetriser s.t.: $c_\lambda^2 = c_\lambda$, $c_\lambda^* = c_\lambda$, $c_\lambda \cdot \mathbb{C}S_{|\lambda|} \cdot c_\lambda = \mathbb{C} c_\lambda$.

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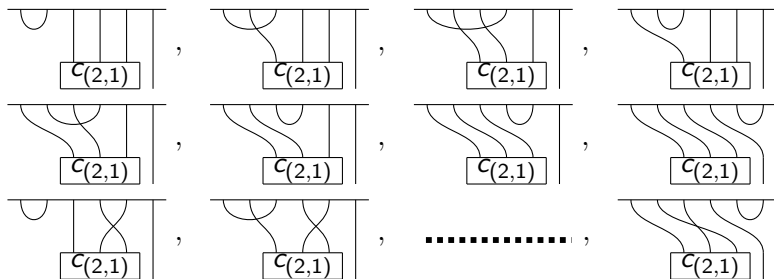
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A basis for $\Delta_{(4,\lambda)}^6$ is:



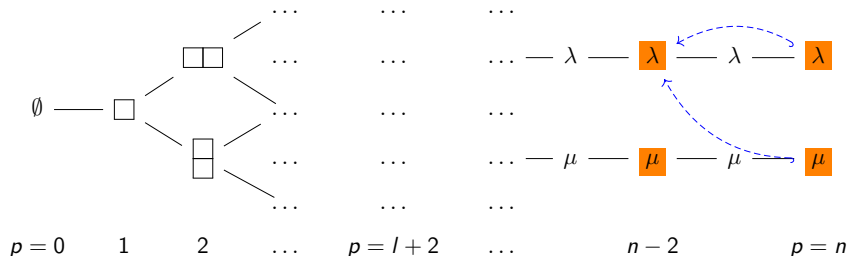
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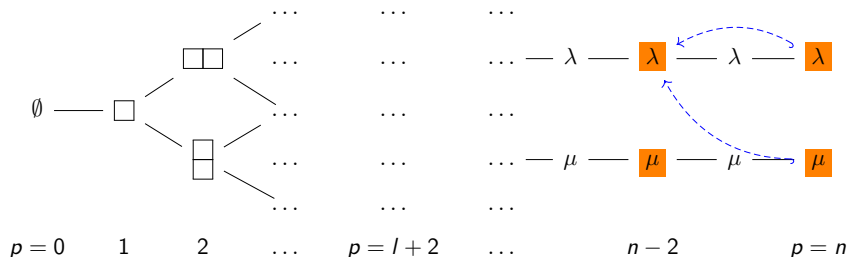
A: When standards = simples. By tower of recollement frame work [Cox et al., 2004], enough to have no maps from “hand” to “wrist”:



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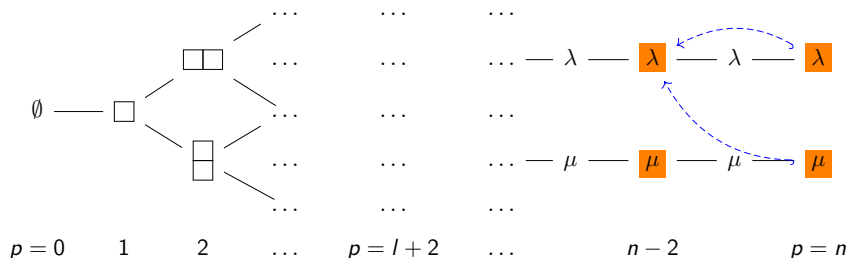


Reduce to looking for (Specht) submodules in $\Delta_{(n-2,\lambda)}^n$.

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Reduce to looking for (Specht) submodules in $\Delta_{(n-2,\lambda)}^n$.

In turn this is reduced to studying non-degeneracy of a bilinear form

$$\Delta_{(n-2,\lambda)}^n \times \Delta_{(n-2,\lambda)}^n \rightarrow \mathbb{C}[\alpha].$$

Representation Theory: Bilinear Form

Q: When is $J_{l,n}$ semisimple?

A: ... study non-degeneracy of $\Delta_{(n-2,\lambda)}^n \times \Delta_{(n-2,\lambda)}^n \rightarrow \mathbb{C}[\alpha]$.

$$\begin{array}{c}
 \text{---} \bigcirc \text{---} \\
 \left| \begin{array}{c} \boxed{c(2,1)} \\ \text{---} \\ \boxed{c(2,1)} \end{array} \right| \\
 \text{---}
 \end{array}
 = \alpha c_{(2,1)}^{(4)} id_4 c_{(2,1)}^{(4)}$$

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 \text{Diagram: A circle on the left connected by a dashed line to a vertical line. To the right of the vertical line are two boxes, both labeled } c_{(2,1)}. \text{ Three vertical lines connect the top box to the bottom box.} \\
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 \hline
 \langle -, - \rangle = -\frac{1}{2}.
 \end{array}$$

Form is symmetric and contravariant with respect to $J_{l,n}$ -action, i.e.

$$\langle \xi, y\zeta \rangle = \langle y^* \xi, \zeta \rangle.$$

Representation Theory: Bilinear Form - TL Case

$$\mathcal{B} = \left\{ \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \vdots \\ \text{diagram n} \end{array} \right\}$$

The diagrams in the set $\mathcal{B}$ are planar diagrams with n inputs and n outputs. Each diagram consists of a box labeled $C_{\ell(\gamma)}$ with n strands entering from the top and n strands exiting from the bottom. The strands are connected in a way that forms a permutation of the inputs to the outputs.

Gram matrix of form:

$$\Gamma(\Delta_{(n-2,(1))}) = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow \begin{aligned} P_n^{(1)}(\alpha) &:= |\Gamma(\Delta_{(n-2,(1))})| \\ &= U_{n-1}(\alpha/2) \end{aligned}$$

Chebyshev recursion: $P_{n+1}^{(1)}(\alpha) = \alpha P_n^{(1)}(\alpha) - P_{n-1}^{(1)}(\alpha),$

$$P_3^{(1)} = \alpha^2 - 1, \quad P_4^{(1)} = \alpha(\alpha^2 - 2), \quad P_4^{(1)} = (\alpha^2 - \alpha - 1)(\alpha^2 + \alpha - 1).$$

Roots: $P_n^{(1)}(\alpha) = 0$ at $\alpha = q + q^{-1}$ and $q^{2n} = 1$ (or $\alpha = 2 \cos(k\pi/n)$).

Representation Theory: Bilinear Form - Det. Factorisation

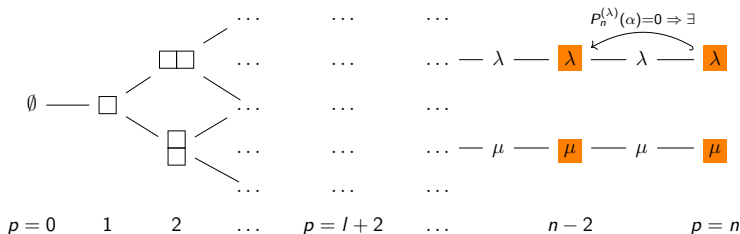
Theorem [Morris and Martin, 2025] - Generalised Roots of Unity

Fix $l, n \geq l + 4, \lambda \vdash l + 2$. $\exists$ polynomials $C^{(\lambda)}(\alpha), P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta_{(n-2,\lambda)}^n))(\alpha) = C^{(\lambda)}(\alpha) \left(P_n^{(\lambda)}(\alpha) \right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**.

Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta_{(n,\lambda)}^n \hookrightarrow \Delta_{(n-2,\lambda)}^n$.

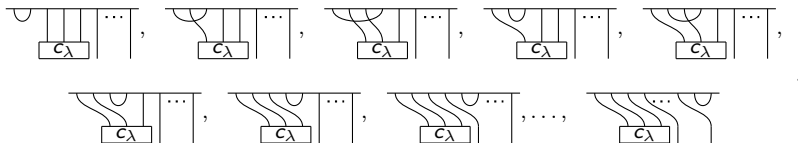


Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$,

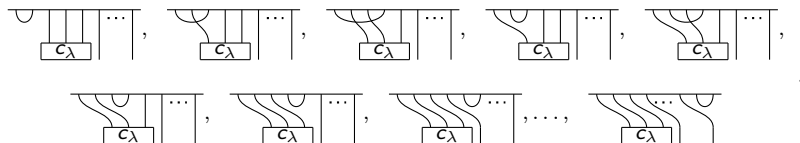
Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, e.g. $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, e.g. $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



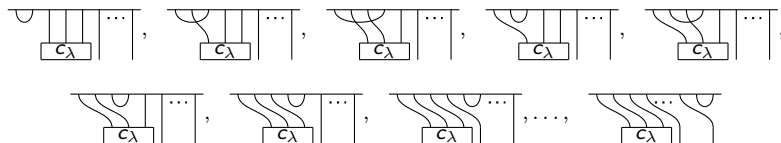
Need to compute determinant of:

$$\left(\begin{array}{cccc|cccc|cccc|cccc} \alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 1 & 0 & \epsilon & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \alpha & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \left. \begin{array}{l} u_{12}b \\ u_{13}b \\ u_{14}b \\ u_{23}b \\ u_{24}b \\ u_{34}b \\ u_{45}b \end{array} \right\} \text{head},$$

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \end{array} \right) \left. \begin{array}{l} u_{56}b \\ u_{67}b \\ u_{78}b \\ u_{89}b \\ u_{9,10}b \\ u_{10,11}b \\ u_{11,12}b \\ u_{12,13}b \end{array} \right\} \text{tail}$$

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, e.g. $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



Need to compute determinant of:

$$\left(\begin{array}{cccc|cccc|cccc|cccc} \alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \left. \begin{array}{l} u_{12}b \\ u_{13}b \\ u_{14}b \\ u_{23}b \\ u_{24}b \\ u_{34}b \\ u_{45}b \end{array} \right\} \text{head},$$

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \end{array} \right) \left. \begin{array}{l} u_{56}b \\ u_{67}b \\ u_{78}b \\ u_{89}b \\ u_{9,10}b \\ u_{10,11}b \\ u_{11,12}b \\ u_{12,13}b \end{array} \right\} \text{tail},$$

By expanding along last row we see the determinant satisfies Chebyshev recursion in n . We therefore, obtain the desired form: $\det = C^{(\lambda)}(\alpha)P_n^{(\lambda)}(\alpha)$.

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, e.g. $l = 1$ and $\lambda = (2, 1)$. Need to compute a determinant like:

$$\left(\begin{array}{cccccccc|cccc|cccc|cccc|cccc|cccc} \alpha & 1 & 1 & 1 & 1 & 0 \\ 1 & \alpha & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 \\ 0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} k_1 \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} k_2$$

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, e.g. $l = 1$ and $\lambda = (2, 1)$. Need to compute a determinant like:

$$\begin{pmatrix} \alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\ 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & \epsilon' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 \end{pmatrix} \begin{matrix} \\ \end{matrix} \left. \begin{matrix} \\ \end{matrix} \right\} k_1 \left. \begin{matrix} \\ \end{matrix} \right\} k_2$$

This has two Chebyshev expansions: $\det = C^{(\lambda)}(\alpha) P_{n_1}^{(\lambda)}(\alpha) = D^{(\lambda)}(\alpha) Q_{n_2}^{(\lambda)}(\alpha).$

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, e.g. $l = 1$ and $\lambda = (2, 1)$. Need to compute a determinant like:

$$\left(\begin{array}{cccccccc|cccc|cccc|cccc|cccc} \alpha & 1 & 1 & 1 & 1 & 0 \\ 1 & \alpha & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 \\ 0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 \\ 0 & 0 & \epsilon & 1 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} k_1 \\ \\ \\ \\ \\ \\ \\ \\ k_2 \end{array}$$

This has two Chebyshev expansions: $\det = C^{(\lambda)}(\alpha) P_{n_1}^{(\lambda)}(\alpha) = D^{(\lambda)}(\alpha) Q_{n_2}^{(\lambda)}(\alpha)$.

Need to

- 1 Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
- 2 Show $P_{n_1}^{(\lambda)}(\alpha) Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

- 1 Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
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To prove 1. we show both series have the same initial conditions by a row reduction procedure...

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

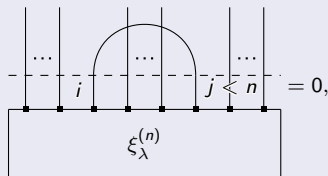
- ① Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
- ② Show $P_{n_1}^{(\lambda)}(\alpha) Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

To prove 1. we show both series have the same initial conditions by a row reduction procedure... uses a nice family of elements $\xi_\lambda^{(n)} \in \Delta_{(n-2, \lambda)}^n$.

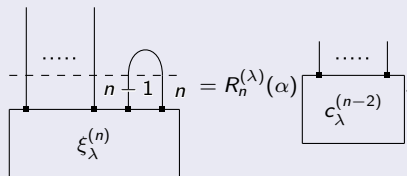
Lemma

For each $n \geq l + 4$ there exists a unique $\xi_\lambda^{(n)} \in \Delta_{(n-2, \lambda)}^n$ (polynomial coefficients) such that:

- ① $\xi_\lambda^{(n)}$ is non-zero for all specialisations $\alpha \in \mathbb{C}$,
- ② We have (in $\Delta_{(n-2, \lambda)}^{n-2}$)



A diagram representing an element in a bilinear form. It consists of a horizontal dashed line with several vertical lines extending upwards from it. A semi-circular arc (bubble) connects two of these vertical lines, labeled i and j from left to right. The label $j < n$ is placed to the right of the bubble. Below the dashed line is a rectangular box labeled $\xi_\lambda^{(n)}$. The entire diagram is followed by an equals sign and a zero, indicating that this element is zero.



A diagram representing an element in a bilinear form. It consists of a horizontal dashed line with several vertical lines extending upwards from it. A semi-circular arc (bubble) connects two of these vertical lines, labeled $n-1$ and n from left to right. Below the dashed line is a rectangular box labeled $\xi_\lambda^{(n)}$. To the right of this box is an equals sign followed by $R_n^{(\lambda)}(\alpha)$. This is followed by another equals sign and a diagram of a bilinear form element with a box. This second element has a horizontal dashed line with vertical lines extending upwards. Below it is a rectangular box labeled $c_\lambda^{(n-2)}$.

When $\alpha \in \mathbb{C}$ s.t. $R_n^{(\lambda)}(\alpha) = 0$, $\xi_\lambda^{(n)}$ generates a submodule $J_{l,n}\xi_\lambda^{(n)} \simeq \mathcal{S}_\lambda \subset \Delta_{(n-2, \lambda)}^n$. □

Representation Theory: Bilinear Form - Det. Factorisation

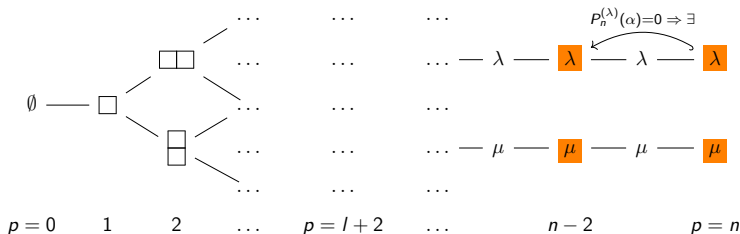
Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix $l, n \geq l + 4, \lambda \vdash l + 2$. $\exists$ polynomials $C^{(\lambda)}(\alpha), P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta_{(n-2,\lambda)}^n))(\alpha) = C^{(\lambda)}(\alpha) \left(P_n^{(\lambda)}(\alpha) \right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**.

Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta_{(n,\lambda)}^n \hookrightarrow \Delta_{(n-2,\lambda)}^n$.



Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

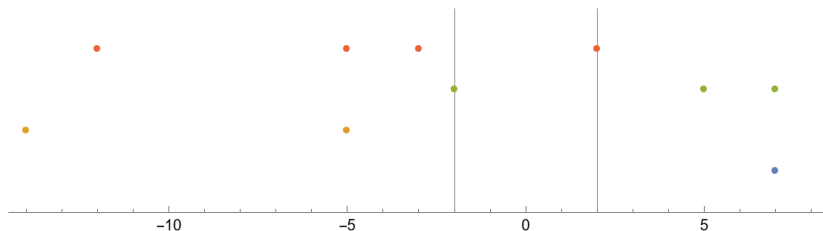


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = -1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

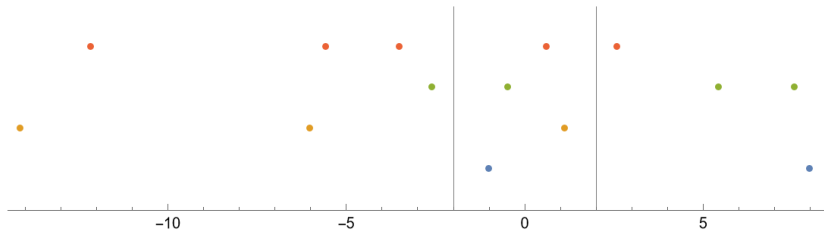


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 0$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

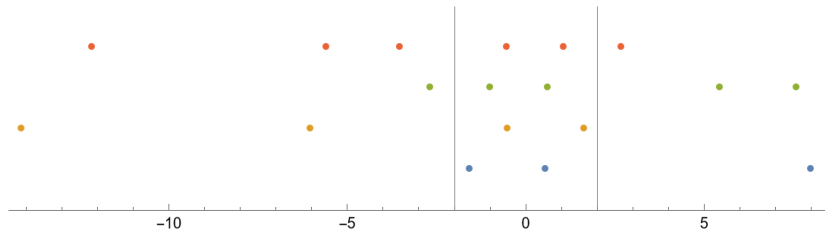


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

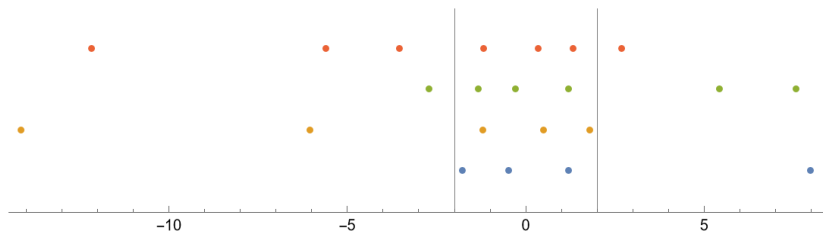


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 2$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

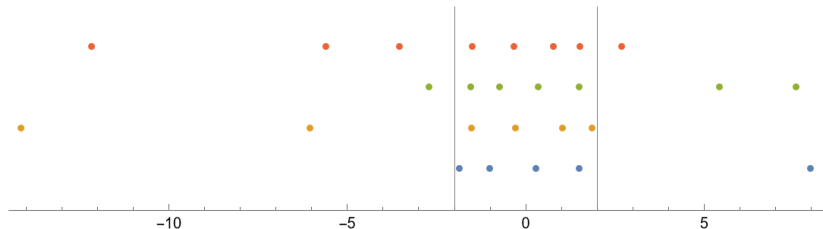


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 3$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

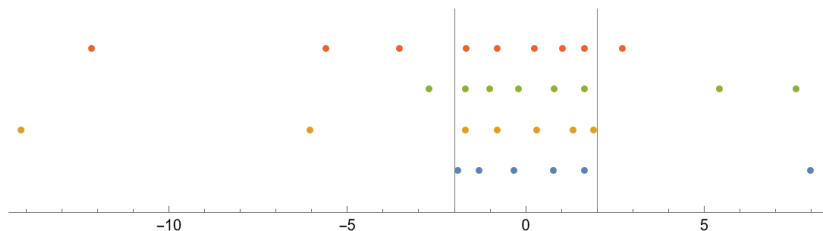


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 4$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

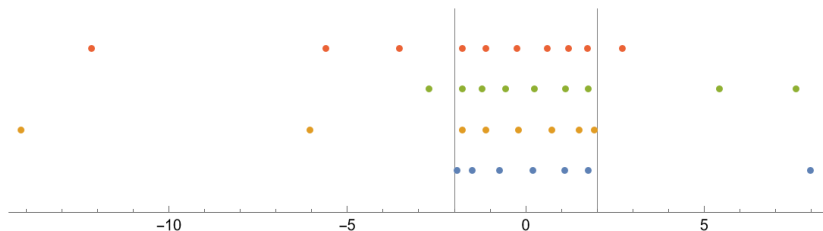


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 5$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

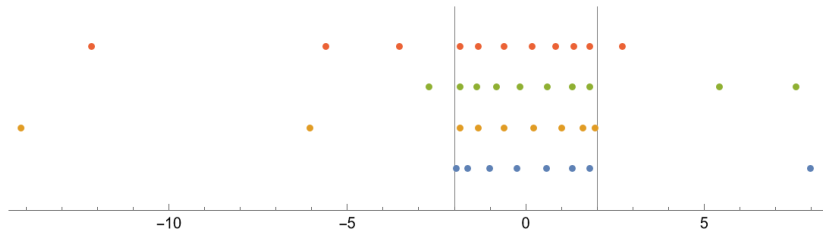


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 6$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

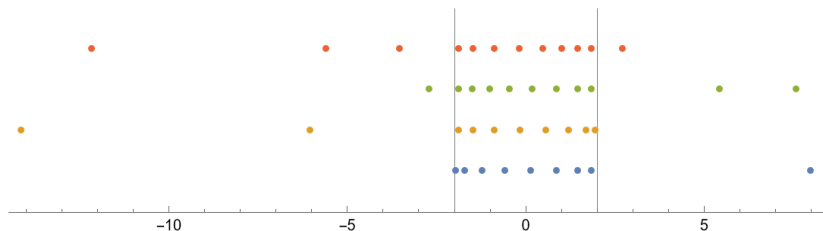


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 7$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

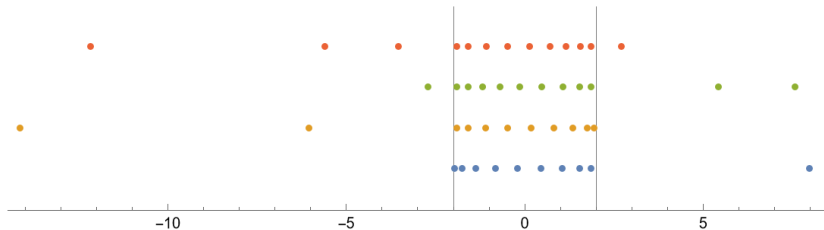


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 8$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

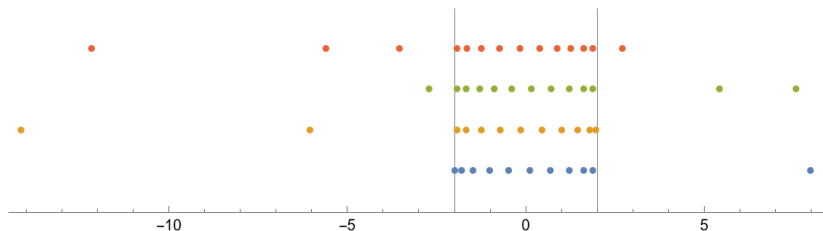


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 9$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

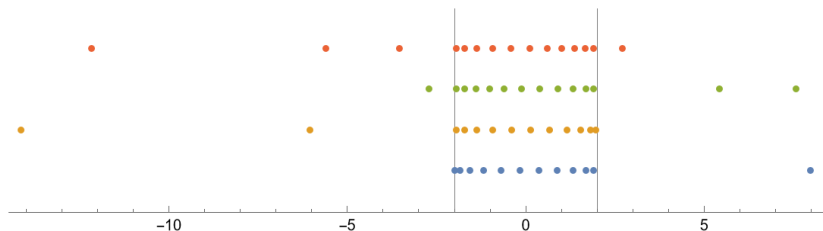


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 10$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

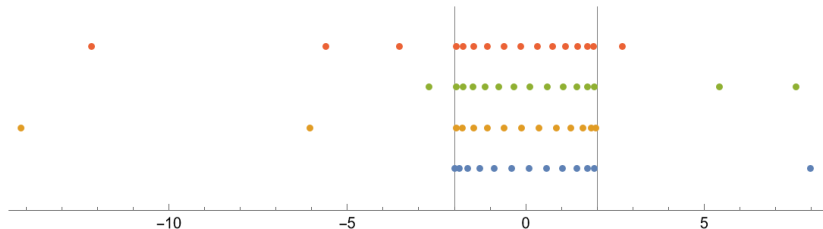


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 11$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

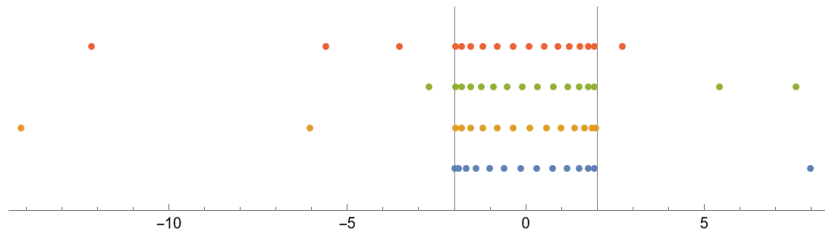


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 12$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

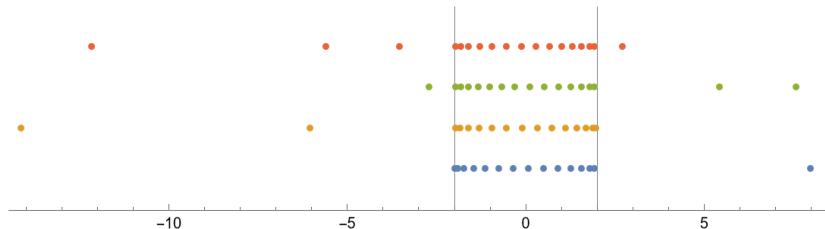


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 13$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

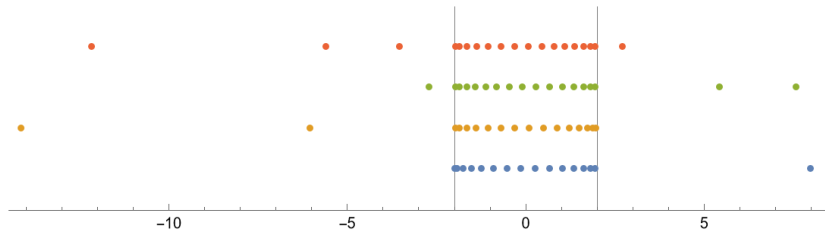


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 14$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

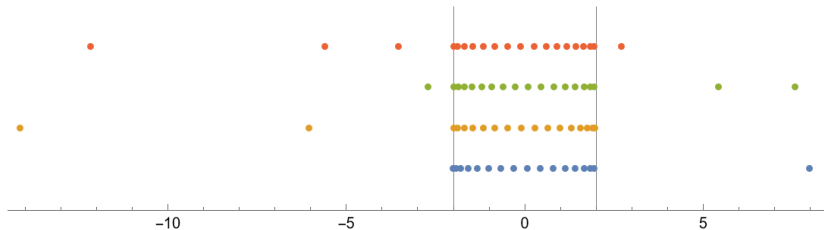


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 15$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

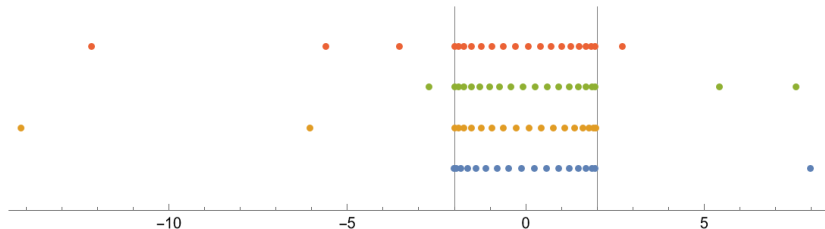


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 16$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

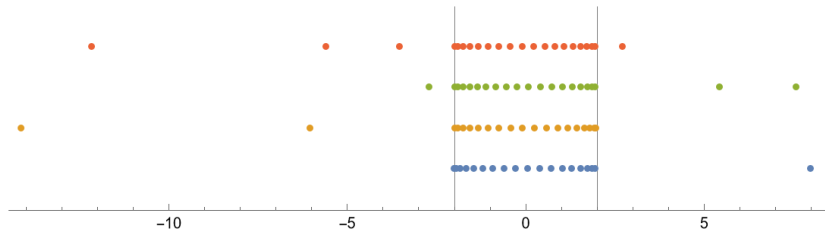


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 17$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

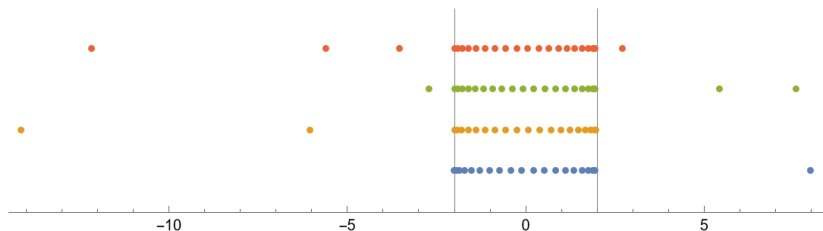


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 18$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

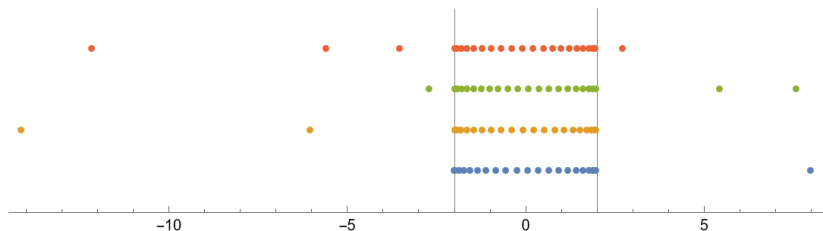


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 19$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

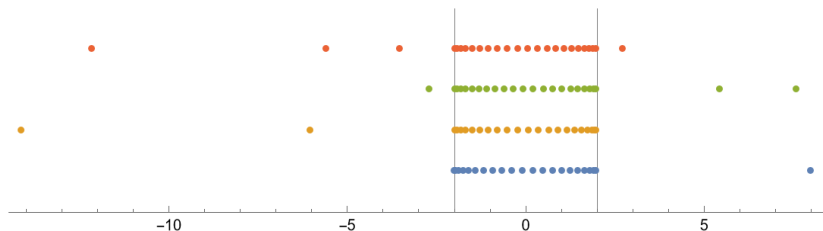


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 20$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

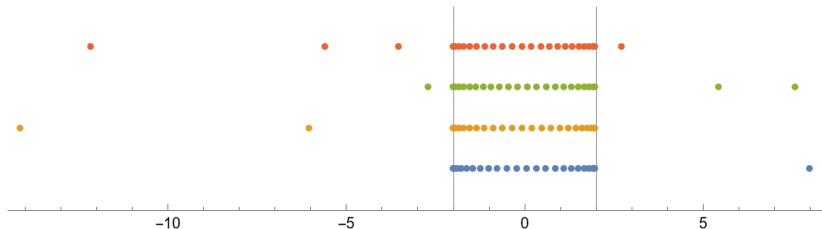


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 21$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

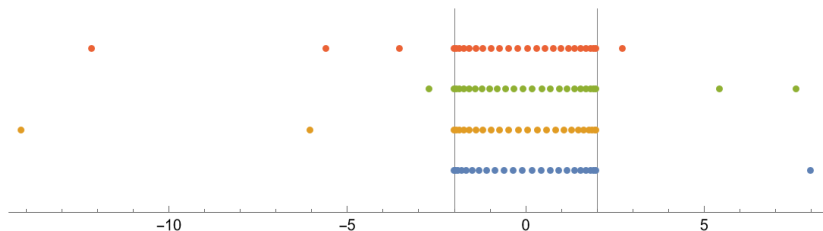


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 22$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

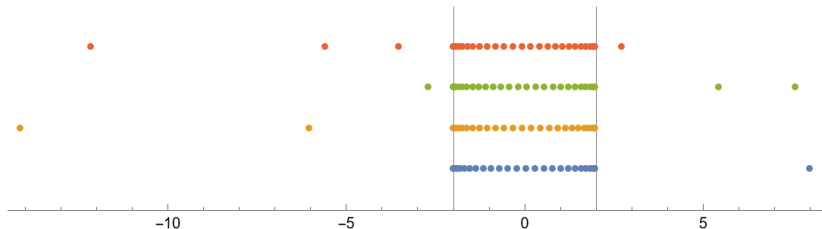


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 23$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

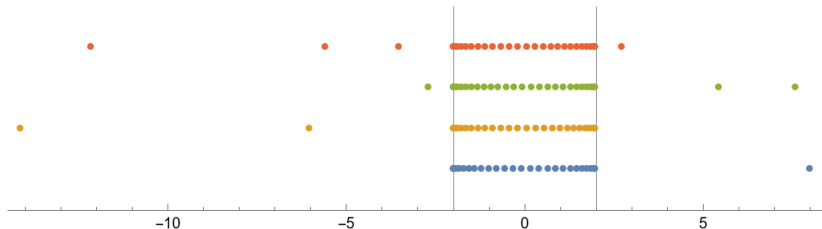


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 24$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

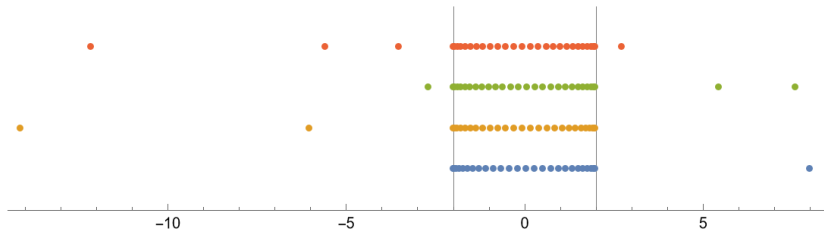


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 25$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

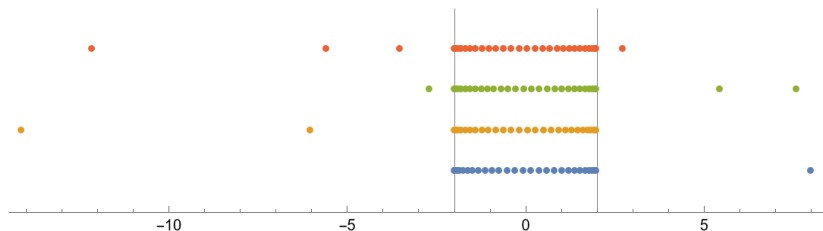


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 26$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

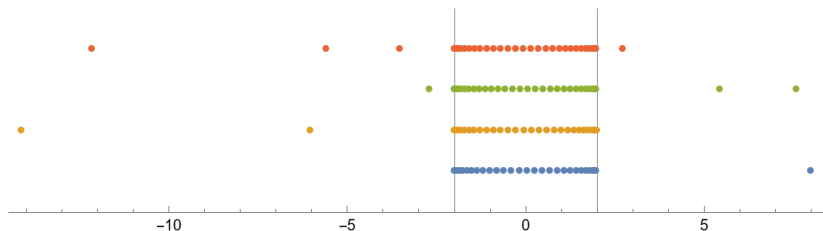


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 27$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

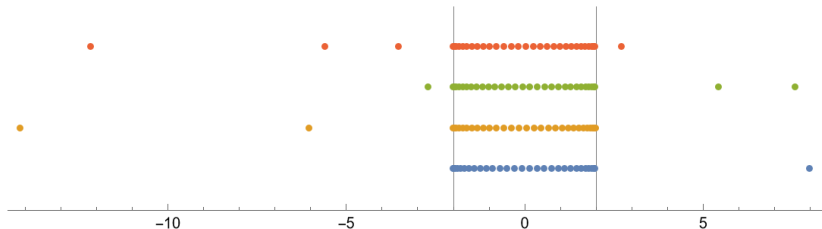


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 28$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

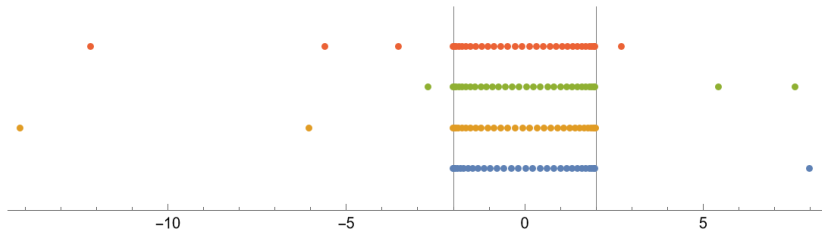


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 29$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

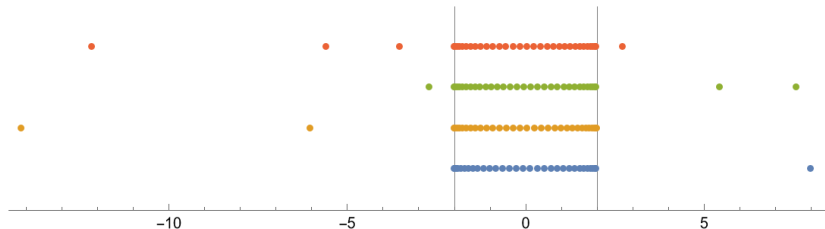


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 30$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

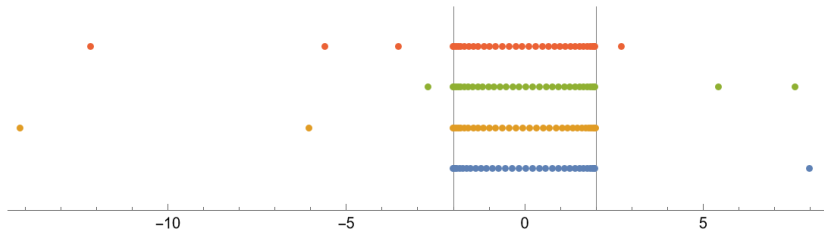


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 31$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

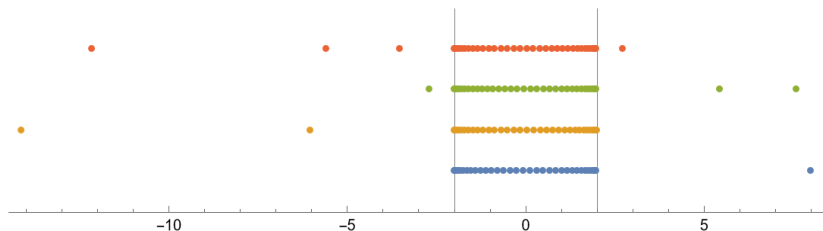


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 32$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

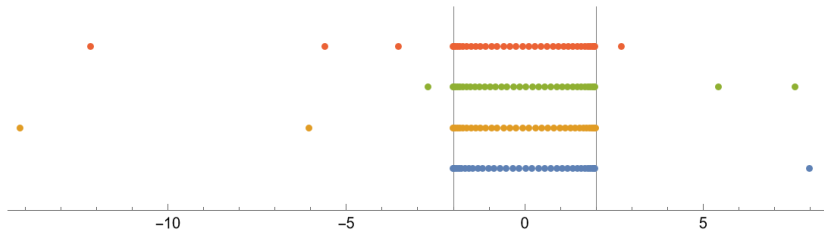


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 33$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \right)^T$$

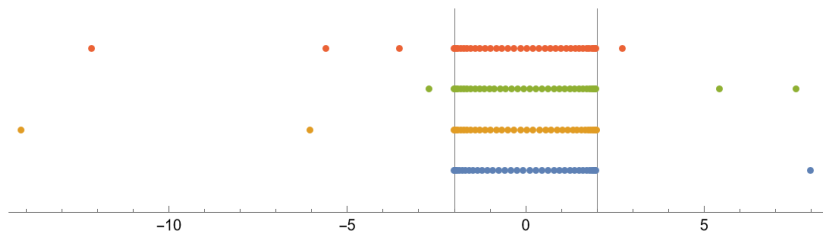


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 34$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

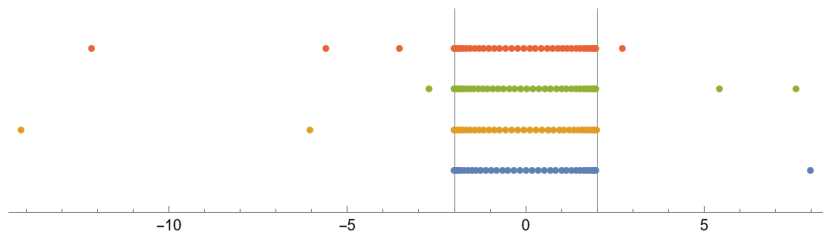


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 35$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

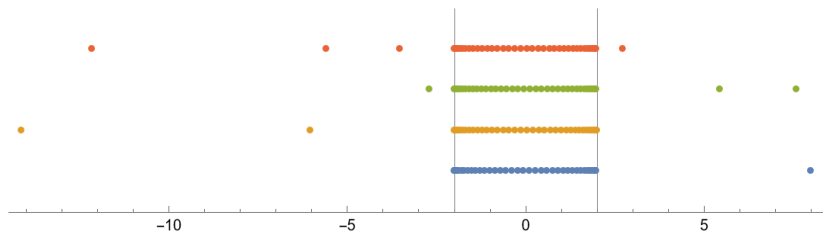


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 36$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

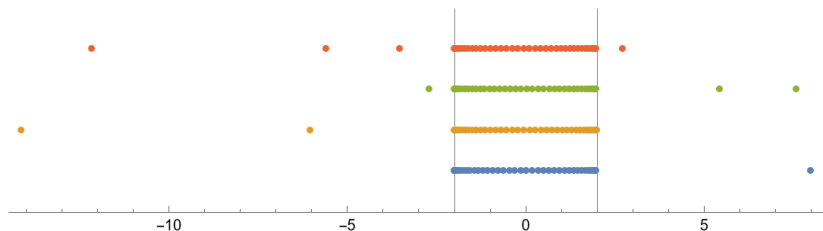


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 37$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

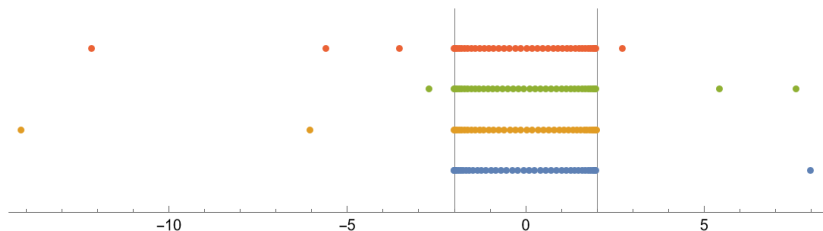


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 38$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

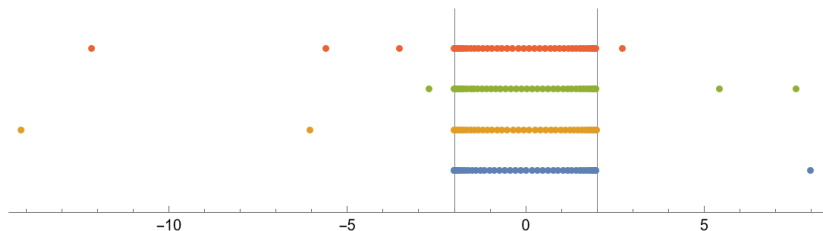


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 39$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right)^T$$

$$\bullet = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array}$$

$$\bullet = \left(\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & \\ \hline \end{array} \right)^T$$

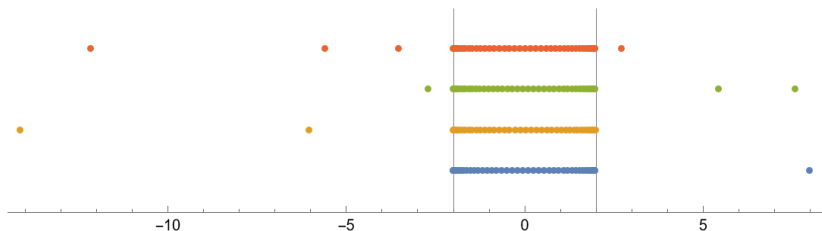


Figure: Roots of $P_{l+4+k}^{(\lambda)}$, with $k = 40$

Rep. Theory - Generalised Roots of Unity

Lets take a look at a series and run it backwards...

Rep. Theory - Generalised Roots of Unity

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k	$P_{l+4+k}^{(5,1)}$
$\vdots$	$\vdots$
-3	$5\alpha^4 + 32\alpha^3 + \alpha^2 - 50\alpha + 48$
-2	$6(\alpha + 2)(\alpha^2 + 5\alpha - 8)$
-1	$(\alpha - 2)(\alpha + 1)(\alpha + 3)(\alpha + 8)$
0	$\alpha^5 + 10\alpha^4 + 5\alpha^3 - 88\alpha^2 - 60\alpha + 96$
1	$\alpha^6 + 10\alpha^5 + 4\alpha^4 - 98\alpha^3 - 71\alpha^2 + 142\alpha + 48$
2	$(\alpha + 1)(\alpha^6 + 9\alpha^5 - 6\alpha^4 - 102\alpha^3 + 26\alpha^2 + 204\alpha - 96)$
$\vdots$	$\vdots$

Rep. Theory - Generalised Roots of Unity

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k	$P_{l+4+k}^{(5,1)}$
$\vdots$	$\vdots$
-3	$5\alpha^4 + 32\alpha^3 + \alpha^2 - 50\alpha + 48$
-2	$6(\alpha + 2)(\alpha^2 + 5\alpha - 8)$
-1	$(\alpha - 2)(\alpha + 1)(\alpha + 3)(\alpha + 8)$
0	$\alpha^5 + 10\alpha^4 + 5\alpha^3 - 88\alpha^2 - 60\alpha + 96$
1	$\alpha^6 + 10\alpha^5 + 4\alpha^4 - 98\alpha^3 - 71\alpha^2 + 142\alpha + 48$
2	$(\alpha + 1)(\alpha^6 + 9\alpha^5 - 6\alpha^4 - 102\alpha^3 + 26\alpha^2 + 204\alpha - 96)$
$\vdots$	$\vdots$

Coincidences?

Rep. Theory - Generalised Roots of Unity

Lets take a look at a series and run it backwards...

k	$P_{l+4+k}^{(5,1)}$
$\vdots$	$\vdots$
-3	$5\alpha^4 + 32\alpha^3 + \alpha^2 - 50\alpha + 48$
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1	$\alpha^6 + 10\alpha^5 + 4\alpha^4 - 98\alpha^3 - 71\alpha^2 + 142\alpha + 48$
2	$(\alpha + 1)(\alpha^6 + 9\alpha^5 - 6\alpha^4 - 102\alpha^3 + 26\alpha^2 + 204\alpha - 96)$
$\vdots$	$\vdots$

Coincidences? All data contained in the elements $\xi_\lambda^{(n)} \dots$

Representation Theory: Bilinear Form - Det. Factorisation

Computed for all $\lambda \vdash l+2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).

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Example: For $l = 0$ we have $\lambda = (2), (1^2)$, $d_\lambda = 1$

$$C^{(2)}(\alpha) = (\alpha - 1), \quad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

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Computed for all $\lambda \vdash l+2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).

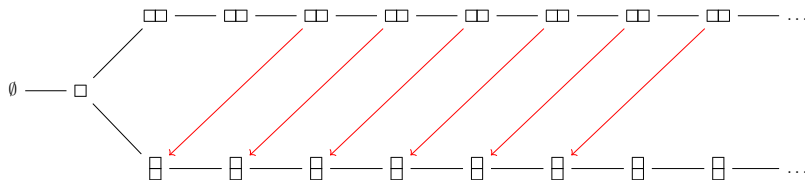
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Example: For $l = 0$ we have $\lambda = (2), (1^2)$, $d_\lambda = 1$

$$C^{(2)}(\alpha) = (\alpha - 1), \quad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

Zeros correspond to maps “across arms” i.e. at $\alpha = -2$ we have

$$\Delta_{(n,(2))}^n \hookrightarrow \Delta_{(n-2,(1^2))}^n.$$



Representation Theory: Bilinear Form - Det. Factorisation

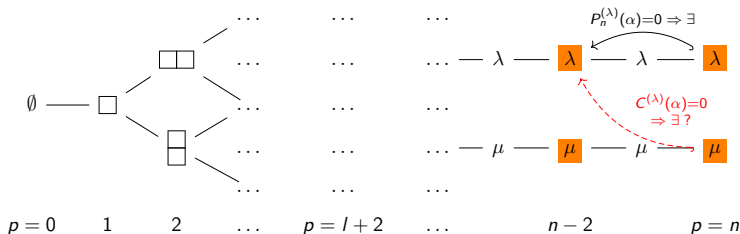
Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix $l, n \geq l + 4, \lambda \vdash l + 2$. $\exists$ polynomials $C^{(\lambda)}(\alpha), P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta_{(n-2,\lambda)}^n))(\alpha) = C^{(\lambda)}(\alpha) \left(P_n^{(\lambda)}(\alpha) \right)^{d_\lambda} \dots$$

Conjecture

The $C^{(\lambda)}(\alpha)$ have integer roots which correspond to “non-TL behaviour”.



Representation Theory: Reflection Geometry

In the TL and Brauer cases maps between standard modules can be described by reflection geometries.

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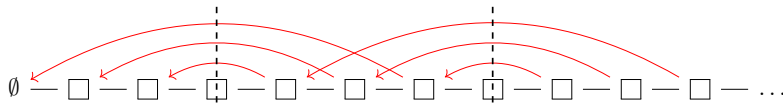
Representation Theory: Reflection Geometry

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The remaining maps can be determined by Frobenius reciprocity:



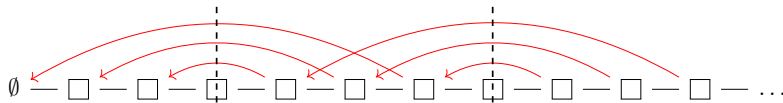
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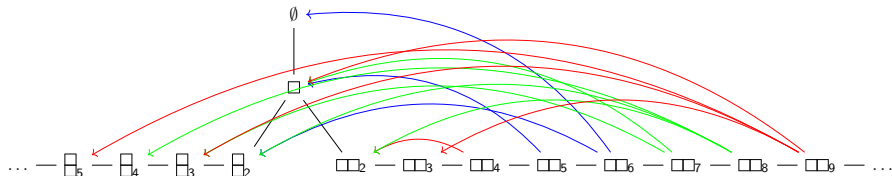
The Block structure can be realised by the action of affine reflections on the Rollet graph! (In this case it is 1D)

Rep. Theory: Reflection Geometry beyond TL?

Let's consider $l = 0$. We have $P_4^{(2)} = \alpha(\alpha^2 + \alpha - 4) = 0$ - we take α such that $(\alpha^2 + \alpha - 4) = 0$:

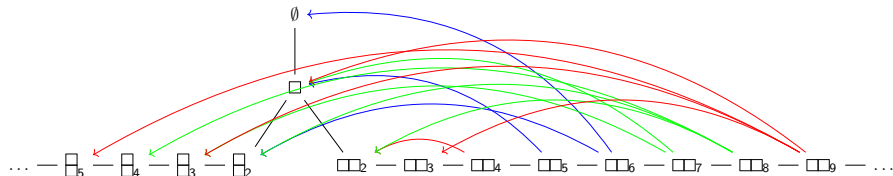
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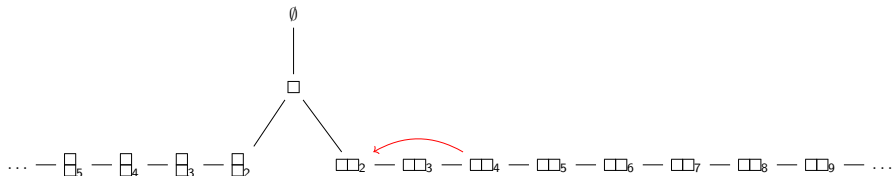


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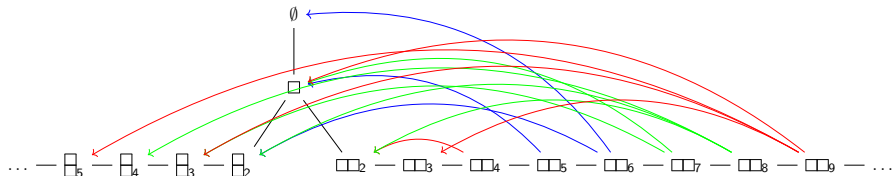


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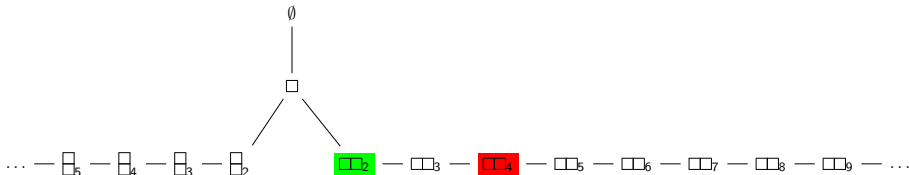


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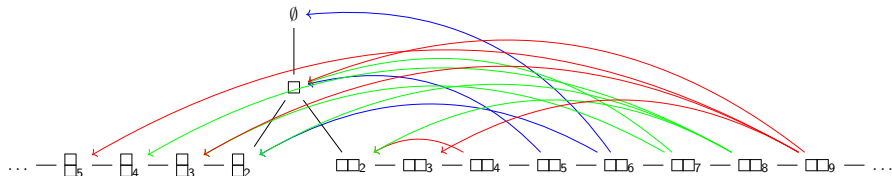


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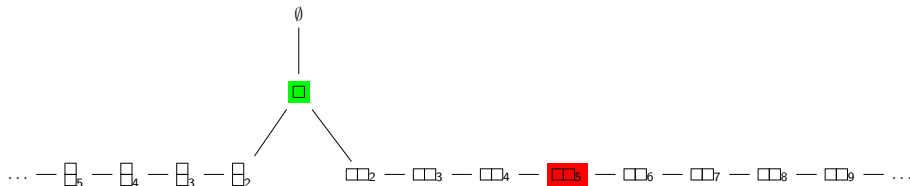


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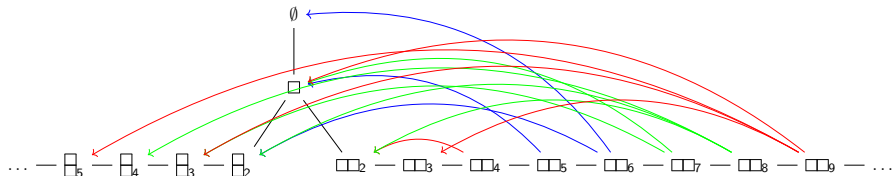


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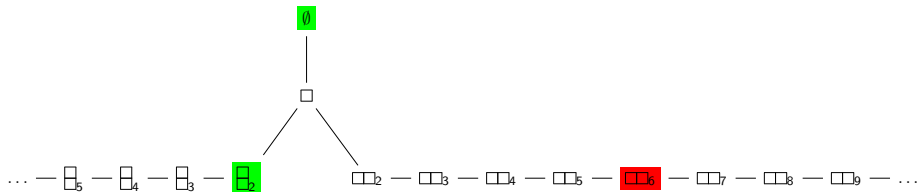


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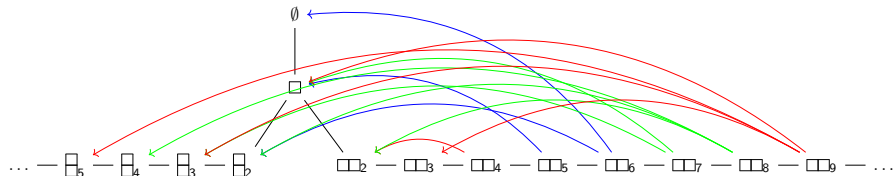


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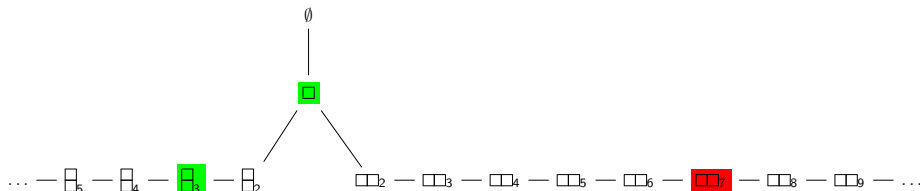


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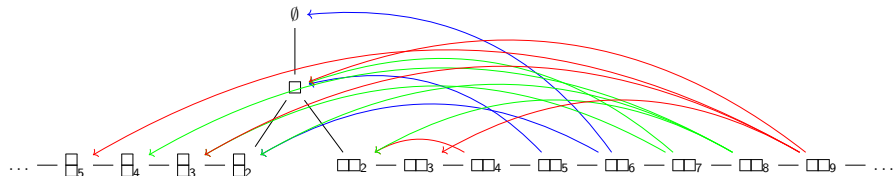


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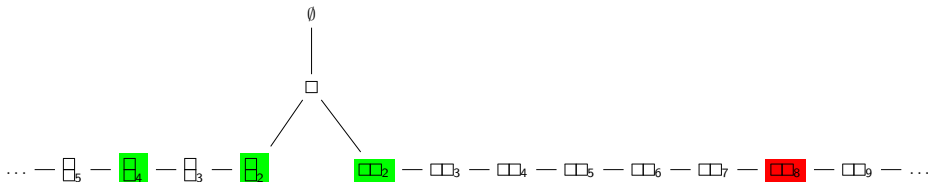


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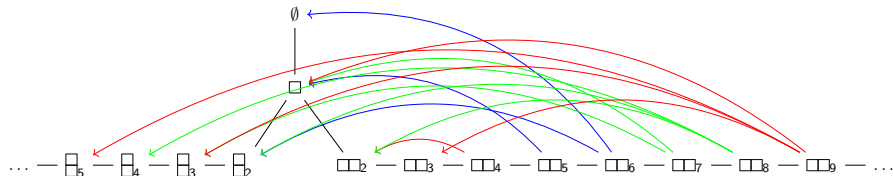


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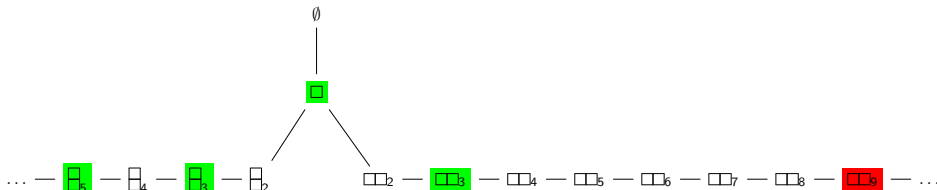


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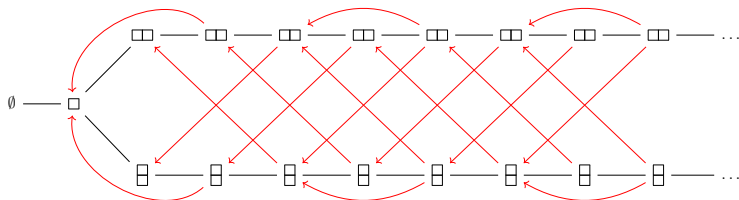


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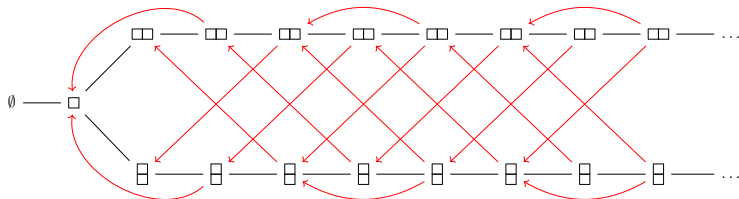
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Let's again consider $l = 0$. When $\alpha = 1$ we have:

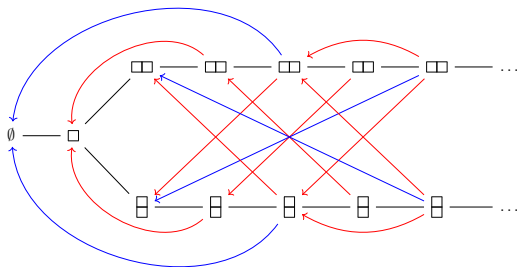


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Let's run frobenius reciprocity...

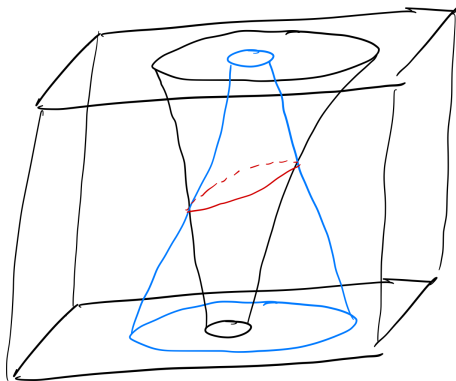


Overview

- 1 Motivation
- 2 Kadar-Martin-Yu Categories/Algebras
- 3 Representation Theory by “Tower of Recollment”
 - Simple/standard modules
 - Semisimplicity Criteria: Generalised Roots of Unity
 - Maps by Reflection Geometry
- 4 Higher Dimensional “Height” Conditions?

Higher Dimensional “Height” Conditions

IDEA: Study nested cobordism categories 1-dimension up, defined by higher “height” conditions, *i.e.* topological restrictions on where embedded surfaces can intersect. *e.g.*



DANKE SCHÖN!

Questions?

References:

-  Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv 2407.07028. arXiv: 2407.07028 [math.RT]. URL: <https://arxiv.org/abs/2407.07028>.
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-  Kadar, Zoltan, Paul P. Martin, and Shona Yu (2014). *On geometrically defined extensions of the Temperley-Lieb category in the Brauer category*. arXiv 1401.1774. DOI: 10.48550/ARXIV.1401.1774. URL: <https://arxiv.org/abs/1401.1774>.
-  Morris, Benjamin and Paul P. Martin (2025). *On semisimplicity criteria and non-semisimple representation theory for the Kadar-Yu algebras*. arXiv: 2512.24535 [math.RT]. URL: <https://arxiv.org/abs/2512.24535>.