An Unorientable Extension of the Temperley Lieb Category

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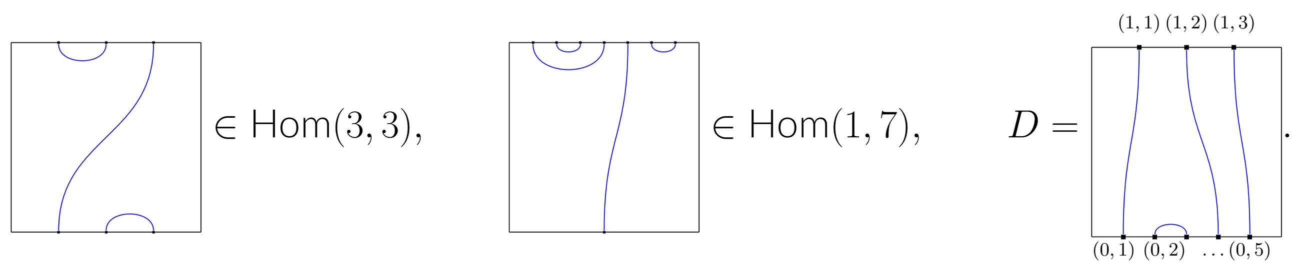
Motivation

Temperley-Lieb algebras and related diagrammatic algebras are common tools in mathematical physics, enjoying applications in statistical mechanics, knot theory, low-dimensional topology, and topological quantum systems [1, 4, 3]. In this work we introduce a combinatorial monoidal category, which generalises the celebrated Temperley-Lieb category, which includes (as morphisms) diagrams on (once bounded) surfaces with a concrete realisation, considered upto a "handlesliding equivalence". This construction grew out of questions related to Skein-modules of unorientable surfaces [2].

The Temperley-Lieb Category

For a commutative ring R and $\alpha \in R$, the Temperley-Lieb category $TL(\alpha)$ is (for us) a combinatorial, R-linear, category with

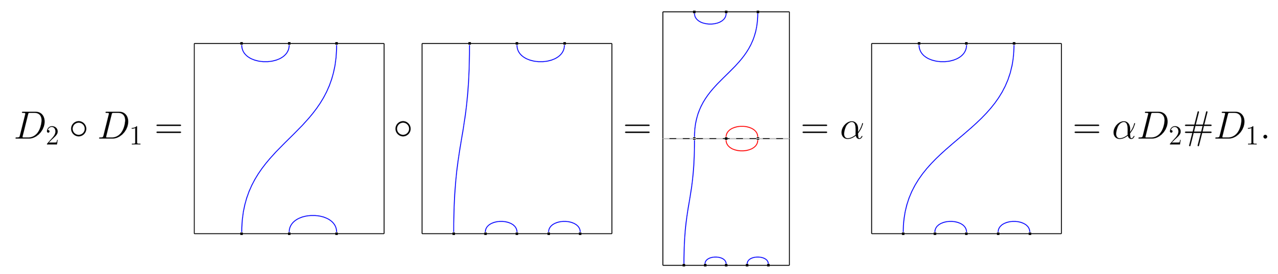
- Objects: Non-negative integers, $\mathbb{Z}_{\geq 0}$.
- Morphisms: A morphism $n \to m$ is an R-linear combination of "TL-diagrams" of type (n, m):



Diagrams are encoded "up to isotopy" by pair partitions of a vertex set, e.g.

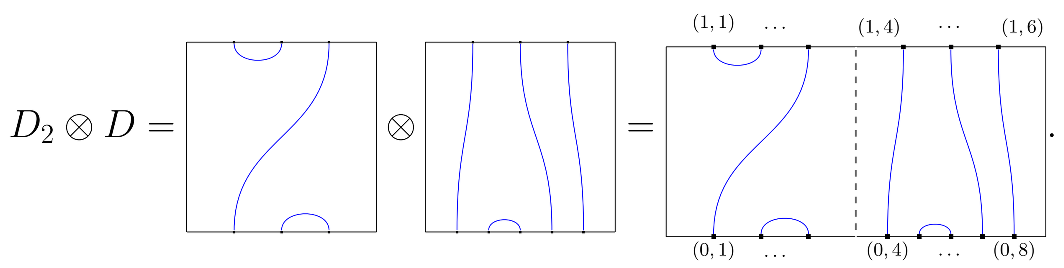
$$D = \{\{(0,1),(1,1)\},\{(0,2),(0,3)\},\{(0,4),(1,2)\},\{(0,5),(1,3)\}\}.$$

- Composition: Composition of diagrams is given by "vertical juxtaposition" e.g.



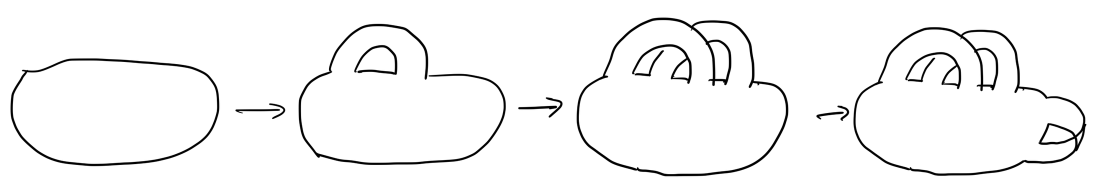
generically, $D_2 \circ D_1 = \alpha^{L_{D_1,D_2}} D_2 \# D_1$.

- Tensor Product: On objects $n_1 \otimes n_2 = n_1 + n_2$, and on diagrams "horizontal juxtaposition"

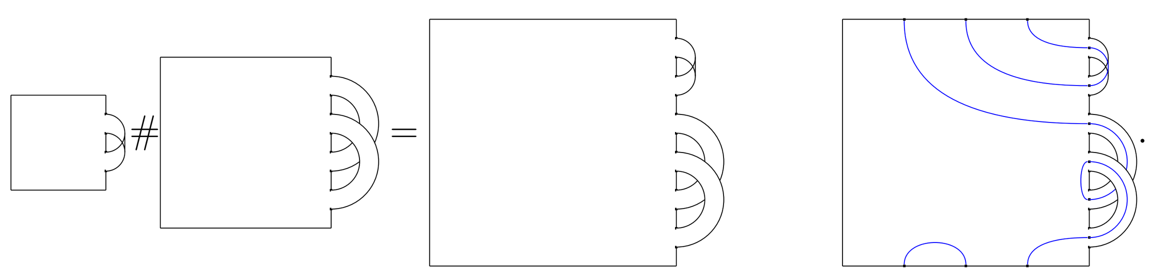


Square with Bands Realisation of Surfaces

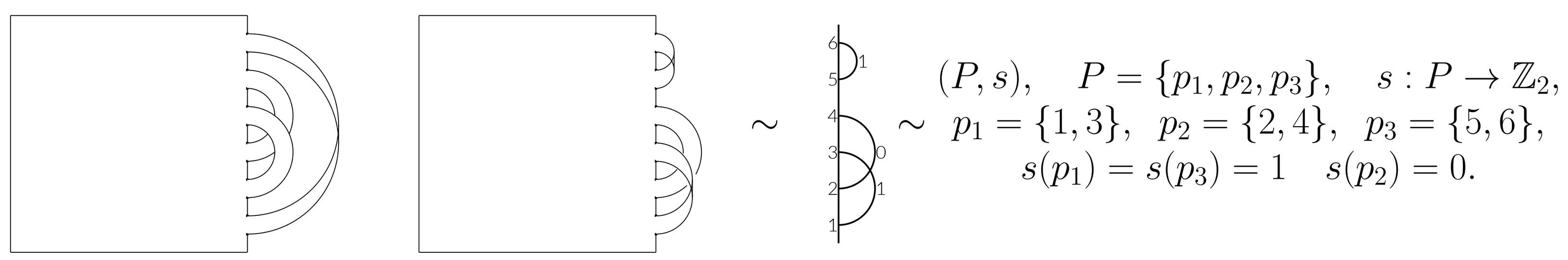
Our first task is to modify the "square frame" in TL-diagrams to accommodate any surface type (with boundary) in a tractable way. To do this, we use the disc with bands model for a surface:



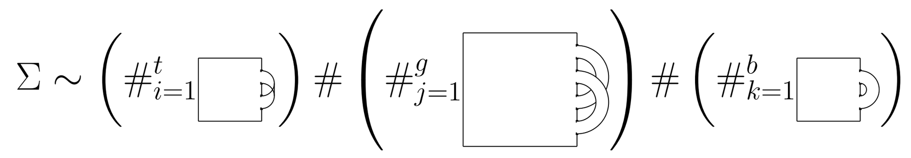
Marrying this with the square frame, we obtain:



PROBLEM: this presentation of a surface is not unique. Below are two equivalent presentations of the surface above. In general two presentations are related by "handlesliding".

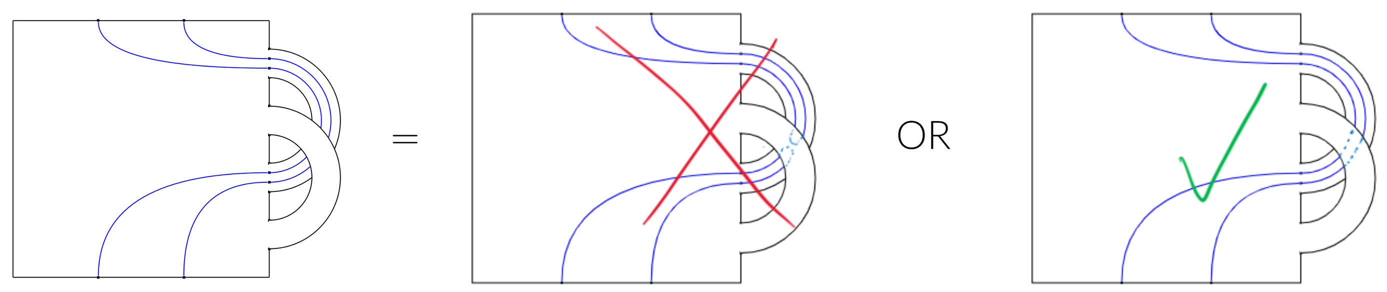


Classification of surfaces: For any surface Σ with boundary, there exist unique $g, b \in \mathbb{Z}_{\geq 0}$ and $t \in \{0,1,2\}$ such that

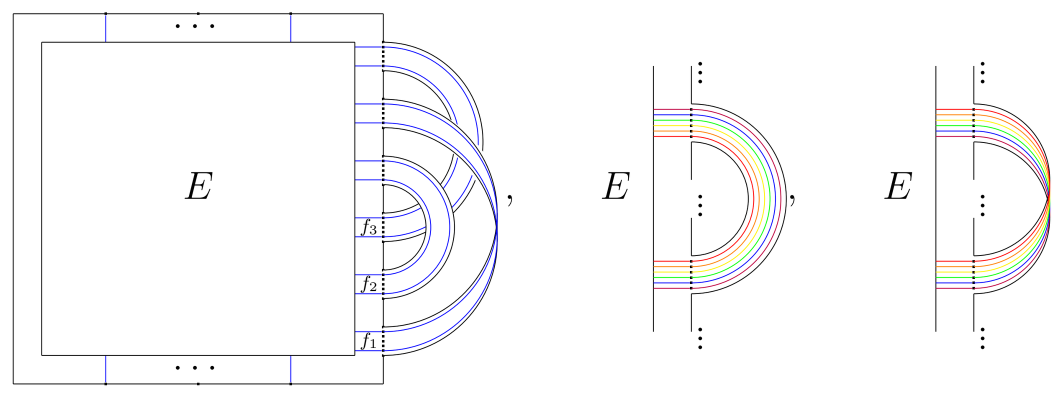


Surface Temperley-Lieb Diagrams

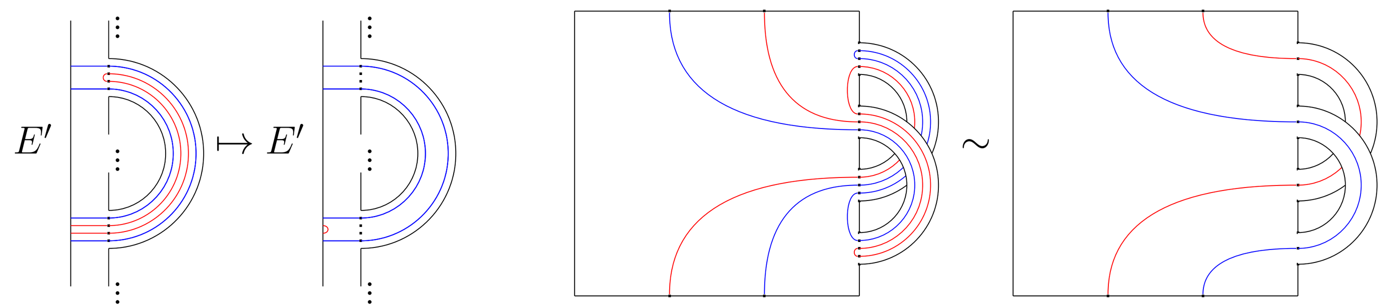
We have seen how to modify our square frames for our diagrams. Let us first deal with a source of ambiguity coming from the "crossings of bands".



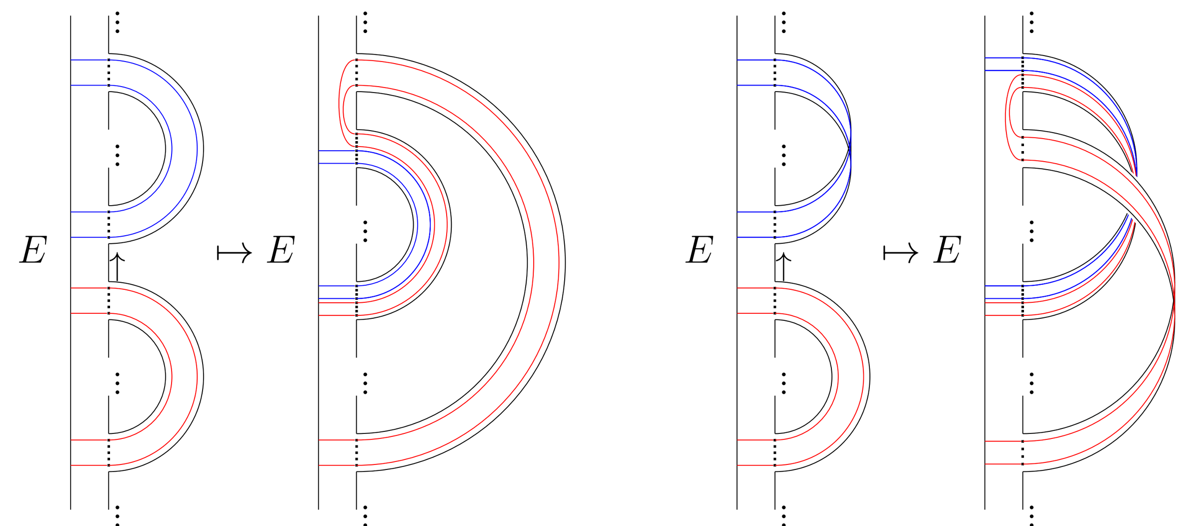
with this requirement on diagrams, our surface TL diagrams are recorded by a tuple (P, s, f, E), where (P, s) is the frame, $f : P \to \mathbb{Z}_{\geq 0}$ gives the number of "through lines" for a band, and E is a catalan state inside the square:



In this set-up, there are two equivalence moves we impose. The first is our analogue of isotopy:



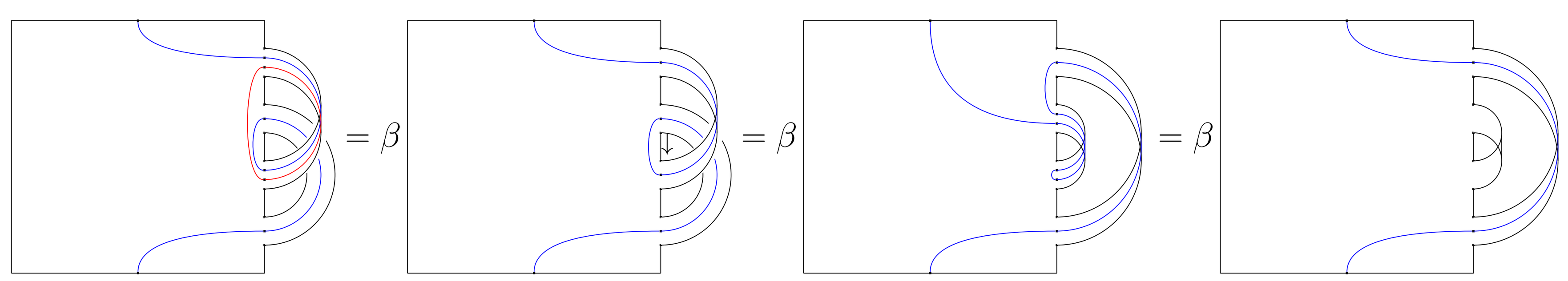
The second, is handlesliding which relates diagrams on different frames (but equivalent surfaces):



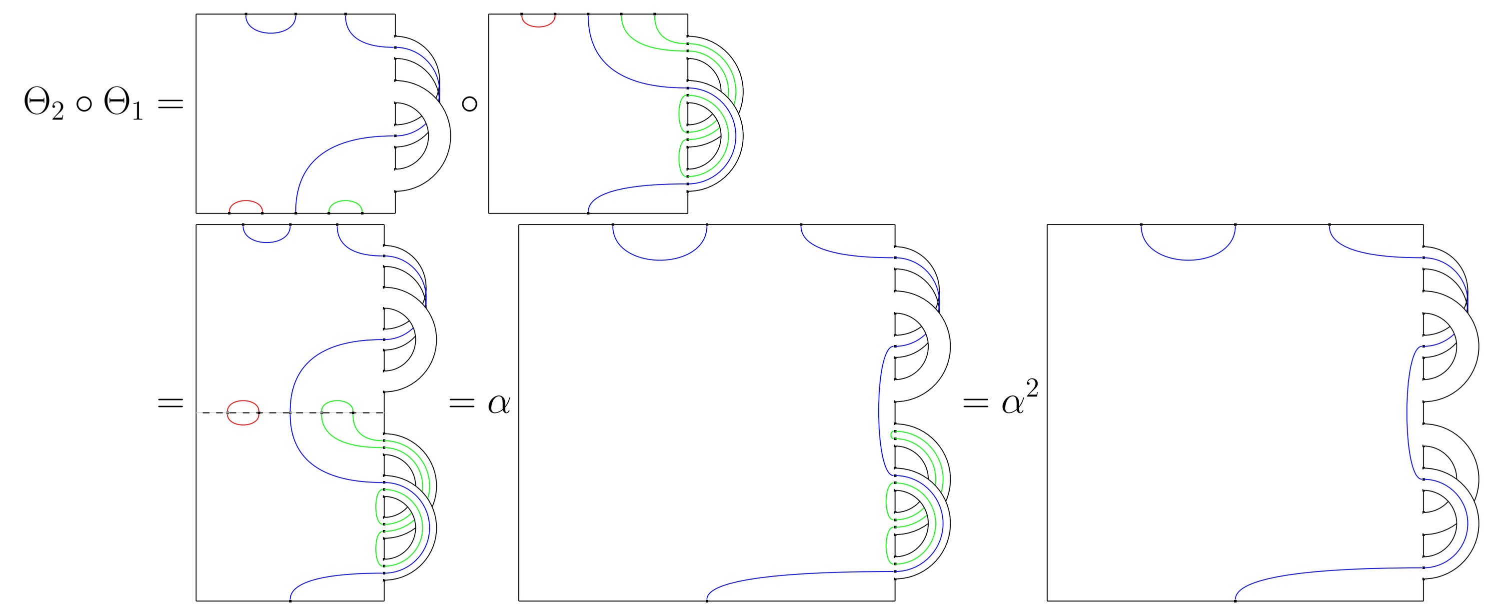
Square with Bands Category

For $\alpha, \beta \in R$, the category $\mathcal{SQ}(\alpha, \beta)$ is:

- Objects: Non-negative integers $\mathbb{Z}_{\geq 0}$.
- Morphisms: Morphisms $n \to m$ are R-linear combinations of surface TL diagrams type (n, m) on surfaces with 1-boundary component, considered up to the above equivalences, and we can remove loops with a factor β if it is twisted and a factor α if it is untwisted e.g.



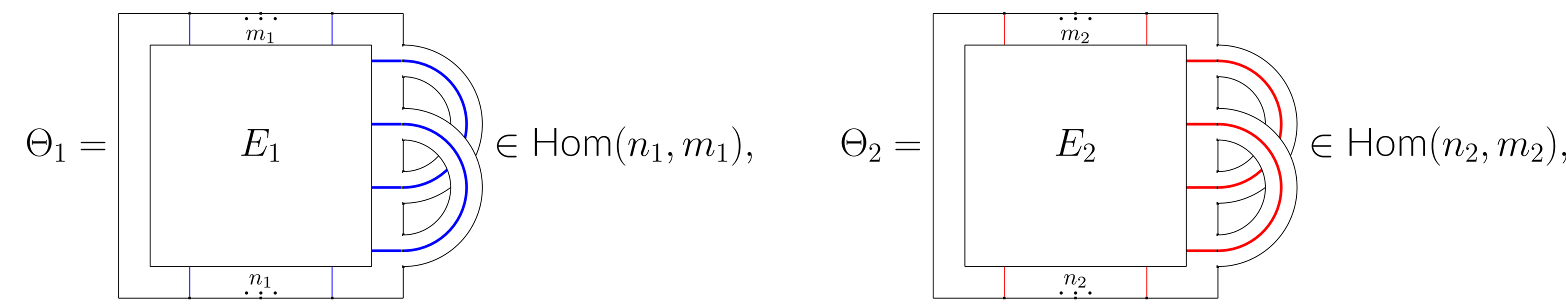
Composition: As in the TL case we have "vertical juxtaposition" $\Theta_2 \circ \Theta_1 = \alpha^{L_{\Theta_1,\Theta_2}} \Theta_2 \# \Theta_1$, e.g.



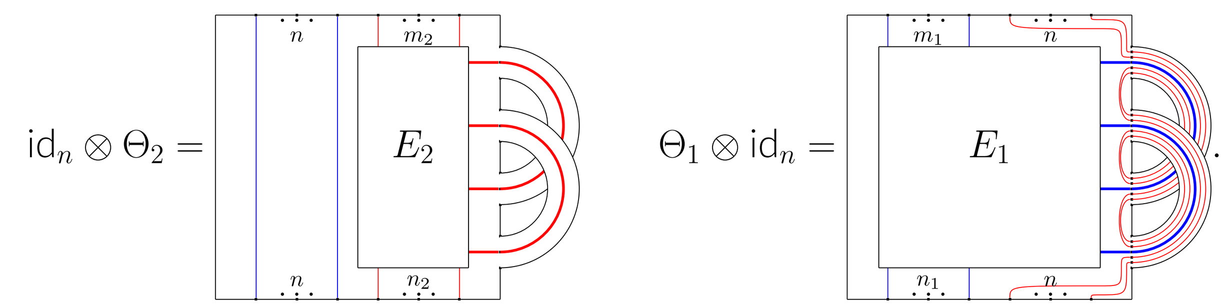
Tensor Product: We aim to define a tensor product such that $n_1 \otimes n_2 = n_1 + n_2$ on objects. To define the tensor product on morphisms we will use a trick. For $\Theta_1 \in \text{Hom}(n_1, m_1)$, $\Theta_2 \in \text{Hom}(n_2, m_2)$ a tensor product needs to satisfy:

$$\Theta_1 \otimes \Theta_2 = (\Theta_1 \otimes \text{id}_{m_2}) \circ (\text{id}_{n_1} \otimes \Theta_2) \overset{?}{=} (\text{id}_{m_1} \otimes \Theta_2) \otimes (\Theta_1 \otimes \text{id}_{n_2})$$

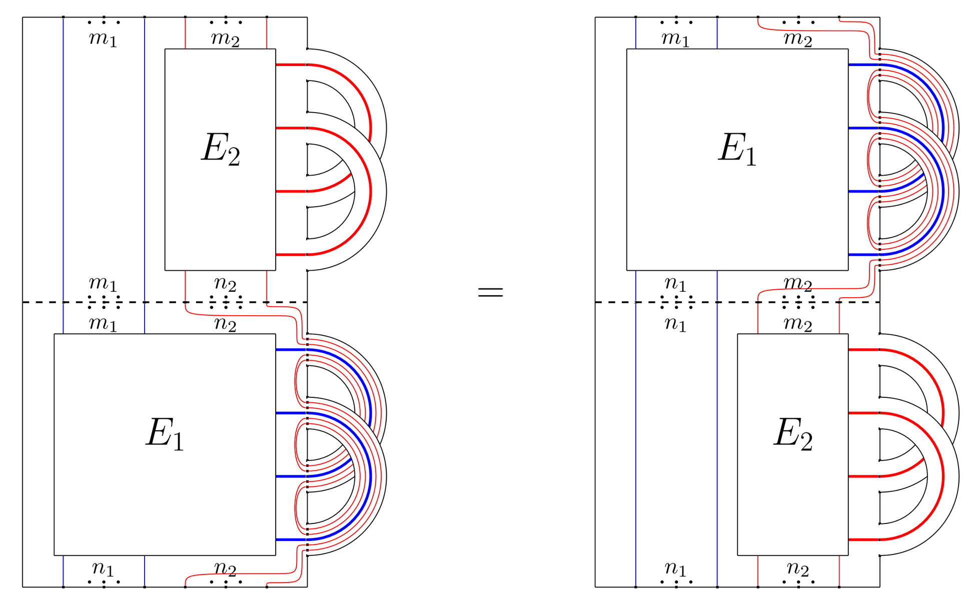
Therefore if we can define a notion of a left/right tensor product with the identity diagram we can define a prospective tensor product where "$\overset{?}{=}$" becomes a non-trivial consistency condition. Let



then after some thinking, we come to



Our non-trivial consistency condition looks like the below. It can be proved (difficultly) with an explicit sequence of handleslides and isotopy moves.



References

[1] Michael H. Freedman et al. "Topological Quantum Computation". In: (Jan. 2001). arXiv: quant-ph/0101025.

[2] Dionne Ibarra and Gabriel Montoya-Vega. *A Study of Gram Determinants in Knot Theory*. 2024. arXiv: 2402.09704 [math.GT].

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