Interpolating between Temperley-Lieb and Brauer representation theory: generalised roots of unity
(based on arXiv:2512.24535 joint with P. P. Martin)

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QTCat Seminar, Univeristät Hamburg,
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Overview

1. Motivation
2. Kadar-Martin-Yu Categories/Algebras
3. Representation Theory by "Tower of Recollment"
   - Simple/standard modules
   - Semisimplicity Criteria: Generalised Roots of Unity
   - Maps by Reflection Geometry
4. Higher Dimensional "Height" Conditions?

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Representation Theory: We define a family of algebras

$$TL_n = J_{-1,n} \subset J_{0,n} \subset J_{1,n} \subset \ldots \subset J_{n-2,n} = J_{\infty,n} = Br_n.$$

Temperley-Lieb and Brauer algebras are well understood and have a rich representation theory, which are significantly different. The algebras $J_{l,n}$ give the opportunity to "interpolate" between them in a discrete way.

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Kadar-Martin-Yu Categories: Context

For $\alpha \in \mathbb{K}$, we are going to define a discrete family of nested $\mathbb{K}$-linear subcategories of the diagrammatic Brauer category $Br(\alpha)$:

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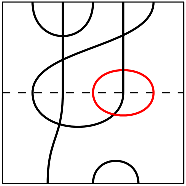

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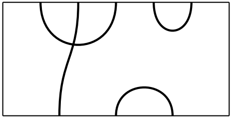

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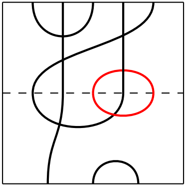
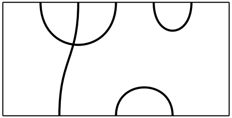
$Br(5,5) \ni$ / $Br(5,3) \ni$  $= \alpha$  $\in Br(5,3)$

- $\otimes : Br \times Br \to Br$ given by horizontal juxtaposition of diagrams.

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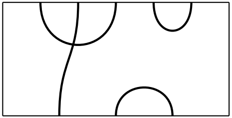
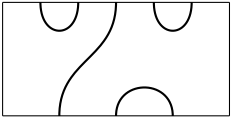
 $\notin TL(5,3)$,  $\in TL(5,3)$

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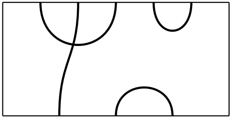
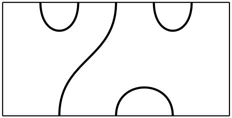
 $\notin TL(5,3)$,  $\in TL(5,3)$

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$$\frac{\dim(TL_n)}{\dim(Br_n)} = \frac{C_n}{(2n-1)!\,!} = \frac{2^n}{(n+1)!} \to 0, \qquad \text{as} \qquad n \to \infty.$$

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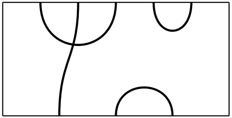
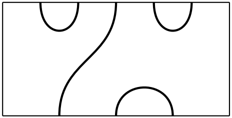
 $\notin TL(5,3)$,  $\in TL(5,3)$

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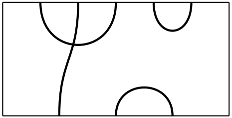
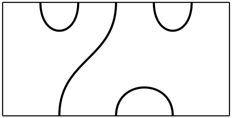
 $\notin TL(5,3)$,  $\in TL(5,3)$

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Context summary: We have $TL(\alpha) \subset Br(\alpha)$ diagrammatic categories and their endomorphism algebras $TL_n(\alpha) \subset Br_n(\alpha)$. The TL "crossingless" condition kills almost everything...

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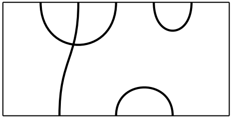

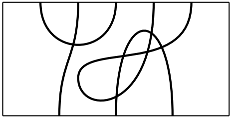

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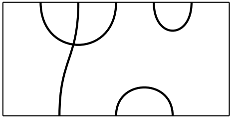

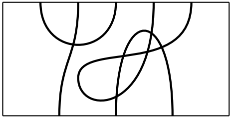

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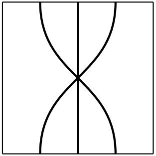

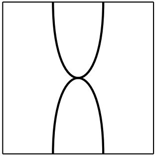

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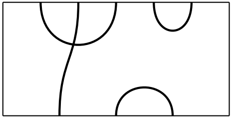

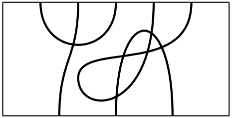

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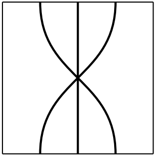
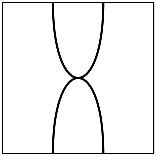
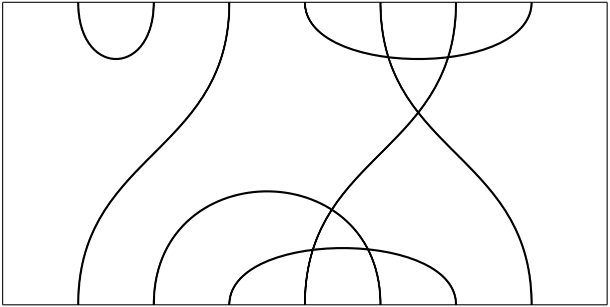




We will pretty much always draw diagrams with **minimal crossings**.

Kadar-Martin-Yu Categories: The Height Condition



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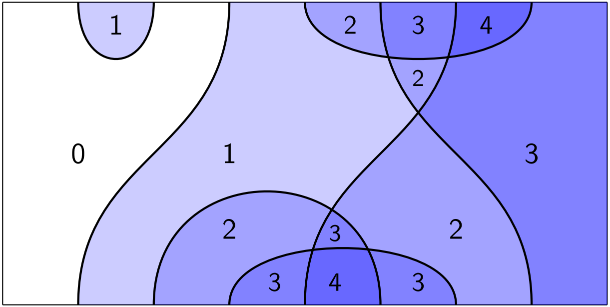

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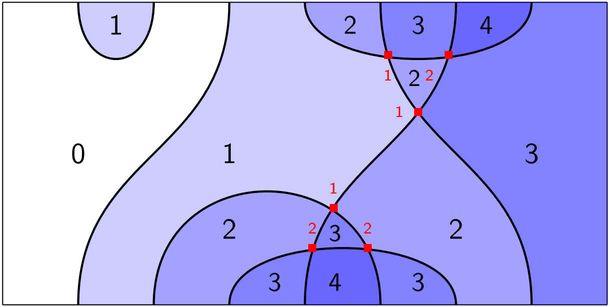

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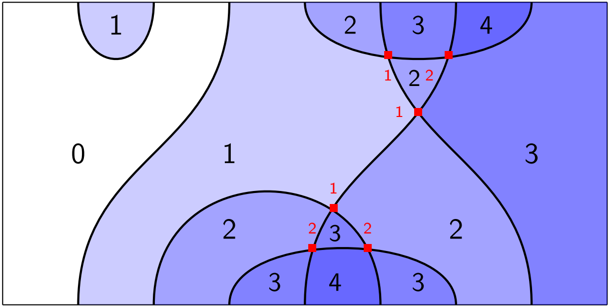

The height of a **diagram**: $ht(\text{diag.}) = \max_{\times_k \in \text{diag.}} {\color{red}k}$, *e.g.* above $ht = 2$.

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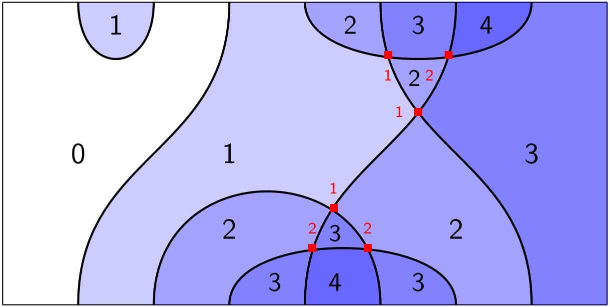

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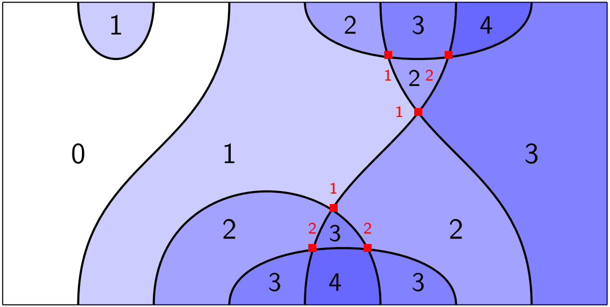

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Gadget*: Minimal crossing number diagrams are minimal height.

Kadar-Martin-Yu Categories: The Height Condition (ctd)

Lemma (Height is non-increasing)

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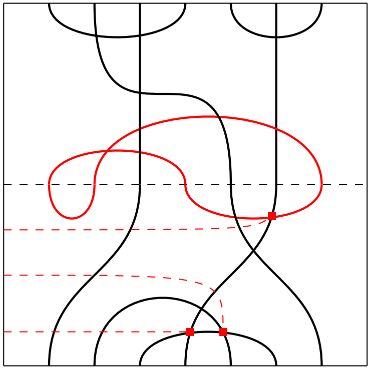
Proof: $(p' \circ p = \alpha^{\#\text{loops}} p'\#p)$



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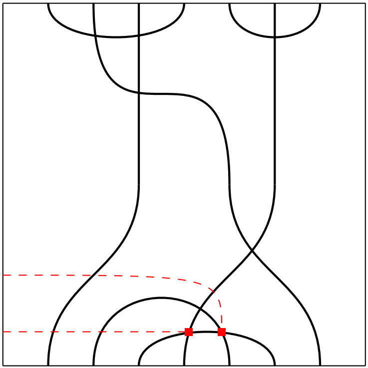

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For $l = -1, 0, 1, \ldots, \infty$ define $J_l(\alpha) \subset Br(\alpha)$ to be the wide, $\mathbb{K}$-linear subcategory where now, $J_l(n, m)$ is spanned by pairings with height $\leq l$.

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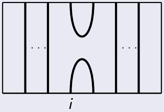
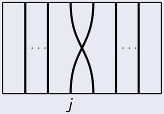
Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}(\alpha)$ is generated by:

$e_i =$ , $s_j =$ ,

for $i = 1, \ldots, n-1$, and $j = 1, \ldots, \min(l+1, n-1)$. *i.e.* $\mathbb{K}S_n \subset Br_n$ whereas $\mathbb{K}S_{\min(l+2,n)} \subset J_{l,n}$.

Overview

Representation Theory: Globalisation/Localisation

Use hom-spaces as bimodules:

$$\begin{tikzcd}
M \arrow[r, mapsto] & J_l(n+2,n) \otimes_{J_{l,n}} M \\
\mathrm{mod}(J_{l,n}) \arrow[r, bend left, "G"] & \mathrm{mod}(J_{l,n+2}) \arrow[l, bend left, "F"] \\
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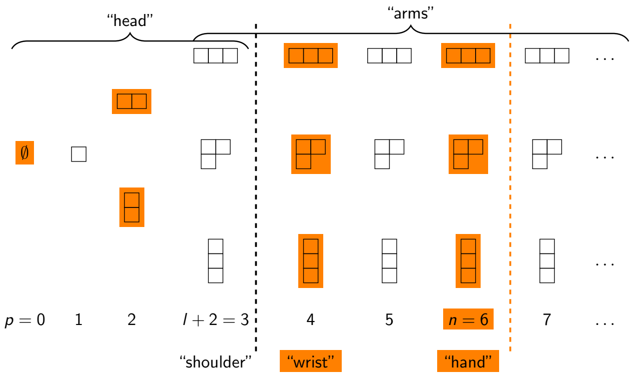
Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$. Use to label simples inductively:

$$\begin{aligned} \Lambda_{l,n} &= \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_{l,n}^{\leq n-2}\right) \\ &= \Lambda_{l,n-2} \sqcup \Lambda(\mathbb{K}\, S_{\min(n,l+2)}) = \Lambda_{l,n-2} \sqcup \{\lambda \vdash \min(n, l+2)\}, \end{aligned}$$

$$\Rightarrow \qquad \Lambda_{l,n} = \{(p,\lambda) \mid p = n, n-2, \ldots, 0/1, \quad \lambda \vdash \min(p, l+2)\}$$

Representation Theory: Anatomy of mod($J_{l,\_}$)



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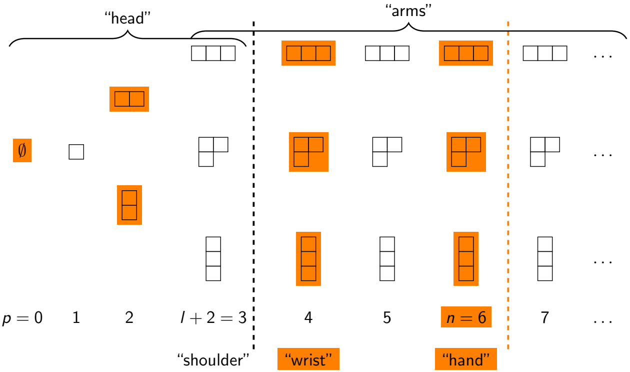

For each label have a **standard module**, constructed inductively:

$$\Delta^n_{(n,\lambda)} \simeq \mathcal{S}_\lambda, \qquad \Delta^n_{(n-2k,\lambda)} = G(\Delta^{n-2}_{(n-2k,\lambda)}).$$

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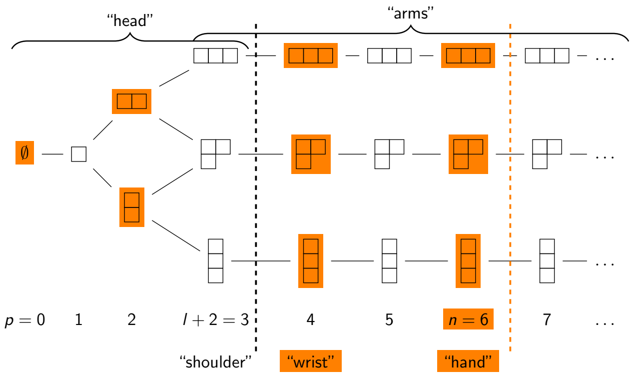

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Encode induction/restriction rules on **Rollet diagram**, *e.g.* above.

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Standard modules have simple heads $L_{(p,\lambda)}$ and determine $[L_{(p,\lambda)} : P_{(p,\lambda)}]$.

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Example: $n = 6$ and label $(4,(2,1))$. Write

$$\mathcal{S}_{(2,1)} = \mathbb{C}S_3 \cdot c_{(2,1)} = \mathbb{C}\{c_{(2,1)}, (2\,3)c_{(2,1)}\},$$

c_λ is a young symmetriser s.t.: $c_\lambda^2 = c_\lambda,\ c_\lambda^* = c_\lambda,\ c_\lambda \cdot \mathbb{C}S_{|\lambda|} \cdot c_\lambda = \mathbb{C}\, c_\lambda.$

Representation Theory: Standard Modules

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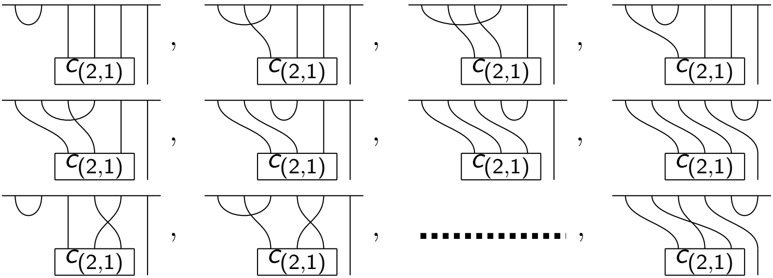
A basis for $\Delta^6_{(4,\lambda)}$ is:



Representation Theory: Semisimplicity

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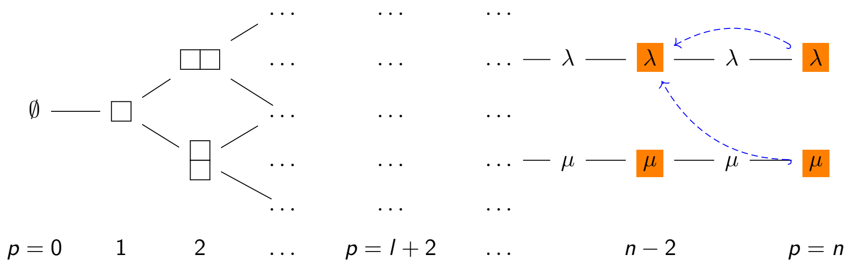
A: When standards = simples. By tower of recollment frame work [Cox et al., 2004], enough to have no maps from "hand" to "wrist":



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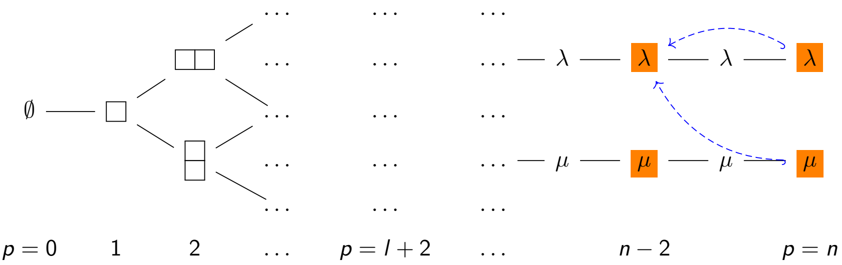

Reduce to looking for (Specht) submodules in $\Delta^{n}_{(n-2,\lambda)}$.

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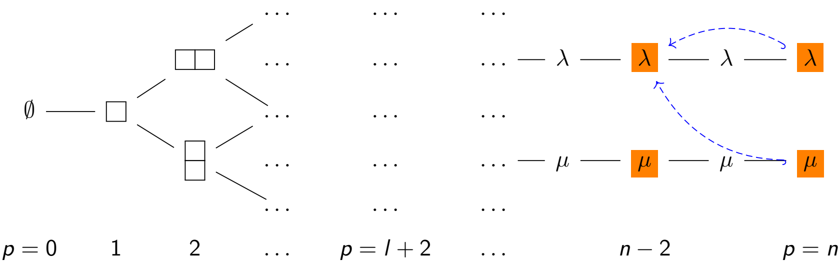
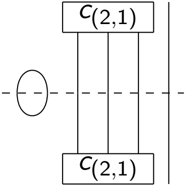


Reduce to looking for (Specht) submodules in $\Delta^n_{(n-2,\lambda)}$.
In turn this is reduced to studying non-degeneracy of a bilinear form

$$\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha].$$

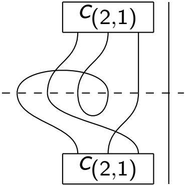
Representation Theory: Bilinear Form

Q: When is $J_{l,n}$ semisimple?
A: … study non-degeneracy of $\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha]$.



$$= \alpha\, c^{(4)}_{(2,1)}\, id_4\, c^{(4)}_{(2,1)}$$
$$= \alpha\, c^{(4)}_{(2,1)}$$

$$\langle \_, \_ \rangle = \alpha,$$

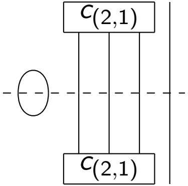


$$= c^{(4)}_{(2,1)}\,(1\ 2\ 3)\,c^{(4)}_{(2,1)}$$
$$= -\frac{1}{2}\, c^{(4)}_{(2,1)}$$

$$\langle \_, \_ \rangle = -\frac{1}{2}.$$

Representation Theory: Bilinear Form

Q: When is $J_{l,n}$ semisimple?
A: ... study non-degeneracy of $\Delta^{n}_{(n-2,\lambda)} \times \Delta^{n}_{(n-2,\lambda)} \to \mathbb{C}[\alpha]$.



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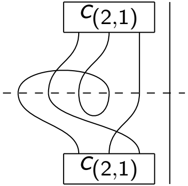


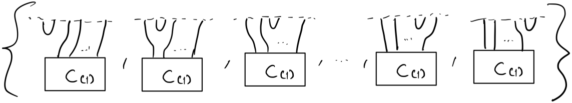
$$= c^{(4)}_{(2,1)}\,(1\ 2\ 3)\, c^{(4)}_{(2,1)}$$
$$= -\frac{1}{2}\, c^{(4)}_{(2,1)}$$
$$\langle\_,\_\rangle = -\frac{1}{2}.$$

Form is symmetric and contravariant with respect to $J_{l,n}$-action, *i.e.*

$$\langle \xi, y\zeta \rangle = \langle y^{*}\xi, \zeta \rangle.$$

Representation Theory: Bilinear Form - TL Case

$\mathcal{B}$ =



Gram matrix of form:

$$\Gamma(\Delta_{(n-2,(1))}) = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow \begin{aligned} P_n^{(1)}(\alpha) \quad &:= |\Gamma(\Delta_{(n-2,(1))})| \\ &= U_{n-1}(\alpha/2) \end{aligned}$$

Chebyshev recursion: $P_{n+1}^{(1)}(\alpha) = \alpha P_n^{(1)}(\alpha) - P_{n-1}^{(1)}(\alpha)$,

$$P_3^{(1)} = \alpha^2 - 1, \qquad P_4^{(1)} = \alpha(\alpha^2 - 2), \qquad P_4^{(1)} = \left(\alpha^2 - \alpha - 1\right)\left(\alpha^2 + \alpha - 1\right).$$

Roots: $P_n^{(1)}(\alpha) = 0$ at $\alpha = q + q^{-1}$ and $q^{2n} = 1$ (or $\alpha = 2\cos(k\pi/n)$).

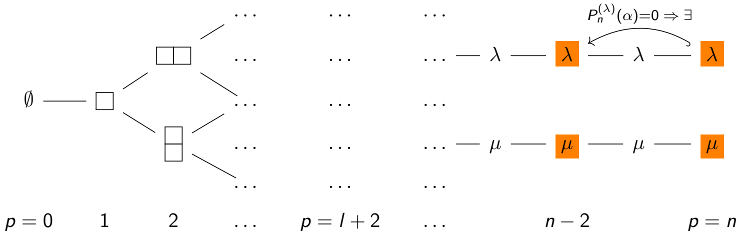
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Morris and Martin, 2025] - Generalised Roots of Unity

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**. Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.



Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$,

Representation Theory: Bilinear Form - Det. Factorisation

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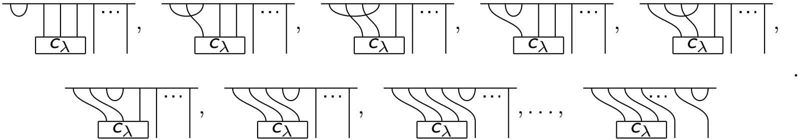

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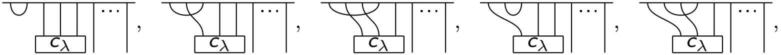

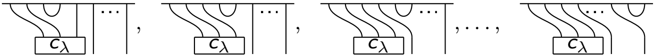

Need to compute determinant of:

$$
\left(\begin{array}{ccccccc|cccccccc}
\alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \mathrm{u}_{12}b \\ \mathrm{u}_{13}b \\ \mathrm{u}_{14}b \\ \mathrm{u}_{23}b \\ \mathrm{u}_{24}b \\ \mathrm{u}_{34}b \\ \mathrm{u}_{45}b \end{array}\right\}\text{head} \\
\left.\begin{array}{l} \mathrm{u}_{56}b \\ \mathrm{u}_{67}b \\ \mathrm{u}_{78}b \\ \mathrm{u}_{89}b \\ \mathrm{u}_{9,10}b \\ \mathrm{u}_{10,11}b \\ \mathrm{u}_{11,12}b \\ \mathrm{u}_{12,13}b \end{array}\right\}\text{tail}
\end{array},
$$

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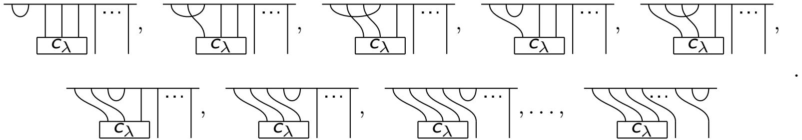

Need to compute determinant of:

$$\left(\begin{array}{ccccccc|cccccccc}
\alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \mathrm{u}_{12}b \\ \mathrm{u}_{13}b \\ \mathrm{u}_{14}b \\ \mathrm{u}_{23}b \\ \mathrm{u}_{24}b \\ \mathrm{u}_{34}b \\ \mathrm{u}_{45}b \end{array}\right\} \text{head} \\
\left.\begin{array}{l} \mathrm{u}_{56}b \\ \mathrm{u}_{67}b \\ \mathrm{u}_{78}b \\ \mathrm{u}_{89}b \\ \mathrm{u}_{9,10}b \\ \mathrm{u}_{10,11}b \\ \mathrm{u}_{11,12}b \\ \mathrm{u}_{12,13}b \end{array}\right\} \text{tail}
\end{array},$$

By expanding along last row we see the determinant satisfies Chebyshev recursion in n. We therefore, obtain the desired form: $\det = C^{(\lambda)}(\alpha) P_n^{(\lambda)}(\alpha)$.

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$
\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_1 \\
\\ \\ \\ \\ \\ \\ \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k_2
\end{array}
$$

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$$
\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \\ \\
\} \, k_1 \\
\\ \\ \\ \\ \\ \\ \\ \\ \\
\} \, k_2 \\
\\
\end{array}
$$

This has two Chebyshev expansions: $\quad \det = C^{(\lambda)}(\alpha)\, P^{(\lambda)}_{n_1}(\alpha) = D^{(\lambda)}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha).$

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$
\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
\\ \\ \\ \\ \\ \\ \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} k_1 \\
\\ \\ \\ \\ \\ \\ \\
\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} k_2
\end{array}
$$

This has two Chebyshev expansions: $\quad \det = C^{(\lambda)}(\alpha)\, P^{(\lambda)}_{n_1}(\alpha) = D^{(\lambda)}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$.

Need to

1. Show $P^{(\lambda)}_n(\alpha) = Q^{(\lambda)}_n(\alpha)$. (Interesting)
2. Show $P^{(\lambda)}_{n_1}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$ divides determinant. (Boring)

Proof Sketch: Need to

1. Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
2. Show $P_{n_1}^{(\lambda)}(\alpha)\, Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

To prove 1. we show both series have the same initial conditions by a row reduction procedure...

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

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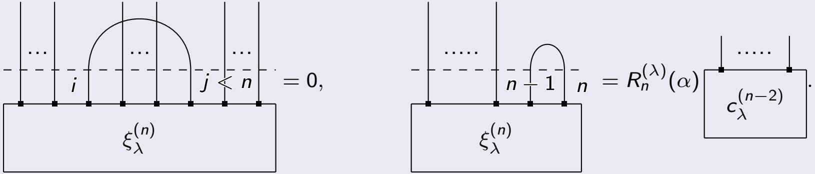
2. Show $P_{n_1}^{(\lambda)}(\alpha)\, Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

To prove 1. we show both series have the same initial conditions by a row reduction procedure... uses a nice family of elements $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$.

Lemma

For each $n \geq l + 4$ there exists a unique $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$ (polynomial coefficients) such that:

1. *$\xi_\lambda^{(n)}$ is non-zero for all specialisations $\alpha \in \mathbb{C}$,*
2. *We have (in $\Delta^{n-2}_{(n-2,\lambda)}$)*



When $\alpha \in \mathbb{C}$ s.t. $R_n^{(\lambda)}(\alpha) = 0$, $\xi_\lambda^{(n)}$ generates a submodule $J_{l,n}\xi_\lambda^{(n)} \simeq \mathcal{S}_\lambda \subset \Delta^n_{(n-2,\lambda)}$. □

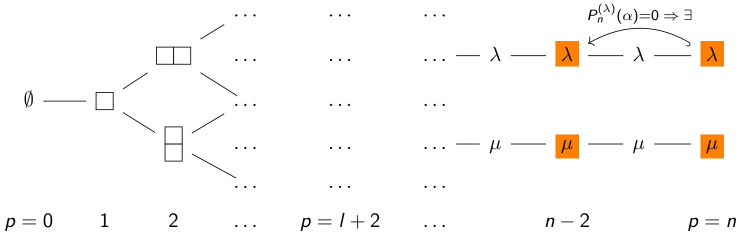
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**.
Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.



We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

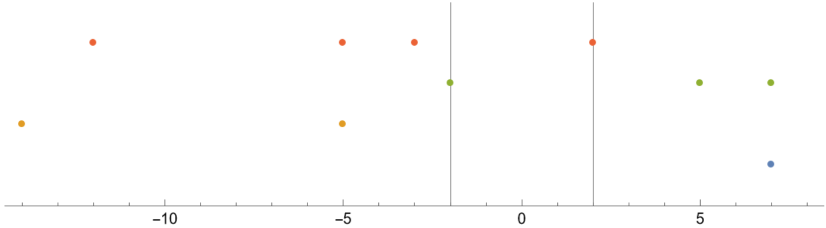


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = -1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

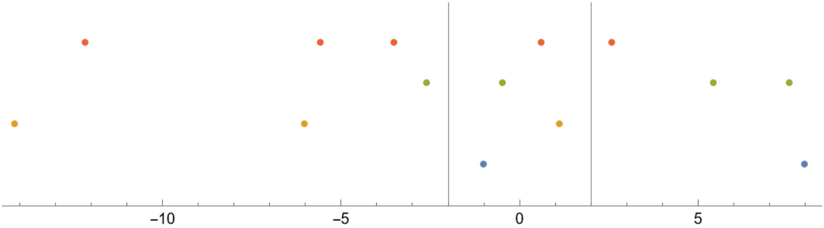


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 0$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

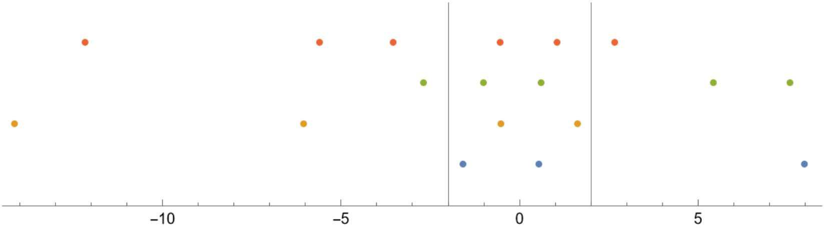


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

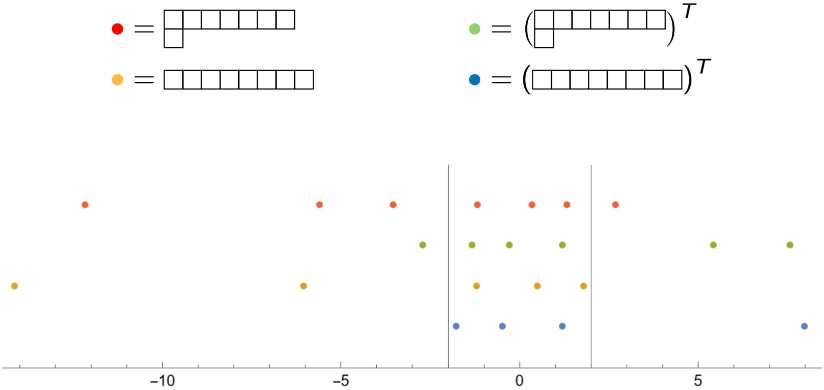


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 2$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l+2=8$ in the cases below:

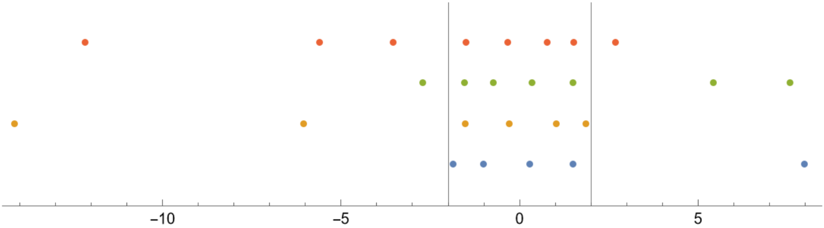


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k=3$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

- ● (red) $= \yng(7,1)$
- ● (orange) $= \yng(8)$
- ● (green) $= \left(\yng(7,1)\right)^T$
- ● (blue) $= \left(\yng(8)\right)^T$

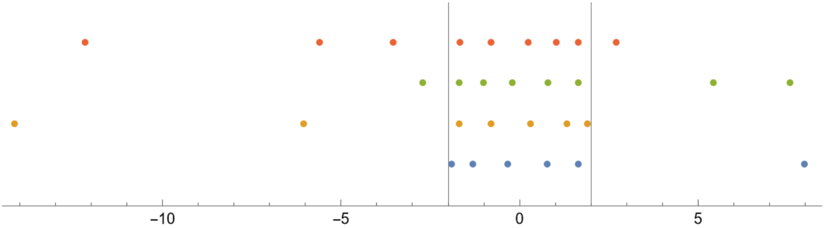


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 4$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

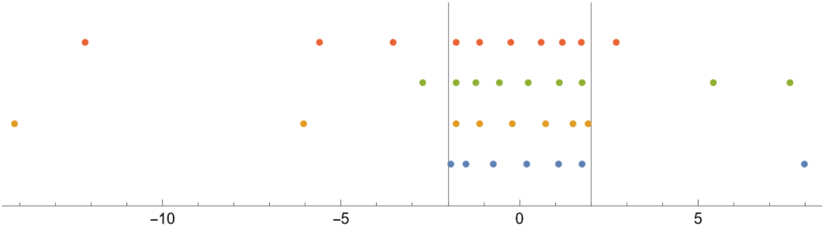


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 5$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l+2 = 8$ in the cases below:

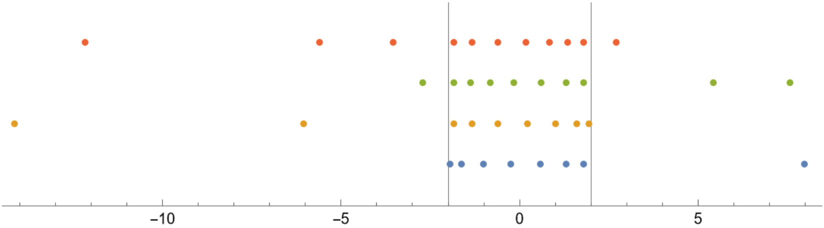


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 6$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

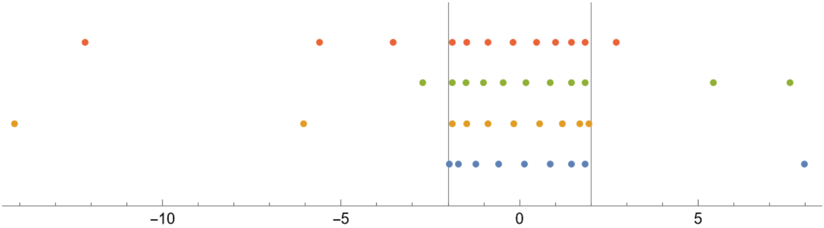


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 7$

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

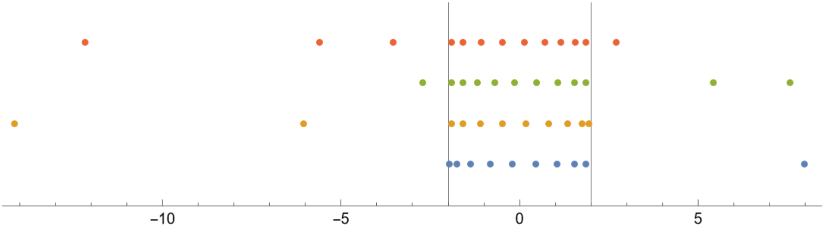


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 8$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

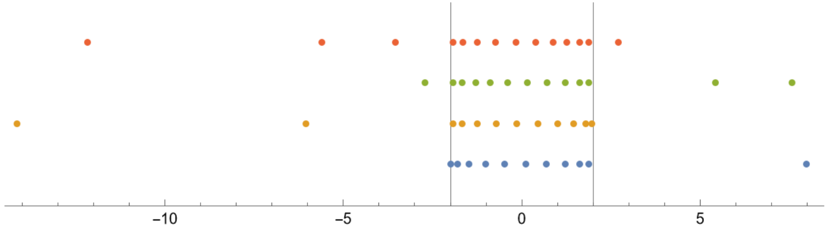


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 9$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

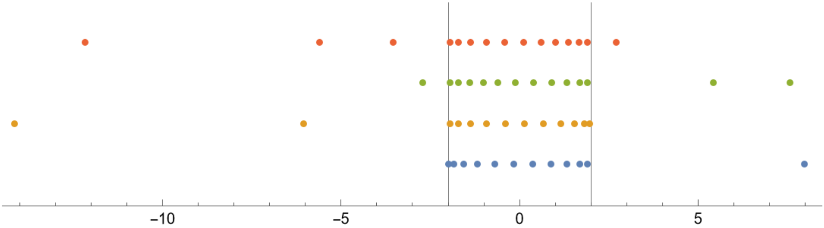


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 10$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l+2 = 8$ in the cases below:

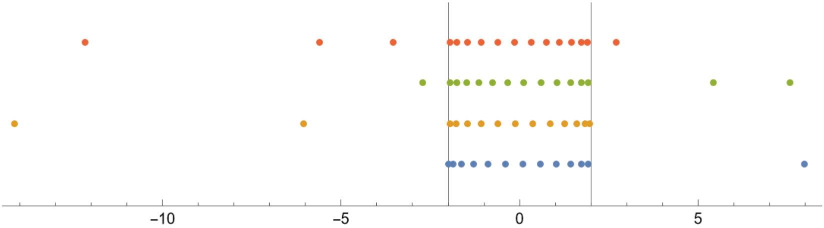


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 11$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

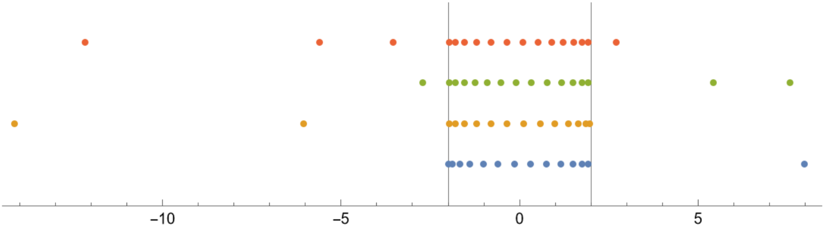


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 12$

We plot roots for $\lambda \vdash l+2 = 8$ in the cases below:

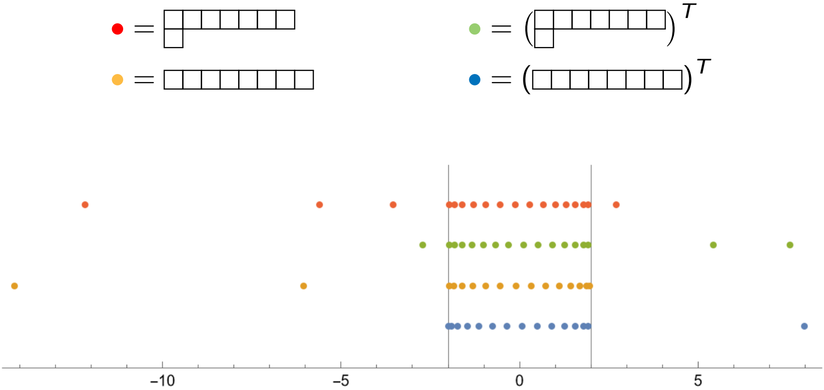


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 13$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

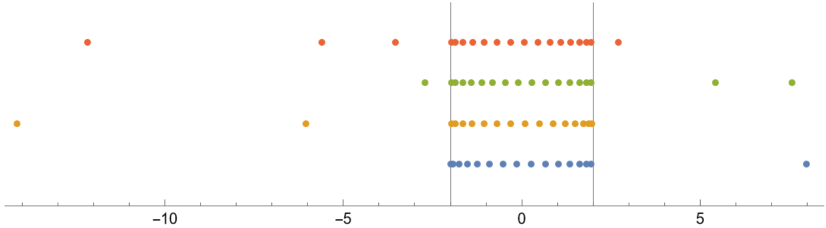


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 14$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$\textcolor{red}{\bullet} = \ydiagram{7,1}$ $\qquad$ $\textcolor{green}{\bullet} = \left(\ydiagram{7,1}\right)^T$

$\textcolor{orange}{\bullet} = \ydiagram{8}$ $\qquad$ $\textcolor{blue}{\bullet} = \left(\ydiagram{8}\right)^T$

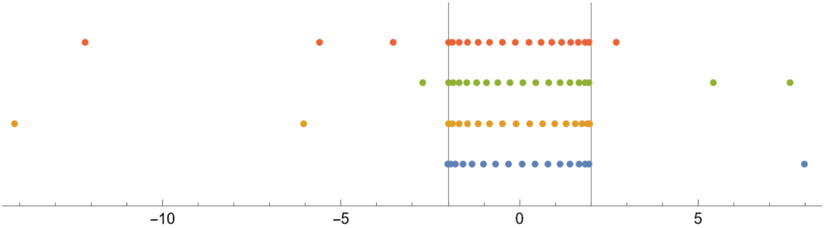


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 15$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

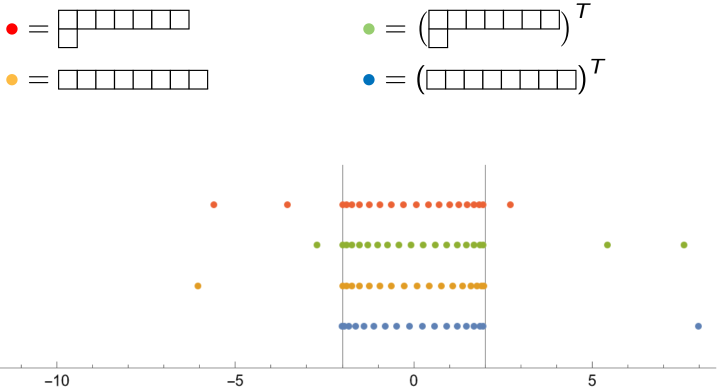


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 16$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

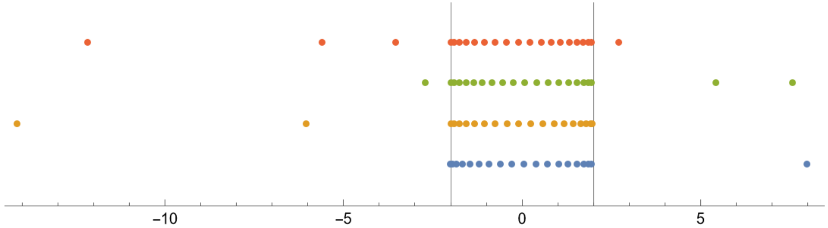
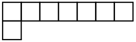


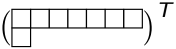
Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 17$

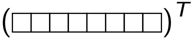
Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

● = 

● = 

● = 

● = 

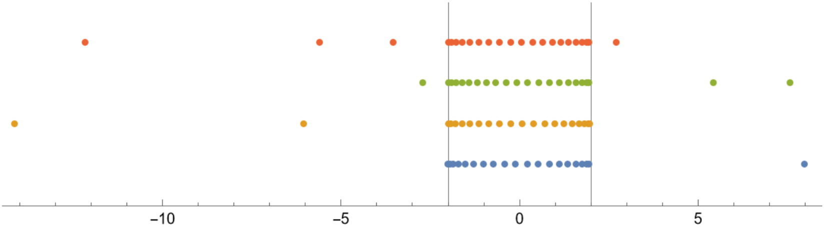


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 18$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

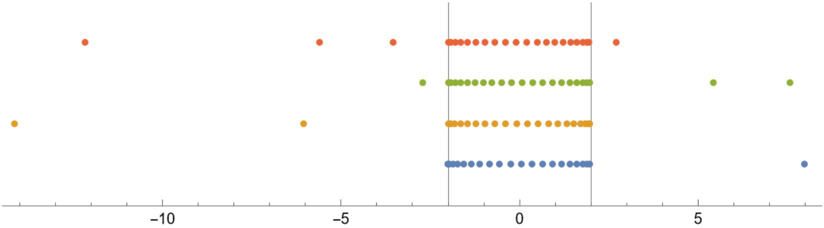


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 19$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

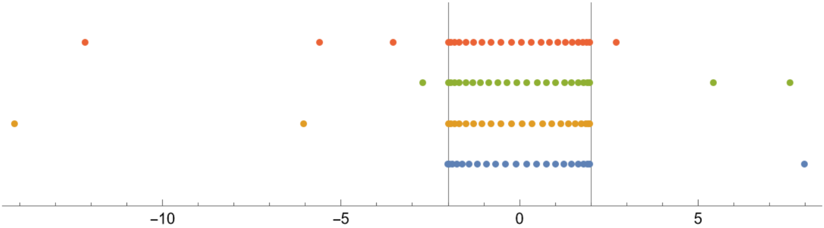


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 20$

We plot roots for $\lambda \vdash l+2=8$ in the cases below:

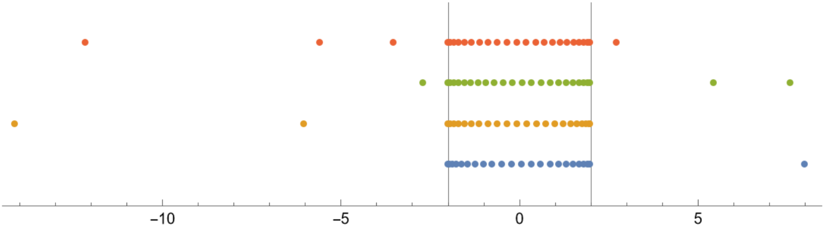


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k=21$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

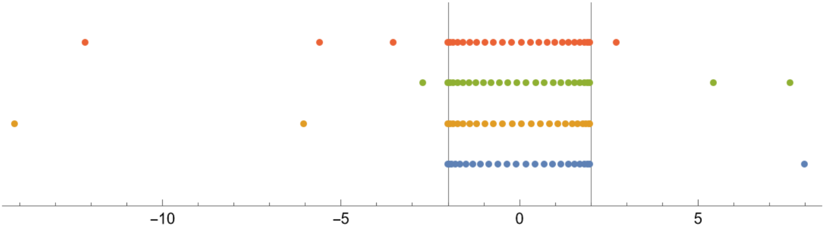


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 22$

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

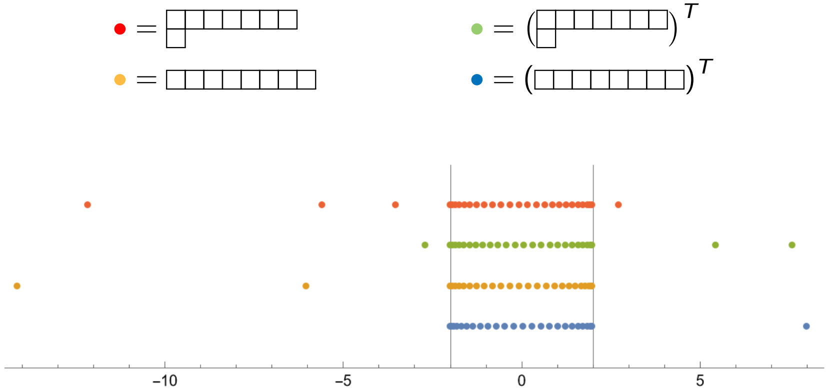


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 23$

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

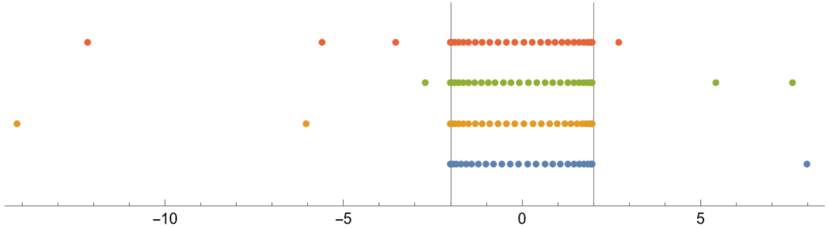


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 24$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

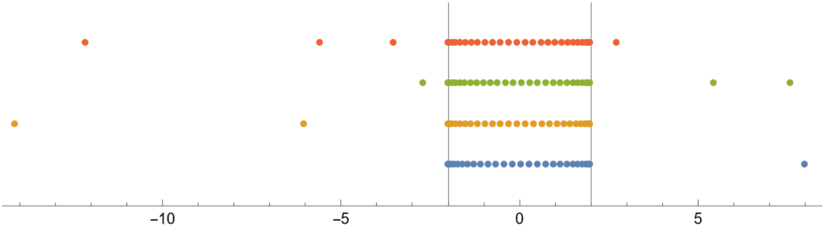


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 25$

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

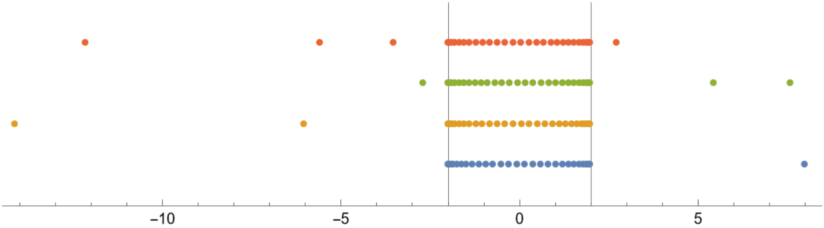


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 26$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

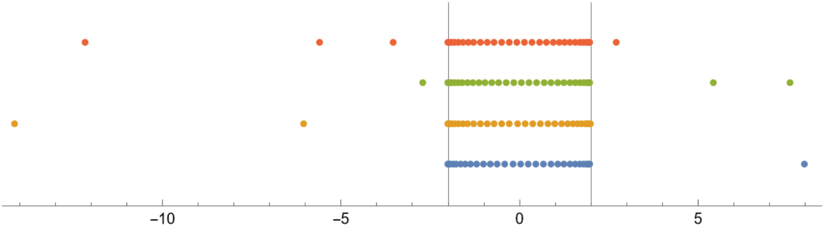


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 27$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

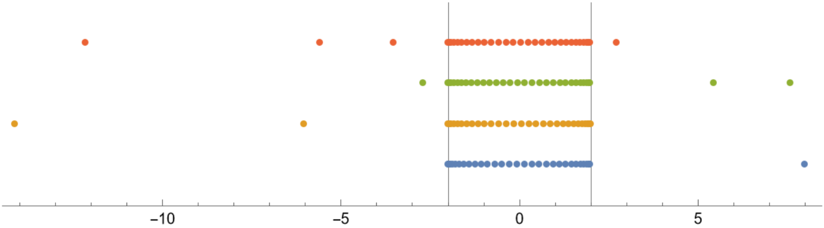


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 28$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$\bullet = \ydiagram{7,1}$ $\qquad$ $\bullet = \left(\ydiagram{7,1}\right)^T$

$\bullet = \ydiagram{8}$ $\qquad$ $\bullet = \left(\ydiagram{8}\right)^T$

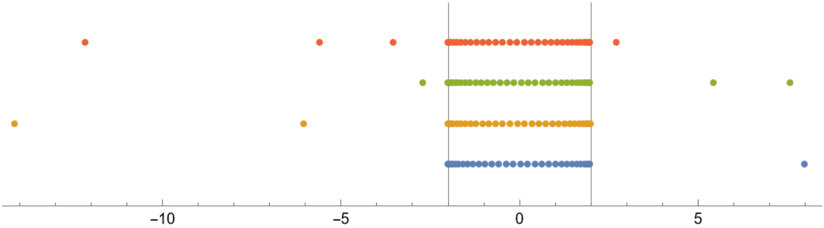


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 29$

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

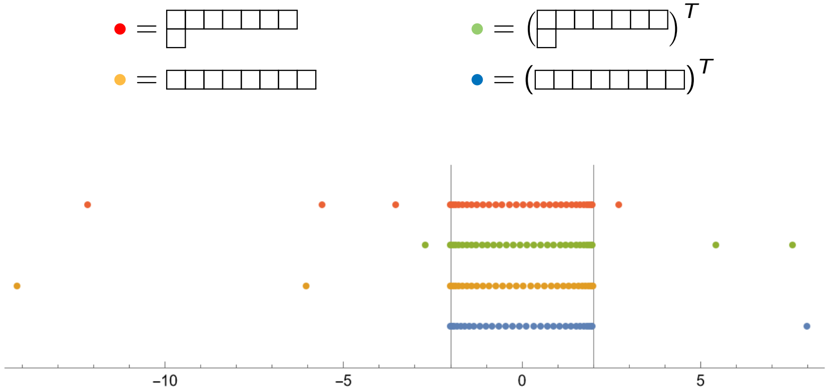


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 30$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

$\bullet = \ydiagram{7,1}$ $\qquad$ $\bullet = \left(\ydiagram{7,1}\right)^T$

$\bullet = \ydiagram{8}$ $\qquad$ $\bullet = \left(\ydiagram{8}\right)^T$

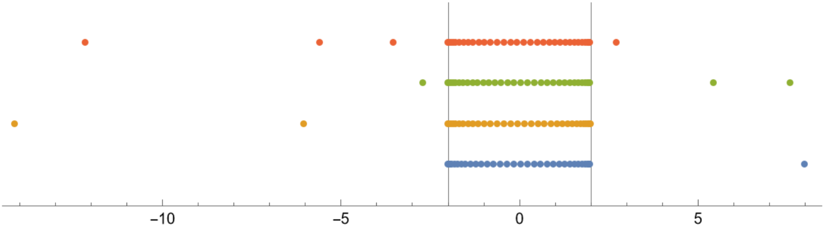


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 31$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

● = $\ydiagram{7,1}$ ● = $\left(\ydiagram{7,1}\right)^T$

● = $\ydiagram{8}$ ● = $\left(\ydiagram{8}\right)^T$

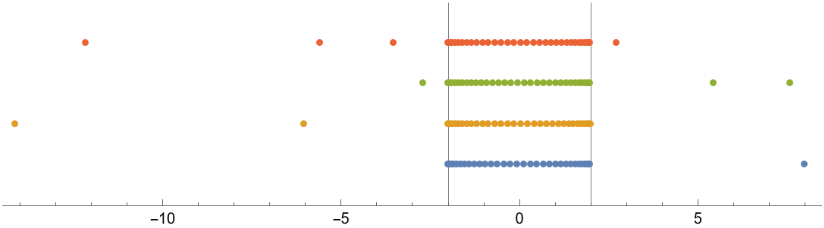


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 32$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

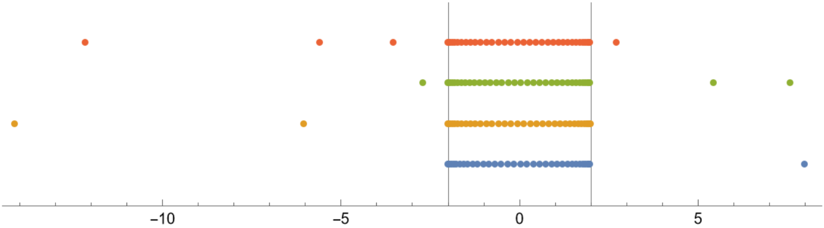


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 33$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

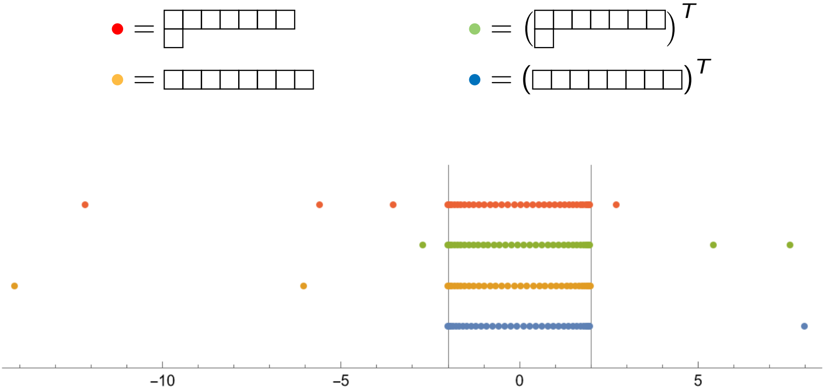


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 34$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

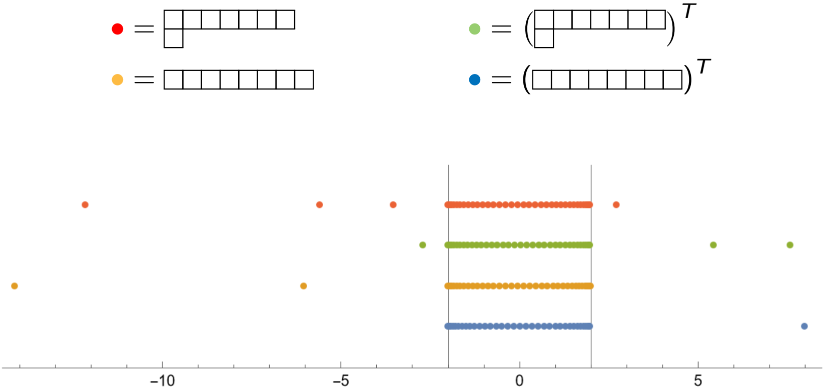


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 35$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

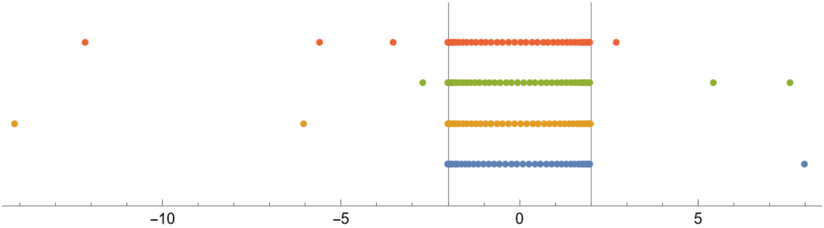


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 36$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

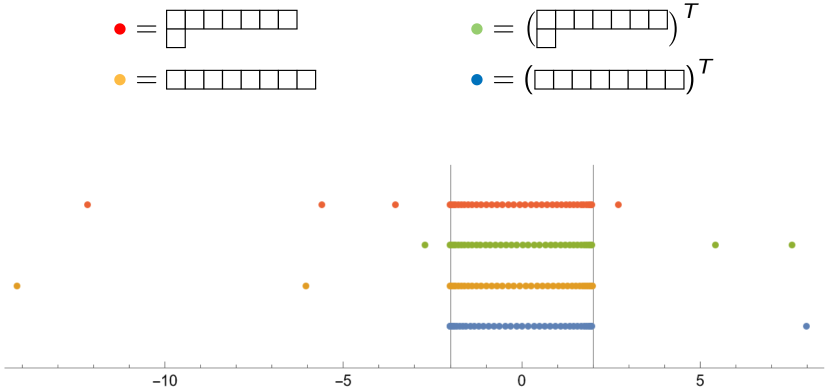


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 37$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

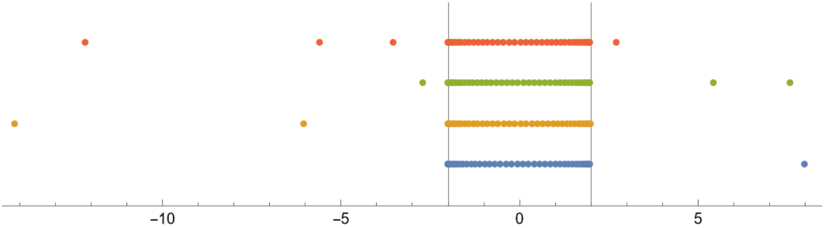


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 38$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

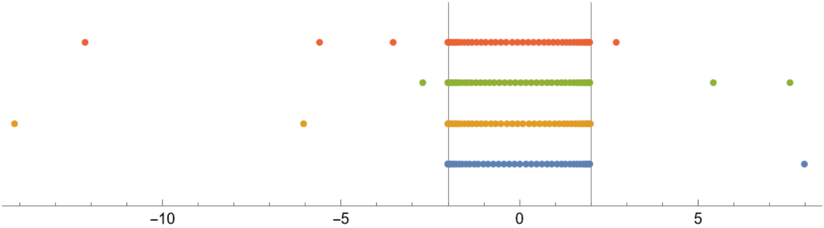


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 39$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:

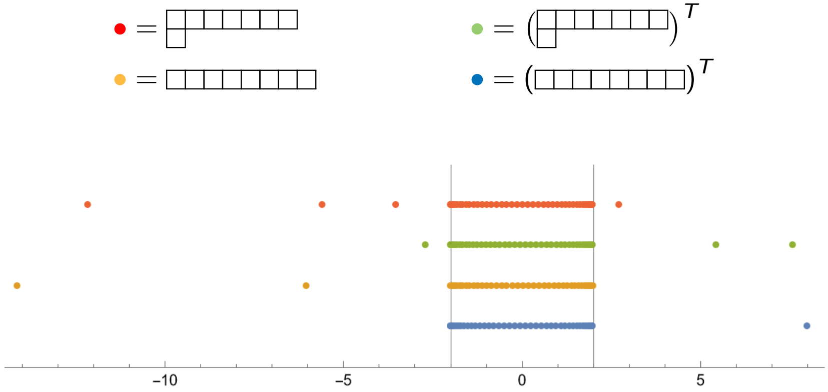


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 40$

Lets take a look at a series and run it backwards...

Rep. Theory - Generalised Roots of Unity

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k	$P^{(5,1)}_{l+4+k}$
$\vdots$	$\vdots$
-3	$5\alpha^4 + 32\alpha^3 + \alpha^2 - 50\alpha + 48$
-2	$6(\alpha+2)\left(\alpha^2 + 5\alpha - 8\right)$
-1	$(\alpha-2)(\alpha+1)(\alpha+3)(\alpha+8)$
0	$\alpha^5 + 10\alpha^4 + 5\alpha^3 - 88\alpha^2 - 60\alpha + 96$
1	$\alpha^6 + 10\alpha^5 + 4\alpha^4 - 98\alpha^3 - 71\alpha^2 + 142\alpha + 48$
2	$(\alpha+1)\left(\alpha^6 + 9\alpha^5 - 6\alpha^4 - 102\alpha^3 + 26\alpha^2 + 204\alpha - 96\right)$
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Coincidences? All data contained in the elements $\xi^{(n)}_\lambda$...

Representation Theory: Bilinear Form - Det. Factorisation

Computed for all $\lambda \vdash l + 2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).

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$$C^{(2)}(\alpha) = (\alpha - 1), \qquad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

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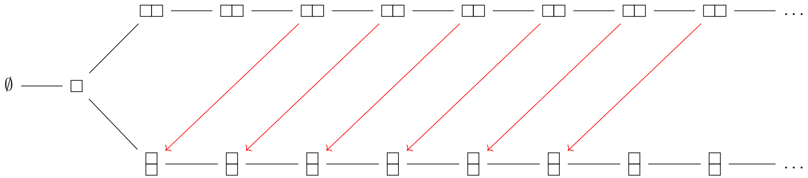
Zeros correspond to maps "across arms" i.e. at $\alpha = -2$ we have $\Delta^n_{(n,(2))} \hookrightarrow \Delta^n_{(n-2,(1^2))}$.



Representation Theory: Bilinear Form - Det. Factorisation

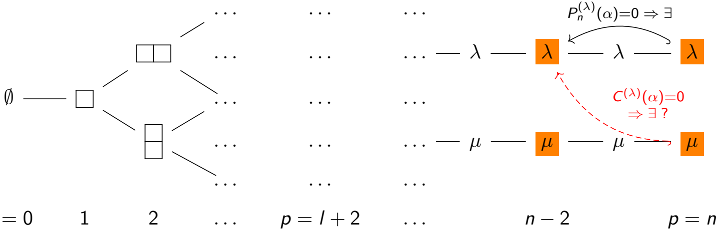
Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda}\dots$$

Conjecture

The $C^{(\lambda)}(\alpha)$ have integer roots which correspond to "non-TL behaviour".



Representation Theory: Reflection Geometry

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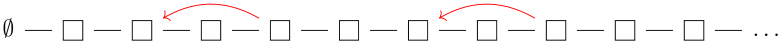
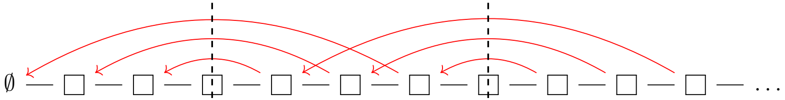


The remaining maps can be determined by frobenius reciprocity:



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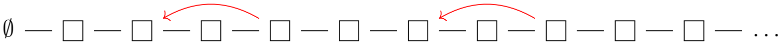

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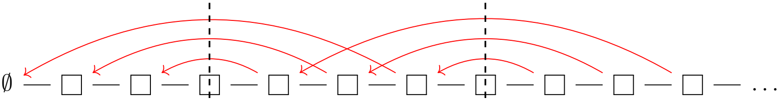

The Block structure can be realised by the action of affine reflections on the Rollet graph! (In this case it is 1D)

Rep. Theory: Reflection Geometry beyond TL?

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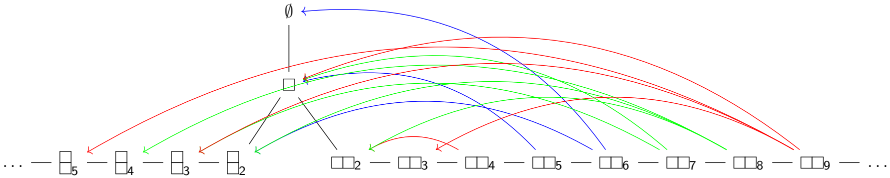

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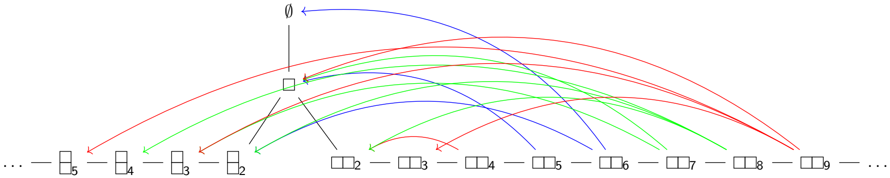

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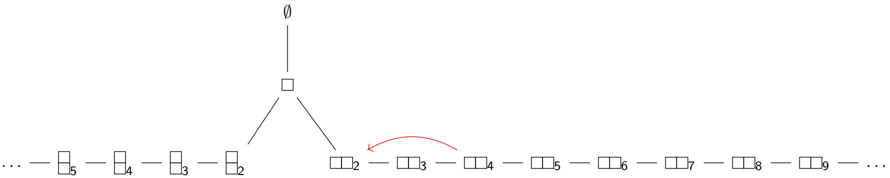

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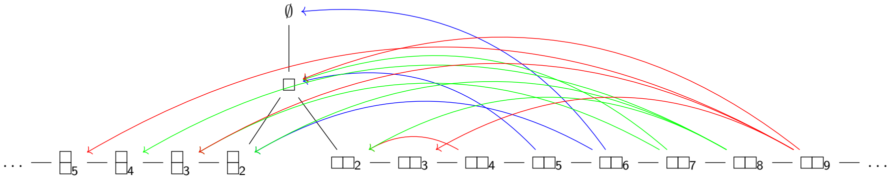

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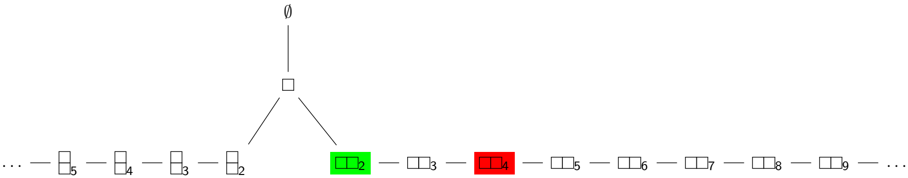

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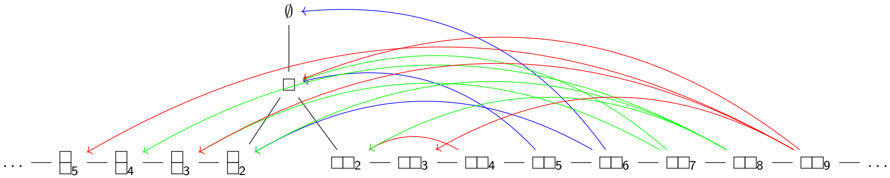

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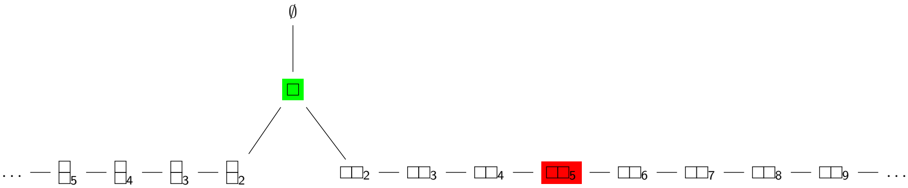

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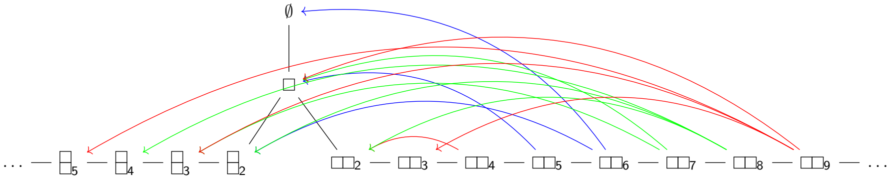

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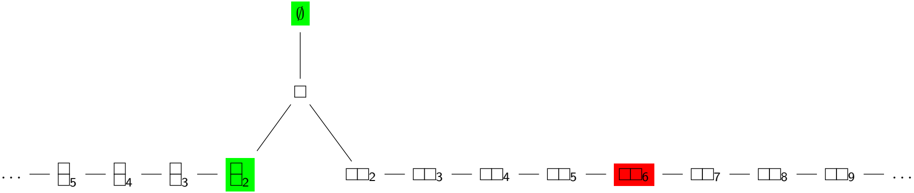

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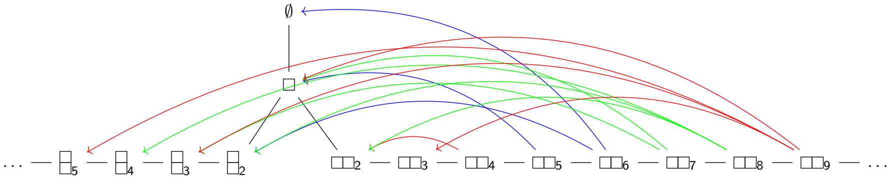

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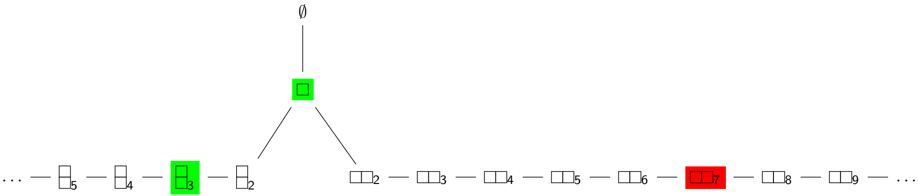

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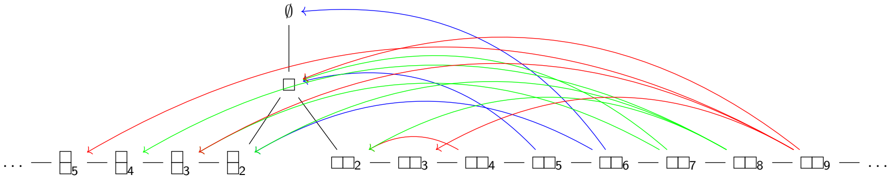

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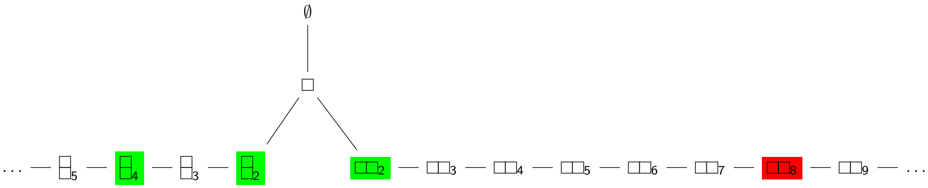

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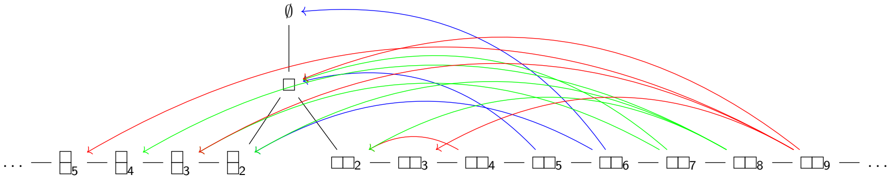

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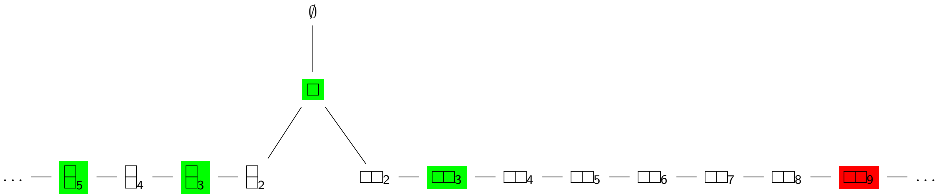

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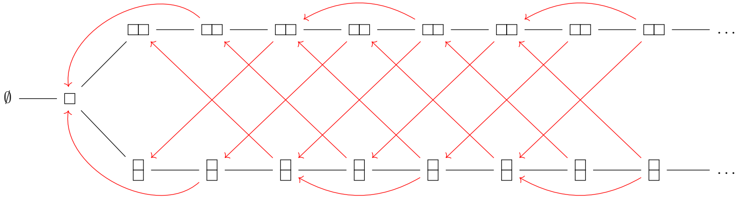
Let's again consider $l = 0$. When $\alpha = 1$ we have:



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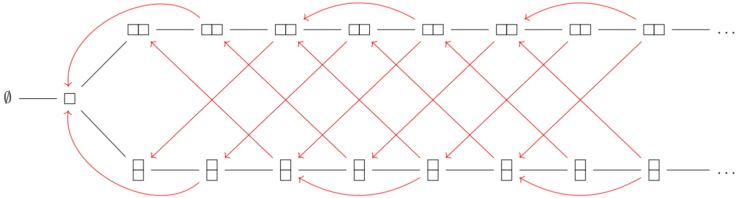
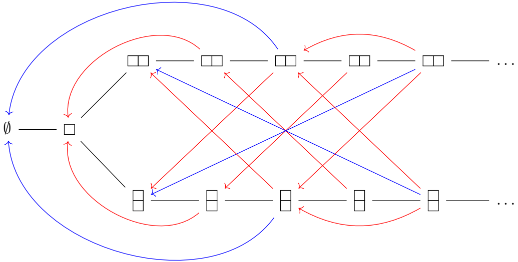


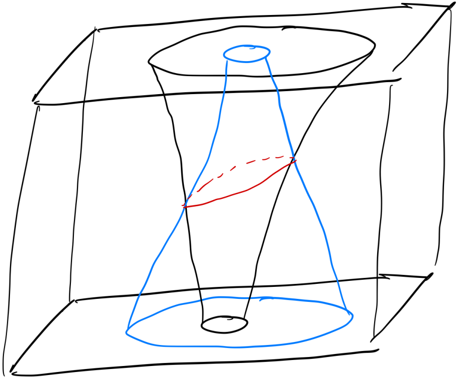
Let's run frobenius reciprocity...



Overview

Higher Dimensional "Height" Conditions

IDEA: Study nested cobordism categories 1-dimension up, defined by higher "height" conditions, *i.e.* topological restrictions on where embedded surfaces can intersect. *e.g.*



DANKE SCHÖN!

Questions?

References:

Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv 2407.07028. arXiv: 2407.07028 [math.RT]. URL: https://arxiv.org/abs/2407.07028.

Cox, Anton et al. (2004). *Representation theory of towers of recollement: theory, notes, and examples*. arXiv 0411395. arXiv: math/0411395 [math.RT].

Kadar, Zoltan, Paul P. Martin, and Shona Yu (2014). *On geometrically defined extensions of the Temperley-Lieb category in the Brauer category*. arXiv 1401.1774. DOI: 10.48550/ARXIV.1401.1774. URL: https://arxiv.org/abs/1401.1774.

Morris, Benjamin and Paul P. Martin (2025). *On semisimplicity criteria and non-semisimple representation theory for the Kadar-Yu algebras*. arXiv: 2512.24535 [math.RT]. URL: https://arxiv.org/abs/2512.24535.