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The Yang-Baxter Equation and Quantum Group Symmetry

Benjamin Morris[1]
Based on Honours thesis sup. by Prof. V. Mangazeev[2]

[1]School of Mathematics,
University of Leeds
[2]Department of Theoretical Physics,
Australian National University

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Overview

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1. Background - Yang-Baxter equation in statistical Mechanics

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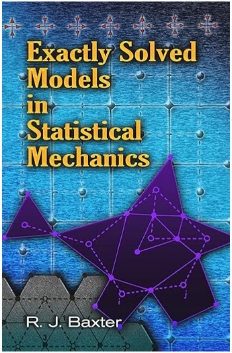
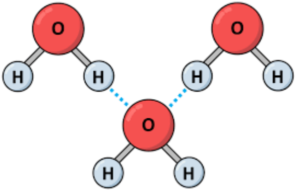


Rodney Baxter in 1999.

Ice type models in statistical mechanics

E.g. 6-vertex model: models hydrogen bonding in (2D crystalline) water:



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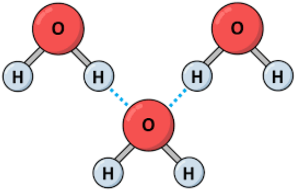

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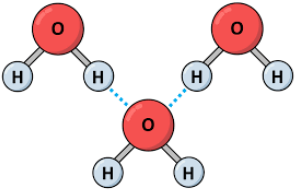

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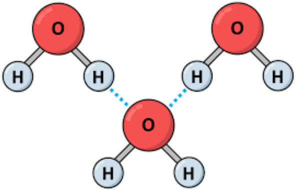

- Each H_2O molecule bonding with 4 others.
- Oxygen atoms regularly arranged, bonds share a hydrogen which is closer to one of the neighbouring oxygens.
- Electronic neutrality condition - each oxygen is near two hydrogens.

Ice type models

More generally on an $N \times M$ square lattice:

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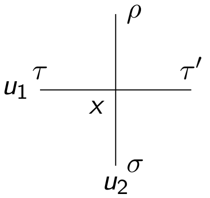

$$= \epsilon(u_1, u_2; \sigma, \rho; \tau, \tau') = \epsilon(x)$$

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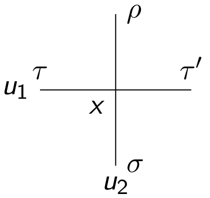

$$= \epsilon(u_1, u_2; \sigma, \rho; \tau, \tau') = \epsilon(x)$$

- Total energy of a configuration Φ is

$$E(\Phi) = \sum_x \epsilon(x)$$

Ice type models

An important quantity to compute is the Partition function:

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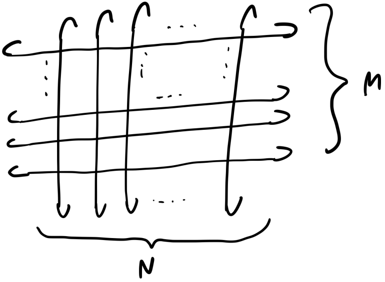
A huge sum ($|S|^{N \cdot M}$)... but finite!



Ice type models

How to compute...?

Ice type models

How to compute...? Define the **transfer matrix** $T(u_1, u_2)$, labelled by pairs $\boldsymbol{\sigma}, \boldsymbol{\rho} \in \mathcal{S}^N$:

$$(T(u_1, u_2))_{\boldsymbol{\sigma},\boldsymbol{\rho}} := \sum_{\boldsymbol{\tau} \in \mathcal{S}^N} \prod_{i=1}^{N} w(u_1, u_2; \sigma_i, \rho_i; \tau_i, \tau_{i+1})$$

where $w(u_1, u_2; \sigma, \rho; \tau, \tau') = e^{-\beta\epsilon(x)}$.

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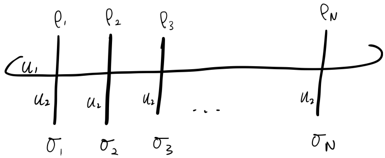

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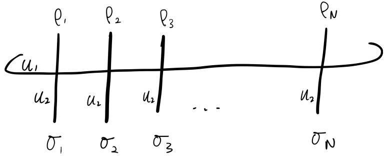

Then

$$\mathcal{Z}_{N,M}(u_1, u_2) = Tr(T(u_1, u_2)^M)$$

Yang-Baxter Equation

Now suppose the Boltzmann weights satisfy:

$$\sum_{\rho'',\sigma'',\tau''} w(u_1,u_2;\rho,\rho'';\sigma,\sigma'')w(u_1,u_3;\rho'',\rho';\tau,\tau')w(u_2,u_3;\sigma'',\sigma';\tau'',\tau')$$

$$=\sum_{\rho'',\sigma'',\tau''} w(u_2,u_3;\sigma,\sigma'';\tau,\tau'')w(u_1,u_3;\rho,\rho'';\tau'',\tau')w(u_1,u_2;\rho'',\rho';\sigma'',\sigma')$$

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Which is the component form of the YBE:

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2),$$

in $(\mathbb{C}\mathcal{S})^{\otimes 3}$.

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in $(\mathbb{C}\mathcal{S})^{\otimes 3}$. Then we have a commuting family of transfer matrices (assuming invertibility of R):

$$[T(u,u'),T(v,u')]=0.$$

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Family of commuting transfer matrices $\Rightarrow$ simultaneously diagonalisable! (Well behaved spectrum)

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E.g. the 6-vertex model: $\mathcal{S} = \{+1, -1\}$ taking $u = u_1 - u_2$,

$$R(u_1, u_2) = R(u) = \rho \begin{pmatrix} \sinh(h+u) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(h) & 0 \\ 0 & \sinh(h) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(h+u) \end{pmatrix}.$$

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E.g. for 6-vertex model

$$R(u) := \rho \sinh(u) \left(I + h \begin{pmatrix} \frac{e^u + e^{-u}}{e^u - e^{-u}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{e^u + e^{-u}}{e^u - e^{-u}} \end{pmatrix} + \mathcal{O}(h^2) \right)$$

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Then $r(u)$ solves the classical YBE:

$$[r_{12}(u - v), r_{13}(u)] + [r_{12}(u - v), r_{23}(u)] + [r_{13}(u), r_{23}(v)] = 0.$$

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Can we pass from solutions of CYBE to solutions of YBE? Where should such solutions live? First guess: in the Universal Enveloping Algebra...

Deformed universal enveloping algebra

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Each solution of the CYBE defines a **deformation** of the UEA, $U_h(\mathfrak{g})$ which is known as **quantum groups**.
$U_h(\mathfrak{g})$ is an algebra (say with multiplication $*$) over $\mathbb{C}[[h]]$, such that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \simeq U(\mathfrak{g})$ and for $x, y \in \mathfrak{g}$

$$x * y - y * x = [x, y] + h\{x, y\}_r + \mathcal{O}(h^2).$$

Yang-Baxter Equation

The YBE on $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ is

$$R_{V_1,V_2}(u_1,u_2)R_{V_1,V_3}(u_1,u_3)R_{V_2,V_3}(u_2,u_3) = R_{V_2,V_3}(u_2,u_3)R_{V_1,V_3}(u_1,u_3)R_{V_1,V_2}(u_1,u_2),$$

($R_{V_i,V_j}(u_i,u_j)$ invertible).

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Additive dependence $\Rightarrow R_{V_i,V_j}(u_i,u_j) = R_{V_i,V_j}(u_i - u_j)$

$$R_{V_1,V_2}(u-v)R_{V_1,V_3}(u)R_{V_2,V_3}(v) = R_{V_2,V_3}(v)R_{V_1,V_3}(u)R_{V_1,V_2}(u-v).$$

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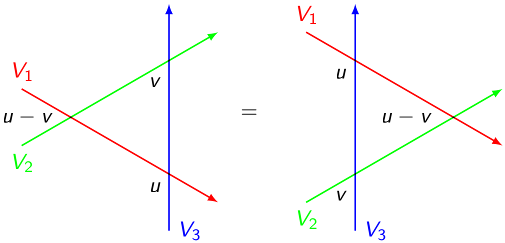
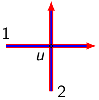


RLL-Method

Our Goal: construct an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$,

$\mathcal{R}_{12}(u) =$  .

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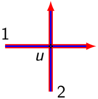
$\mathcal{R}_{12}(u) =$ 

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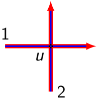
$\mathcal{R}_{12}(u) =$ 

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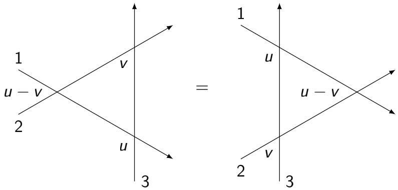

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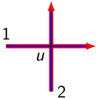
$\mathcal{R}_{12}(u) =$ 

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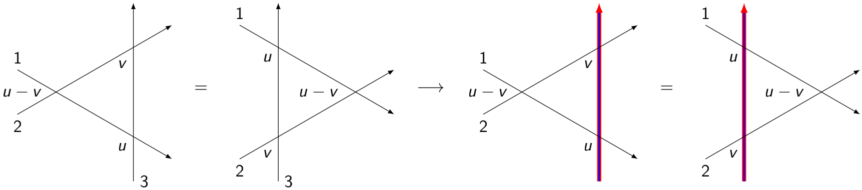

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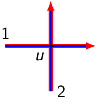
$\mathcal{R}_{12}(u) =$ 

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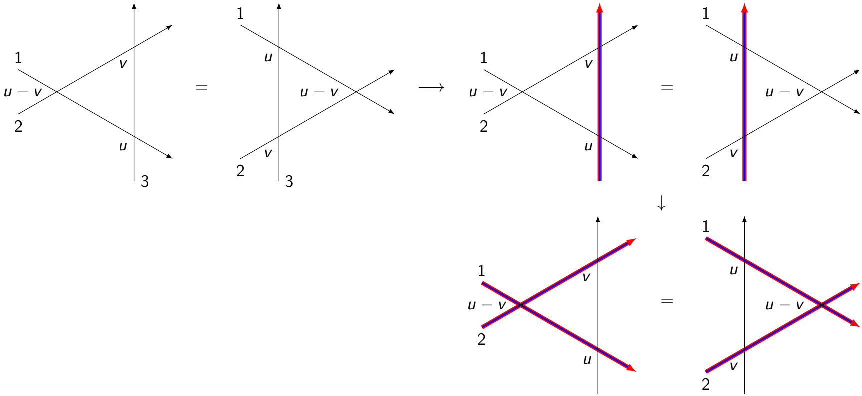

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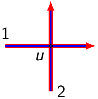
$\mathcal{R}_{12}(u) =$ 

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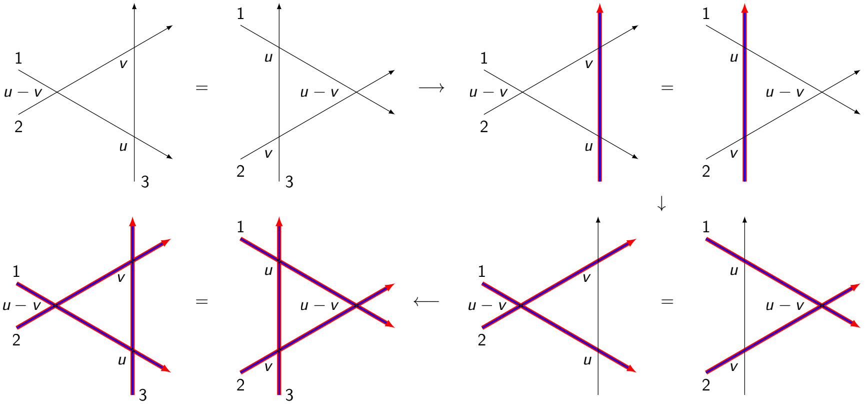
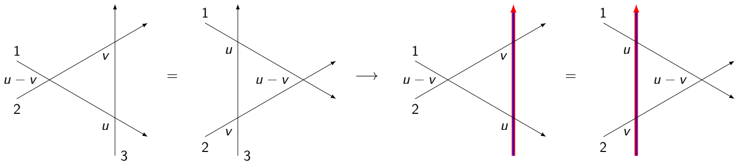


Defining R-Matrix and L-operators



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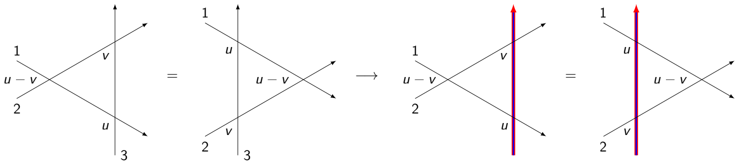

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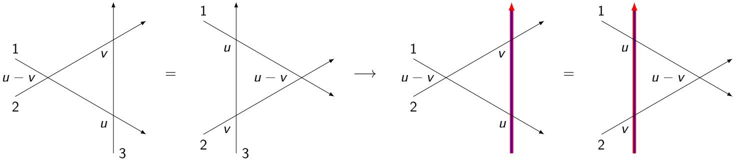

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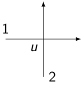
- $R_{12}(u) =$ 

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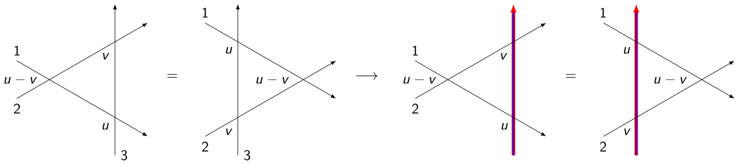
Defining R-Matrix and L-operators



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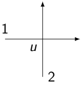
- $R_{12}(u) =$ 

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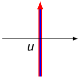
- $L(u) =$ 

$\in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$, where $\mathcal{A} \subset \mathrm{End}(\mathcal{V})$. An $n \times n$ matrix with values in $\mathcal{A}$.

Defining R-Matrix and Universal L-operators

RLL relation in $\mathrm{End}\left(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{V}\right)$:

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v).$$

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Why YBE for R? This is a consistency condition for associativity of $\mathcal{A}$.

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$$R_{12}(u) = u \cdot \mathrm{id}_{n^2} + P_{12} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n,$$

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where e_{ij} is the matrix unit. Here $\{E_{ij}\}$ is the Cartan-Weyl basis for $\mathfrak{sl}_n$:

$$h_i = E_{ii} - E_{i+1,i+1}, \quad \textstyle\sum_i E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i,$$
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Differential Representation of $\mathfrak{sl}_n$

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[Derkachov and Manashov, 2006]

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E.g. $n = 2$ case: Taking $N_x = x\partial x$ and $m = \rho_2 - \rho_1 + 1$,

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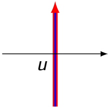
- It has a factorised L-operator!

$$L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1} =$$ 

,

$\boldsymbol{u} = (u_i)$, where $u_i = u - \rho_i$.

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- Notation: $[x]_q = (q^x - q^{-x})/(q - q^{-1})$

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$$L(u) = q^u L^+ + q^{-u} L^- \in \mathrm{End}(\mathbb{C}^n) \otimes U_q(\mathfrak{sl}_n),$$

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Now specialise:

Is there an analogous class of representations for $U_q(\mathfrak{sl}_n)$? How about a factorised L-operator?

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<u>$n = 2$ case:</u> Just one variable $x_{21} = x$

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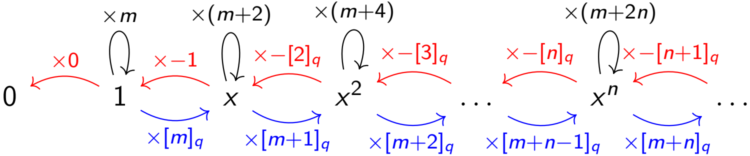

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- An Explicit formula: $m_i = \rho_{n-i} - \rho_{n+1-i} + 1$

$$E_{ii}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji}+1)\,,$$

$$f_i^{(n)} = -D_{i+1,i}\, q^{\sum_{j=1}^{i-1}(N_{ij}-N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j}\, q^{\sum_{k=1}^{j-1}(N_{ik}-N_{i+1,k})}\,,$$

$$e_i^{(n)} = \begin{array}{c} x_{i+1,i}\Big[m_i + N_{i+1,i} + \sum_{j=i+2}^{n} (N_{ji} - N_{j,i+1}) \Big]_q + q^{-m_i} \sum_{j=i+2}^{n} x_{ji} D_{j,i+1}\, q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ -q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij}\, q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{array},$$

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q-Difference Representation of $U_q(\mathfrak{sl}_n)$

- For $\boldsymbol{\rho} \in \mathbb{C}^n$, there is an analogous representation $\mathcal{V}_{\boldsymbol{\rho}}$ of $U_q(\mathfrak{sl}_n)$ [Dobrev, Truini, and Biedenharn, 1994].
- Explicit formula? obtained inductively + not unique!
- An Explicit formula: $m_i = \rho_{n-i} - \rho_{n+1-i} + 1$

$$E_{ii}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji}+1)\,,$$

$$f_i^{(n)} = -D_{i+1,i}\, q^{\sum_{j=1}^{i-1}(N_{ij}-N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j}\, q^{\sum_{k=1}^{j-1}(N_{ik}-N_{i+1,k})}\,,$$

$$e_i^{(n)} = \begin{array}{c} x_{i+1,i}\big[\, m_i + N_{i+1,i} + \sum_{j=i+2}^{n}(N_{ji}-N_{j,i+1})\,\big]_q + q^{-m_i} \sum_{j=i+2}^{n} x_{ji} D_{j,i+1}\, q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ -q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij}\, q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{array},$$

[Awata, Noumi, and Odake, 1994]

Factorised L-operator?

Factorised L-operator?

$\underline{\mathfrak{sl}_n:}$ $L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(\boldsymbol{u}) = \begin{pmatrix} u_n & P_{21} & P_{31} & \dots & P_{n1} \\ & u_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & P_{n,n-1} \\ & & & & u_1 \end{pmatrix},$$

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$\underline{U_q(\mathfrak{sl}_n)}$: Postulate $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1}$

$$D(\boldsymbol{u}) = \begin{pmatrix} [u_n]_q q^{b_1} & P_{21} & \dots & P_{n1} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1}} & P_{n,n-1} \\ & & & [u_1]_q q^{b_n} \end{pmatrix},$$

$$P_{ij} = -D_{ij}q^{b_{ij}} - \sum_{k=i+1}^{n} x_{ki} D_{kj} q^{b_{ijk}}, \quad Z_i(\boldsymbol{u}) = \begin{pmatrix} 1 & & & \\ x_{21}q^{a^{(i)}_{21}} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{n1}q^{a^{(i)}_{n1}} & \dots & x_{n,n-1}q^{a^{(i)}_{n,n-1}} & 1 \end{pmatrix},$$

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n=2: Yes [Derkachov, Karakhanyan, and Kirschner, 2007]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1 - N_x} x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x q^{N_x} \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{u_2 - N_x} x & 1 \end{pmatrix}.$$

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n=3: Yes [Valinevich et al., 2008], $L(u_1, u_2, u_3) = Z_1 D Z_2^{-1}$ with

$$D = \begin{pmatrix} [u_3]_q q^{-N_{21}+N_{31}} & (D_{21} + x_{32} D_{31} q^{N_{31} - N_{32} - 1}) q^{N_{21} + N_{31}} & D_{31} q^{N_{31}} \\ 0 & [u_2]_q q^{N_{21} - N_{32}} & D_{32} q^{u_2 - N_{31} + N_{32}} \\ 0 & 0 & [u_1]_q q^{N_{32} + N_{31}} \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ q^{u_2 - N_{31} + N_{32} - N_{21}} x_{21} & 1 & 0 \\ q^{-u_1 - N_{31} + N_{32}} x_{31} & q^{u_1 - u_2 - N_{32}} x_{32} & 1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ q^{c_{21}} x_{21} & 1 & 0 \\ q^{c_{31}} x_{31} & q^{c_{32}} x_{32} & 1 \end{pmatrix},$$

$$c_{21} = u_3 - N_{21},\ c_{31} = -u_3 - N_{31} - N_{21} - 1,\ c_{32} = N_{21} + N_{31} - N_{32}.$$

Factorised L-operator?

n=4:

Factorised *L*-operator?

n=4: No...

Factorised L-operator?

n=4: No... Our ansatz reduces to a large system of linear equations for the q-shift coefficients (182) which can be shown to be inconsistent.

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"Controlled deformation" breaks - We have "pure quantum phenomena" in the Cartan-Weyl elements:

$$E_{42} = [f_3, f_2]_q = -D_{42} q^{N_{21}-N_{32}-N_{41}-1} - x_{21} D_{41} q^{-(1+N_{31})} + (q - q^{-1}) x_{31} D_{41} D_{32} q^{N_{21}-N_{31}-1}.$$

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Such terms cannot arise from our ansatz.

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Factorised L-operator?

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$$Z_1 = \begin{pmatrix} 1 & & & \\ x_{21}q^{a_{21}} & 1 & & \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ \begin{matrix} x_{21}q^{a_{21}} \\ -(q-q^{-1})x_{31}D_{32}q^{a_{321}} \end{matrix} & 1 & & \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix},$$

$$Z_2 = \begin{pmatrix} 1 & & & \\ x_{21}q^{c_{21}} & 1 & & \\ x_{31}q^{c_{31}} & x_{32}q^{c_{32}} & 1 & \\ x_{41}q^{c_{41}} & x_{42}q^{c_{42}} & x_{43}q^{c_{43}} & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ x_{21}q^{c_{21}} & 1 & & \\ x_{31}q^{c_{31}} & \begin{matrix} x_{32}q^{c_{32}} \\ -(q-q^{-1})x_{21}D_{31}q^{c_{321}} \end{matrix} & 1 & \\ x_{41}q^{c_{41}} & x_{42}q^{c_{42}} & x_{43}q^{c_{43}} & 1 \end{pmatrix}.$$

Factorised L-operator?

General n: Order of highest term in $(q - q^{-1})$

$$\mathcal{O}(L^+(\boldsymbol{u})) \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 1 & 2 & 2 & 2 \\ & & & 0 & 0 & 1 & 2 & 3 \\ & & & & 0 & 0 & 1 & 2 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}$$

$\Rightarrow$ factorisation involves higher terms in $(q - q^{-1})$.

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$\Rightarrow$ factorisation involves higher terms in $(q - q^{-1})$.

Q: Factor L-operator with near diagonal matrices which are only first order in $(q - q^{-1})$.

Parameter Permutations and YBE

For $\check{\mathcal{R}}(u) := P \circ \mathcal{R}(u) \in \mathrm{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$

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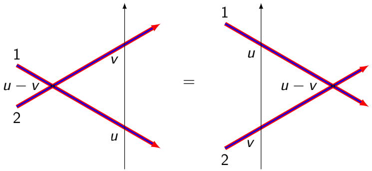

$$\sim \quad \begin{aligned} &\check{\mathcal{R}}(u-v)L_1(\boldsymbol{u})L_2(\boldsymbol{v}) \\ &= L_1(\boldsymbol{v})L_2(\boldsymbol{u})\check{\mathcal{R}}(u-v). \end{aligned}$$

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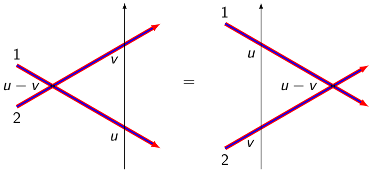

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$\check{\mathcal{R}}$ realises the permutation $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{v}, \boldsymbol{u}) \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}) \simeq S_{2n}$.

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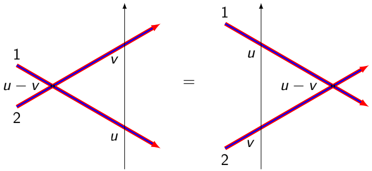

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<u>IDEA:</u> Factorise $\check{\mathcal{R}}(u-v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$

$$\mathcal{S}_i L_{12}(\boldsymbol{u}, \boldsymbol{v}) = L_{12}(s_i(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_i, \quad (L_{12}(\boldsymbol{u}, \boldsymbol{v}) = L_1(\boldsymbol{u})L_2(\boldsymbol{v}))$$

$(s_i(\alpha_1, \ldots \alpha_{2n}) = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots \alpha_{2n}))$ for $i = 1, \ldots, 2n-1$.

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Simplification: Can just find $n-1$-"intertwining" operators $\mathcal{T}_i \in \mathrm{End}(\mathcal{V}_\rho)$:

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and a single "exchange" operator:

$$\mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(u_1,\dots,u_{n-1},v_1,u_n,v_2,\dots,v_n).$$

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These operators should define an action of S_{2n}, *i.e.*,

$$s_{i_j} \dots s_{i_2} s_{i_1} \mapsto \mathcal{S}_{i_j}(s_{i_{j-1}} \dots s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \dots \mathcal{S}_{i_2}(s_{i_1}(\boldsymbol{u}, \boldsymbol{v}))\, \mathcal{S}_{i_1}(\boldsymbol{u}, \boldsymbol{v}),$$

respects the group relations.

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respects the group relations.
YBE then follows from equivalence of the decompositions in $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \stackrel{\check{\mathcal{R}}_{12}}{\longmapsto} (\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \stackrel{\check{\mathcal{R}}_{23}}{\longmapsto} (\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) \stackrel{\check{\mathcal{R}}_{12}}{\longmapsto} (\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}),$$

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \stackrel{\check{\mathcal{R}}_{23}}{\longmapsto} (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \stackrel{\check{\mathcal{R}}_{12}}{\longmapsto} (\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}) \stackrel{\check{\mathcal{R}}_{23}}{\longmapsto} (\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}).$$

Literature

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$U_q(\mathfrak{sl}_2)$ Case: [Derkachov, Karakhanyan, and Kirschner, 2007]

$U_q(\mathfrak{sl}_3)$ Case: [Valinevich et al., 2008]

q-Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ ($|q| < 1$) L-operator are given by

$$\mathcal{T}^{(n)}_{n-i}(\alpha) = \left(\Lambda^{(n)}_{n-i}\right)^{\alpha} \frac{e_{q^2}(q^{2(N_{i+1,i}+1)}\boldsymbol{X}^{(n)}_{n-i})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}^{(n)}_{n-i})},$$

$$e_{q^2}(\boldsymbol{Z}) = ((\boldsymbol{Z};q^2)_\infty)^{-1} = \left[(1-\boldsymbol{Z})(1-q^2\boldsymbol{Z})(1-q^{2\cdot 2}\boldsymbol{Z})\ldots\right]^{-1},$$

$$\frac{e_{q^2}(\boldsymbol{Z})}{e_{q^2}(q^{-\alpha}\boldsymbol{Z})} = \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j}\boldsymbol{Z}^j, \qquad \Lambda^{(n)}_{n-i} = (x_{i+1,i})^{-1}q^{\beta_i}$$

where $\alpha = u_{n-i} - u_{n+1-i}$, and

$$\boldsymbol{X}^{(n)}_{n-i} = 1 + x_{i+1,i}\sum_{j=i+2}^{n} \frac{x_{j,i+1}}{x_{ji}}(q^{N_{ij}} - q^{-N_{ij}})q^{\gamma_i}.$$

q-Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ ($|q|< 1$) L-operator are given by

$$\mathcal{T}^{(n)}_{n-i}(\alpha) = \left(\Lambda^{(n)}_{n-i}\right)^{\alpha} \frac{e_{q^2}(q^{2(N_{i+1,i}+1)}\boldsymbol{X}^{(n)}_{n-i})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}^{(n)}_{n-i})},$$

$$e_{q^2}(\boldsymbol{Z}) = ((\boldsymbol{Z};q^2)_\infty)^{-1} = \left[(1-\boldsymbol{Z})(1-q^2\boldsymbol{Z})(1-q^{2\cdot 2}\boldsymbol{Z})\dots\right]^{-1},$$

$$\frac{e_{q^2}(\boldsymbol{Z})}{e_{q^2}(q^{-\alpha}\boldsymbol{Z})} = \sum_{j=0}^{\infty}\frac{(q^{-\alpha};q)_j}{(q;q)_j}\boldsymbol{Z}^j, \qquad \Lambda^{(n)}_{n-i} = (x_{i+1,i})^{-1}q^{\beta_i}$$

where $\alpha = u_{n-i} - u_{n+1-i}$, and

$$\boldsymbol{X}^{(n)}_{n-i} = 1 + x_{i+1,i}\sum_{j=i+2}^{n}\frac{x_{j,i+1}}{x_{ji}}(q^{N_{ij}} - q^{-N_{ij}})q^{\gamma_i}.$$

Obtained using an approach from [Valinevich et al., 2008].

q-Deformed Case

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The intertwiners for the $U_q(\mathfrak{sl}_n)$ L-operator, $\mathcal{T}_i(\alpha)$, define an action of the symmetric group Perm($\boldsymbol{u}$) $\simeq S_n$.

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Proof.

The only non-trivial relation is the braid relation

$$\mathcal{T}_i(\alpha)\mathcal{T}_{i+1}(\alpha+\beta)\mathcal{T}_i(\beta) = \mathcal{T}_{i+1}(\beta)\mathcal{T}_i(\alpha+\beta)\mathcal{T}_{i+1}(\alpha).$$

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After a series expansion it is reduced to a family of (terminating) q-series identity relating rank $i+1$ and rank $2i-1$ q-Lauricella series. □

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$$\Phi_D^{(n)}[b; a_1, \ldots, a_n; c; q; x_1, \ldots, x_n] = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(b;q)_M (a_1;q)_{m_1} \ldots (a_n;q)_{m_n}}{(c;q)_M (q;q)_{m_1} \ldots (q;q)_{m_n}} x_1^{m_1} \ldots x_n^{m_n}, \qquad (\star)$$

where $M = \sum_{i=1}^{n} m_i$ and

$$(x;q)_m = (1-x)(1-qx)\ldots(1-q^{m-1}x).$$

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[Andrews, 1972] gives a general transformation formula allowing us to rewrite $(\star)$ in terms of a ${}_{n+1}\phi_n$ hypergeometric series.

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For $n \geq 1$ and non-negative integer tuples

$$\boldsymbol{k} = (k_0, \ldots, k_n) = (k_0, \tilde{\boldsymbol{k}}), \quad \boldsymbol{l} = (l_1, \ldots, l_n), \quad \boldsymbol{m} = (m_1, \ldots, m_{n-1}),$$

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The identity we need is the equality $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \frac{(\xi;q)_{L+M}}{(\xi\,\zeta;q)_{L+M}} \Phi_D^{(2n-1)}\left[\,\zeta; q^{-\boldsymbol{l}}, q^{-\boldsymbol{m}}; q^{1-L-M}/\xi; q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{(r_i,\hat{r}_n)+\boldsymbol{m}}\,\right],$$

$$\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \zeta^{k_0} \frac{(\xi;q)_K}{(\xi\,\zeta;q)_K} \Phi_D^{(n+1)}\left[\,\zeta; q^{-\boldsymbol{k}}; q^{1-K}/\xi; q^{1+k_0-K}/(\xi\,\zeta), q^{\boldsymbol{p}+\tilde{\boldsymbol{k}}}\,\right],$$

for arbitrary complex parameters ξ, ζ.

Exchange Operator

Exchange Operator

The defining relation for the exchange operator $\mathcal{S}_n$ is

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Now we can reduce the defining relation to

$$Z_2^{(x,\tilde{\boldsymbol{u}})}(v_1)\left[(D^{(x,\tilde{\boldsymbol{u}})})^{-1}\,\mathcal{S}_n\,D^{(x,\tilde{\boldsymbol{u}})}\right]\left(Z_2^{(x,\tilde{\boldsymbol{u}})}(u_n)\right)^{-1}$$
$$= Z_1^{(y,\tilde{\boldsymbol{v}})}(u_n)\left[D^{(y,\tilde{\boldsymbol{v}})}\,\mathcal{S}_n\,(D^{(y,\tilde{\boldsymbol{v}})})^{-1}\right]\left(Z_1^{(y,\tilde{\boldsymbol{v}})}(v_1)\right)^{-1},$$

if $\mathcal{S}_n^{(x,y)}$ commutes (element wise) with $Z_1^{(x)}$ and $Z_2^{(y)}$.

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Recall in the $n \geq 4$ case the postulated ansatz for the factorisation was inconsistent - the outer most factors will now have q-difference terms.

This seems to represent a serious obstruction to constructing the exchange operator - unclear whether to expect a multiplication operator (by shifted variables) to work or not

Summary

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- We described explicitly all but one of the transposition operators in the $U_q(\mathfrak{sl}_n)$ case, and prove they obey the necessary symmetric group relations.
- We explain how the failure of the factorisation property for the $U_q(\mathfrak{sl}_4)$ L-operator represents an obstruction to constructing the missing "exchange" operator.

Thank You!

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Questions?

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