Interpolating between Temperley-Lieb and Brauer representation theory: generalised roots of unity

(based on arXiv:2512.24535 joint with P. P. Martin)

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8th July 2026

Overview

1. Motivation
2. Kadar-Martin-Yu Categories/Algebras
3. Representation Theory by "Tower of Recollment"
   - Simple/standard modules
   - (Non)semisimplicity Criteria: Generalised Roots of Unity

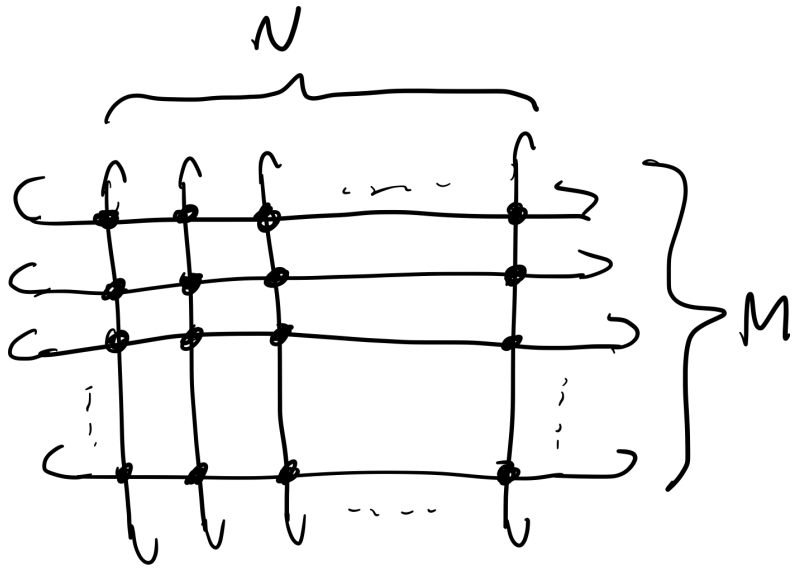
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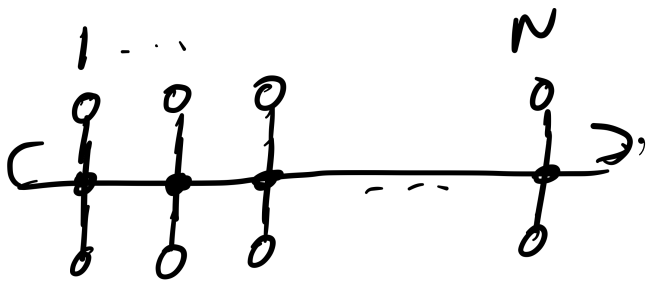
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Transfer matrices are a key calculational tool to compute partition functions in lattice models in statistical mechanics.



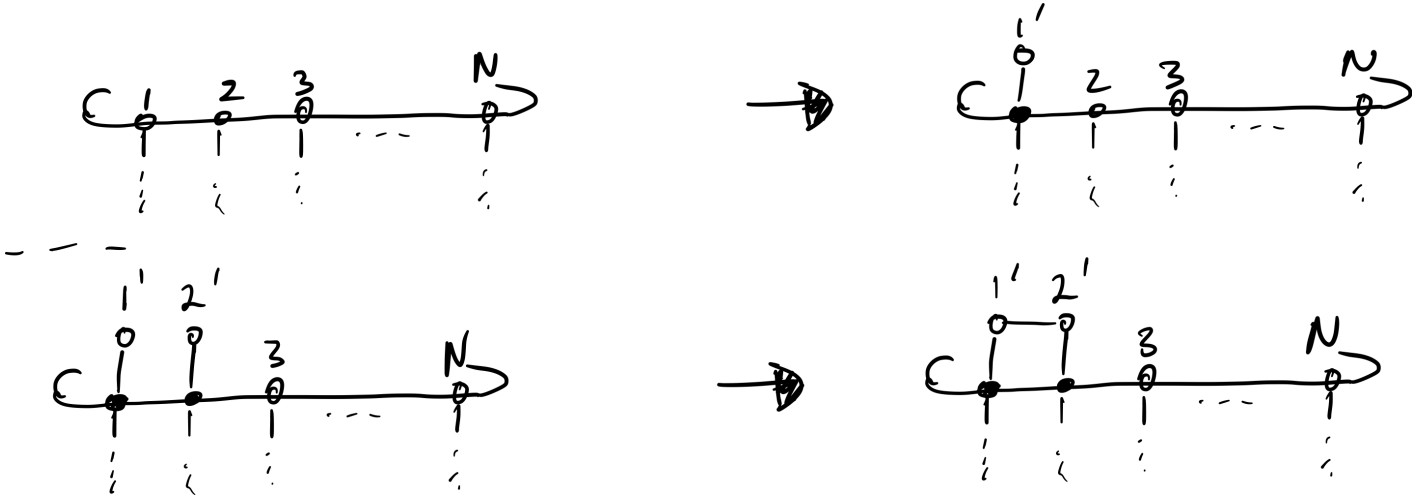
$$Z = \sum_{\text{configs.}} e^{\beta E_{\text{config}}},$$



$$T = \quad , \quad \Rightarrow \quad Z = \mathrm{Tr}(T^M) \sim \lambda_0^M$$

Motivation

In turn, transfer matrices can be decomposed into local operators.

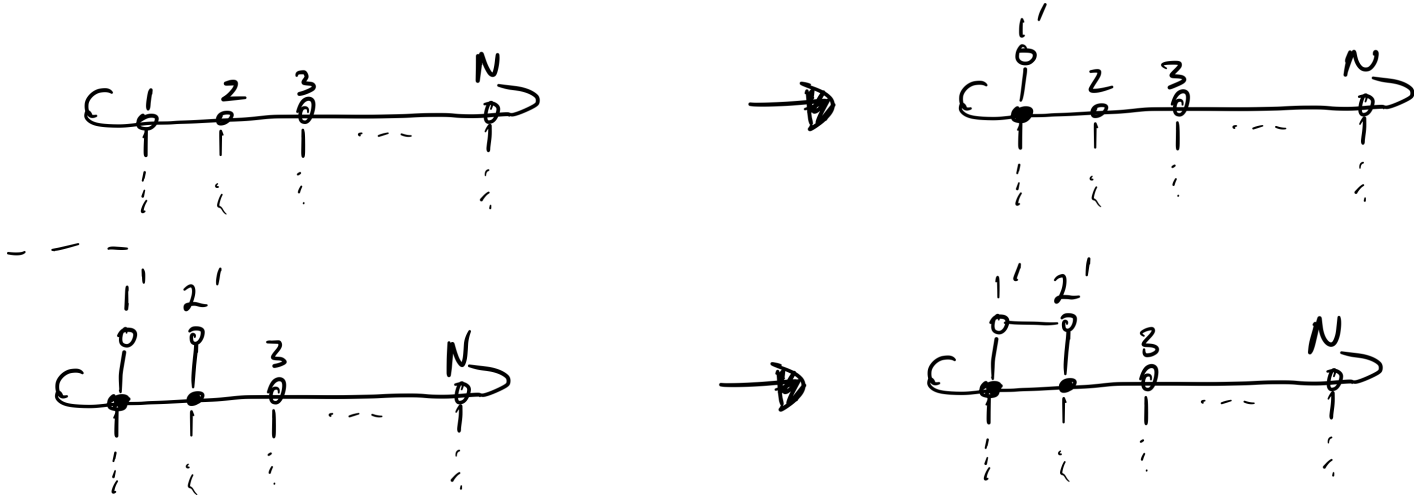


E.g. First case in Ising model ($x = e^{\beta}$)

$$M \otimes \mathrm{id} \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}, \qquad M = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix} = (x-1)\mathrm{id}_2 + U,$$

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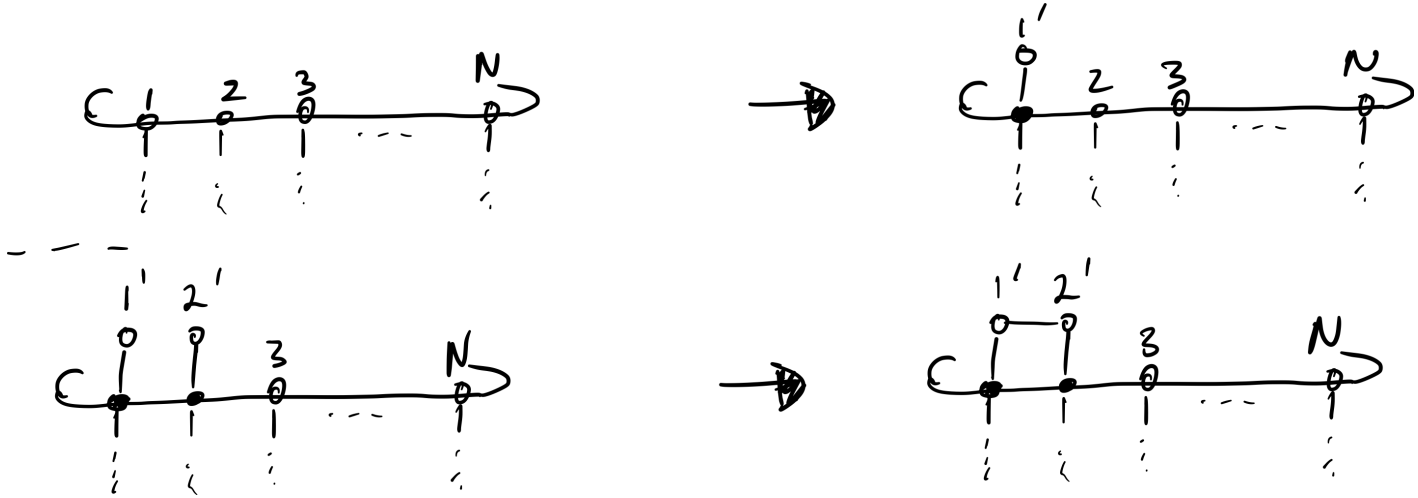
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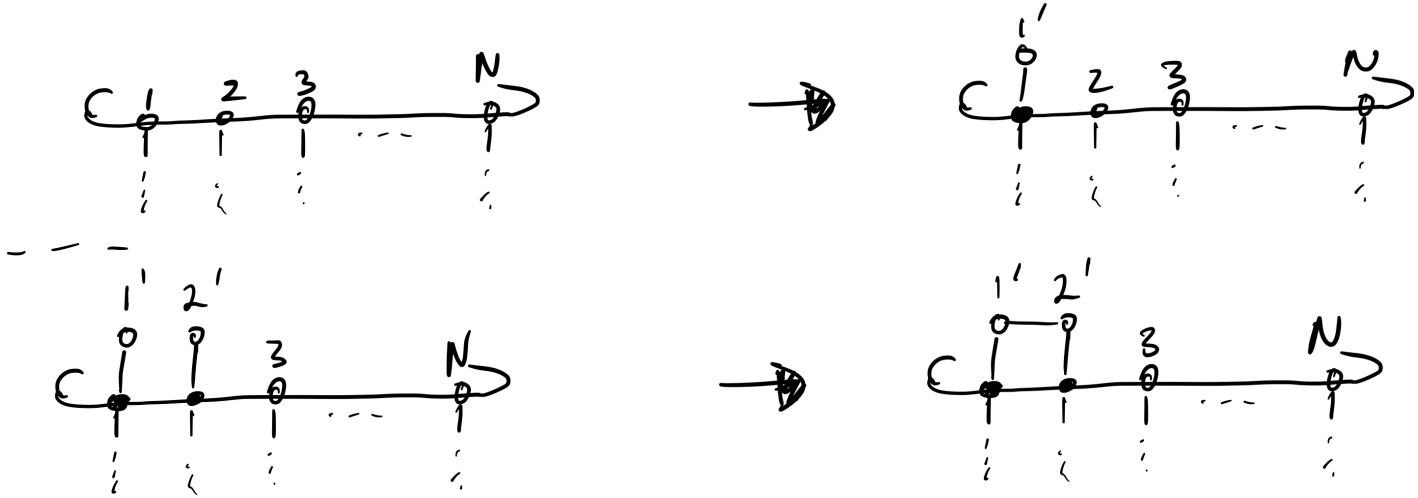
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By physical considerations, can extract an abstract algebra of local operators ind. of Temp. Transfer matrix can be viewed as an element of an abstract algebra evaluated in some representation.
E.g. For Potts models, the underlying **Transfer matrix algebra** is the **Temperley-Lieb algebra**.

Motivation

Temperley-Lieb Algebra: $TL_n(\alpha)$ generated by $U_1, \dots U_{n-1}$ subject to

$$U_i^2 = \alpha U_i, \qquad U_i U_{i\pm 1} U_i, \qquad U_i U_j = U_j U_i,$$

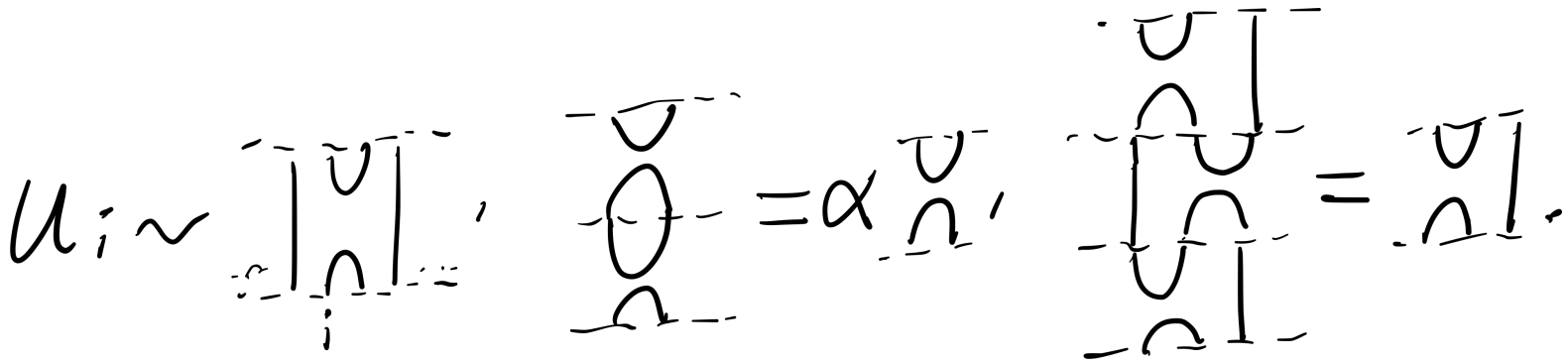
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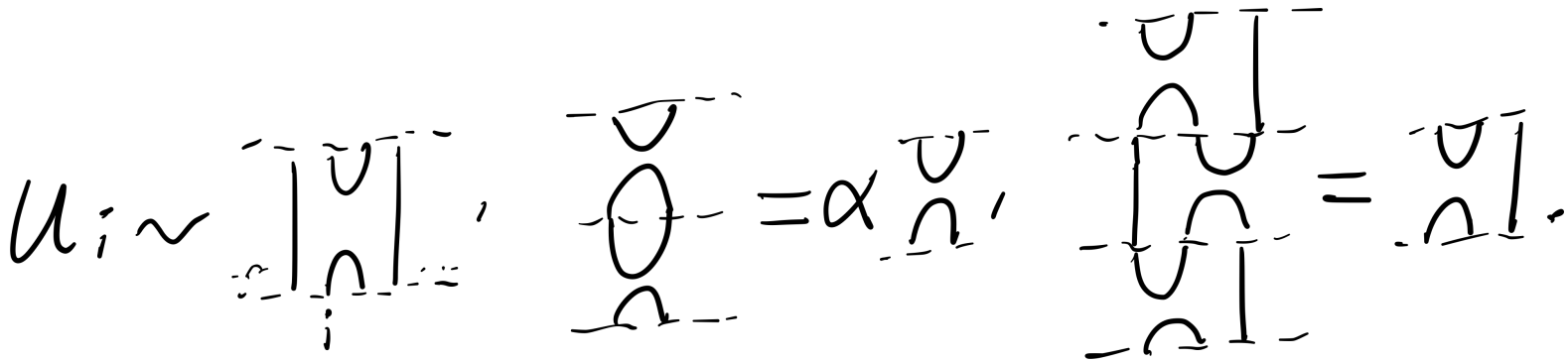


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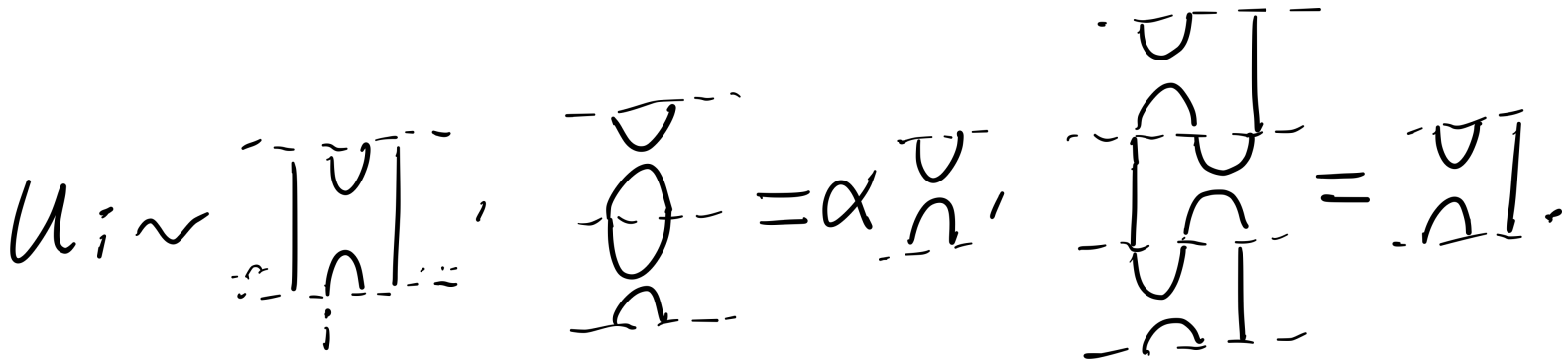
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For today: Diagram algebras often arise as transfer matrix algebras. Can we construct one which captures **inhomogeneity?**

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Kadar-Martin-Yu Categories: Context

For $\alpha \in \mathbb{K}$, we are going to define a discrete family of nested $\mathbb{K}$-linear subcategories of the diagrammatic Brauer category $Br(\alpha)$:

$$TL(\alpha) = J_{-1}(\alpha) \subset J_0(\alpha) \subset \ldots \subset J_l(\alpha) \subset \ldots \subset Br(\alpha) = J_\infty(\alpha).$$

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The Brauer category $Br(\alpha)$ has:

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- $Br(n, m)$ spanned by type (n, m) "diagrams" (pairings) with a diagrammatic composition up to loop replacement:

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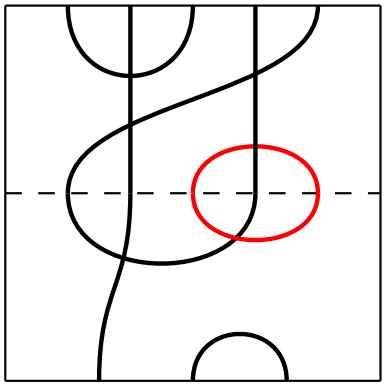
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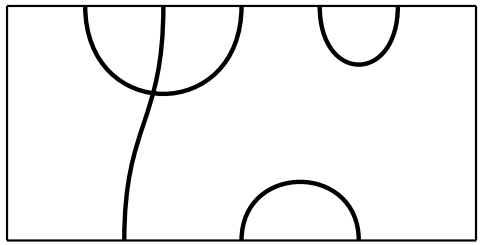
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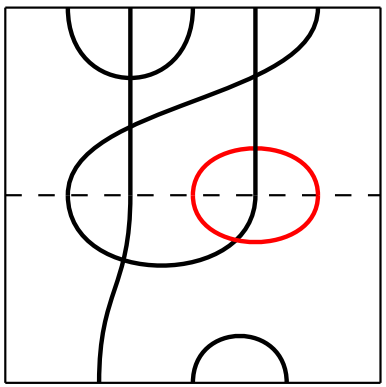
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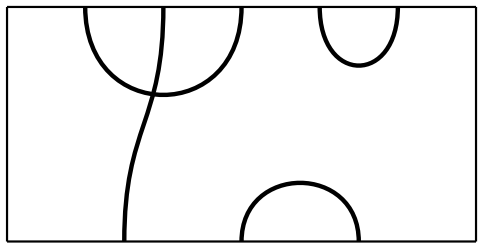
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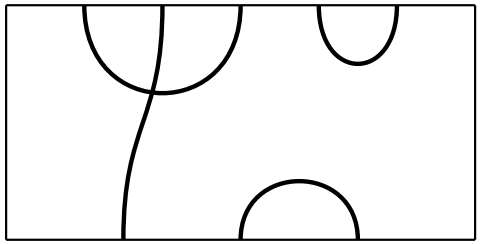


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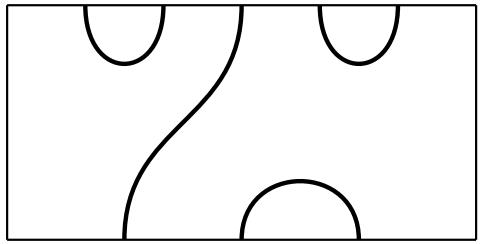
- $\otimes : Br \times Br \to Br$ given by horizontal juxtaposition of diagrams.

Kadar-Martin-Yu Categories: Context (continued)

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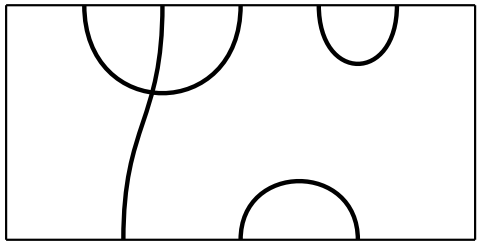
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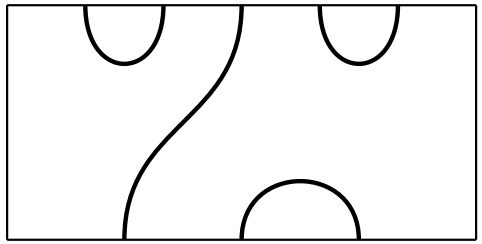
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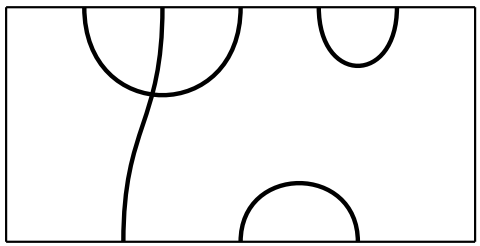
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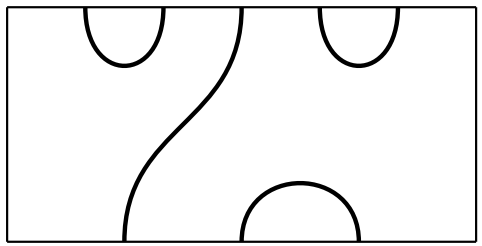
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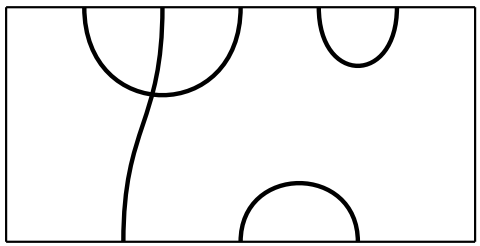
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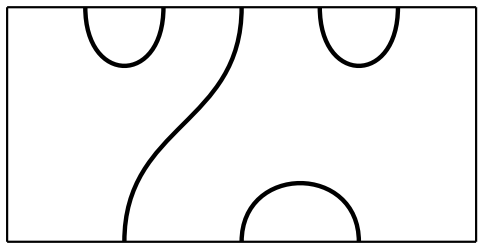
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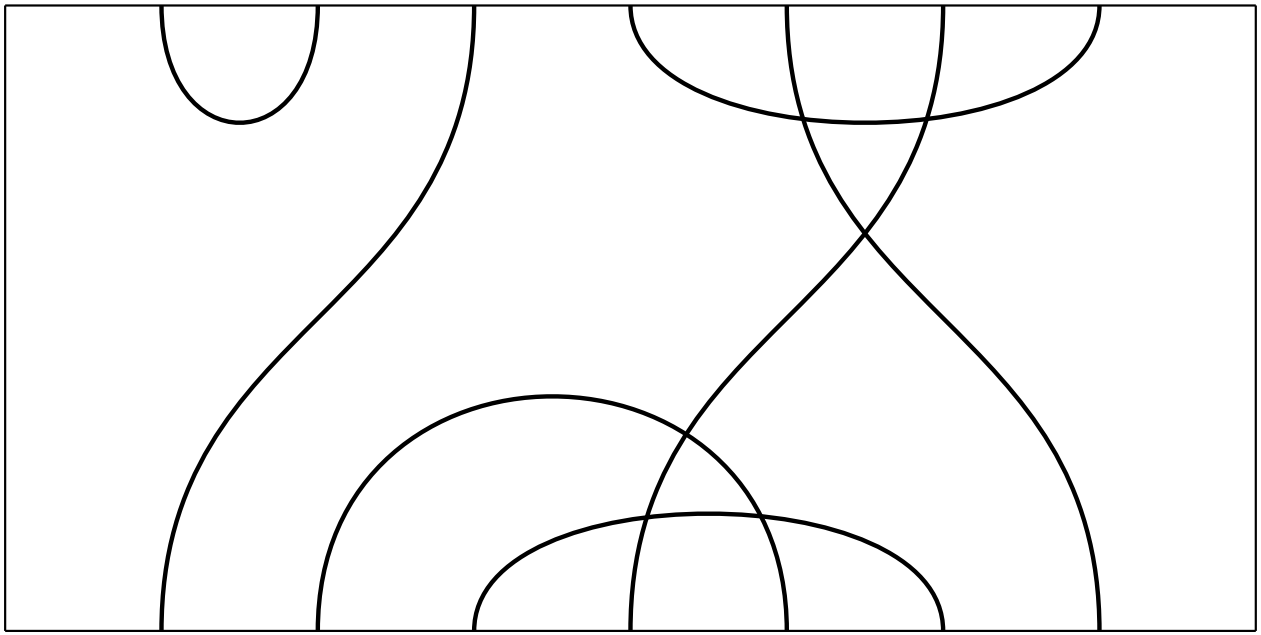
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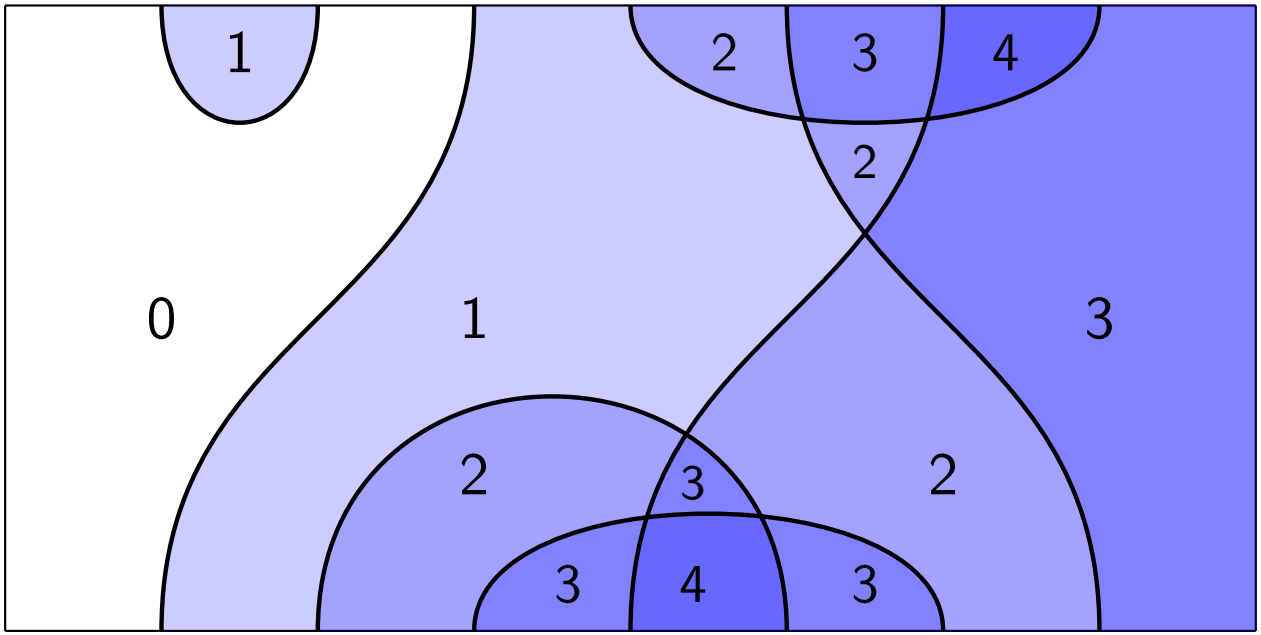
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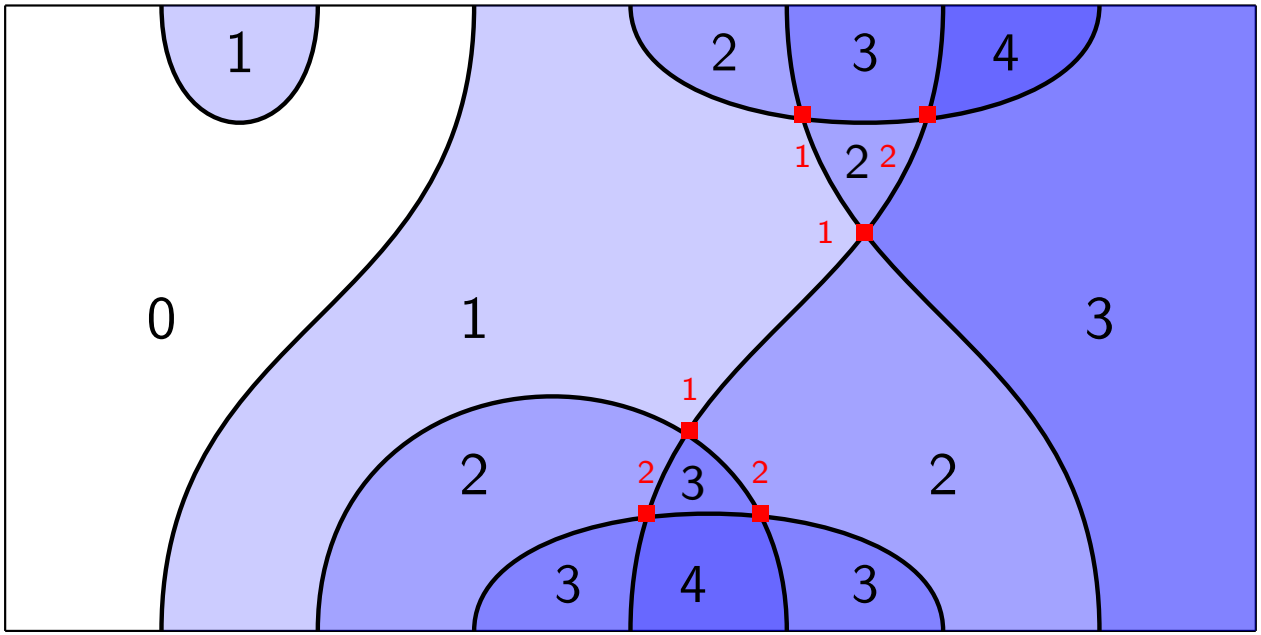
Kadar-Martin-Yu Categories: The Height Condition



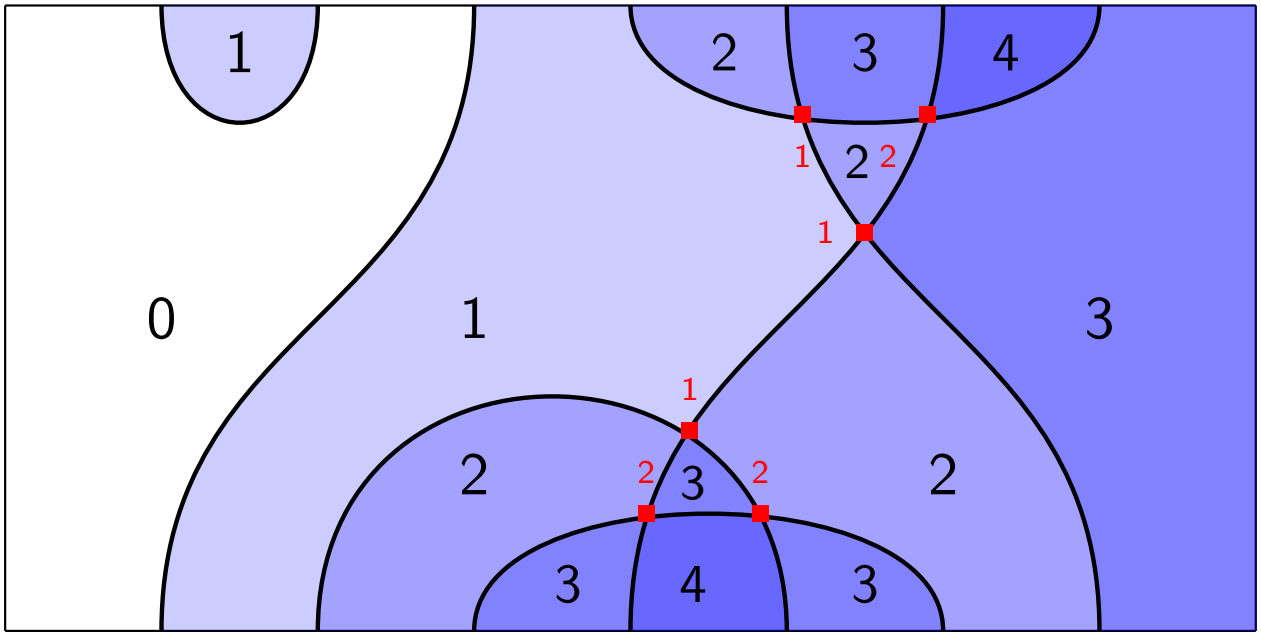
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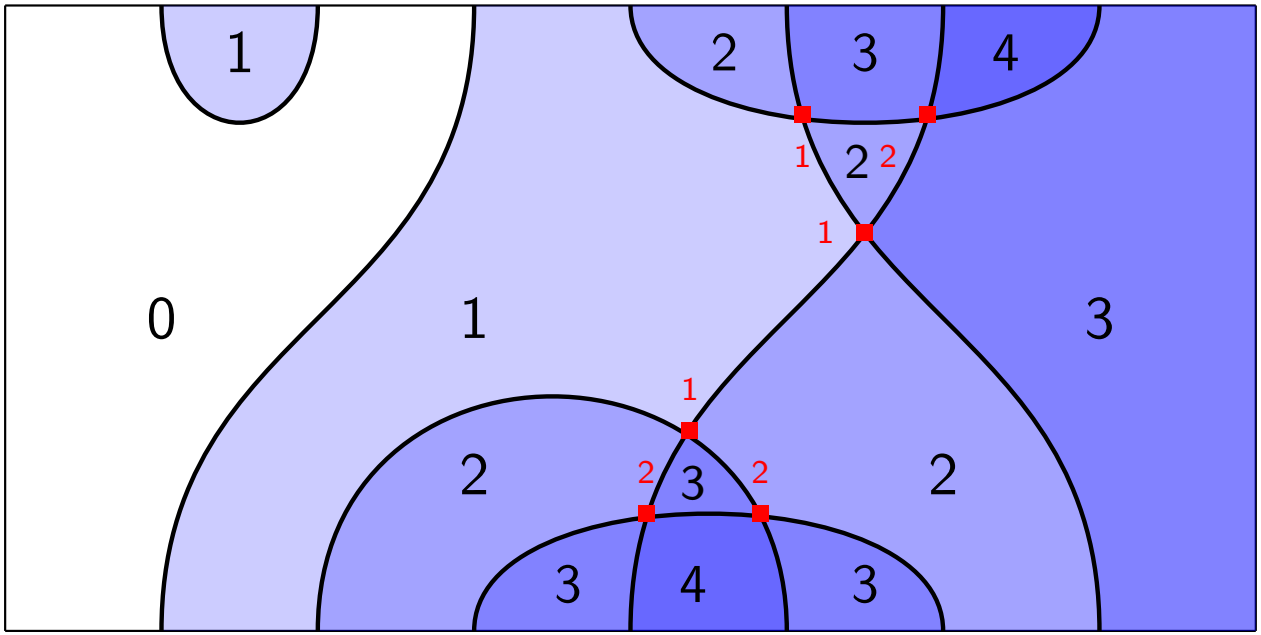
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The height of a **pair partition**:

$$ht(p) = \min_{\text{diag. of } p} ht(\text{diag.}) \quad \text{e.g. above} \quad ht \leq 2$$

Kadar-Martin-Yu Categories: The Height Condition (ctd)

Lemma (Height is non-increasing)

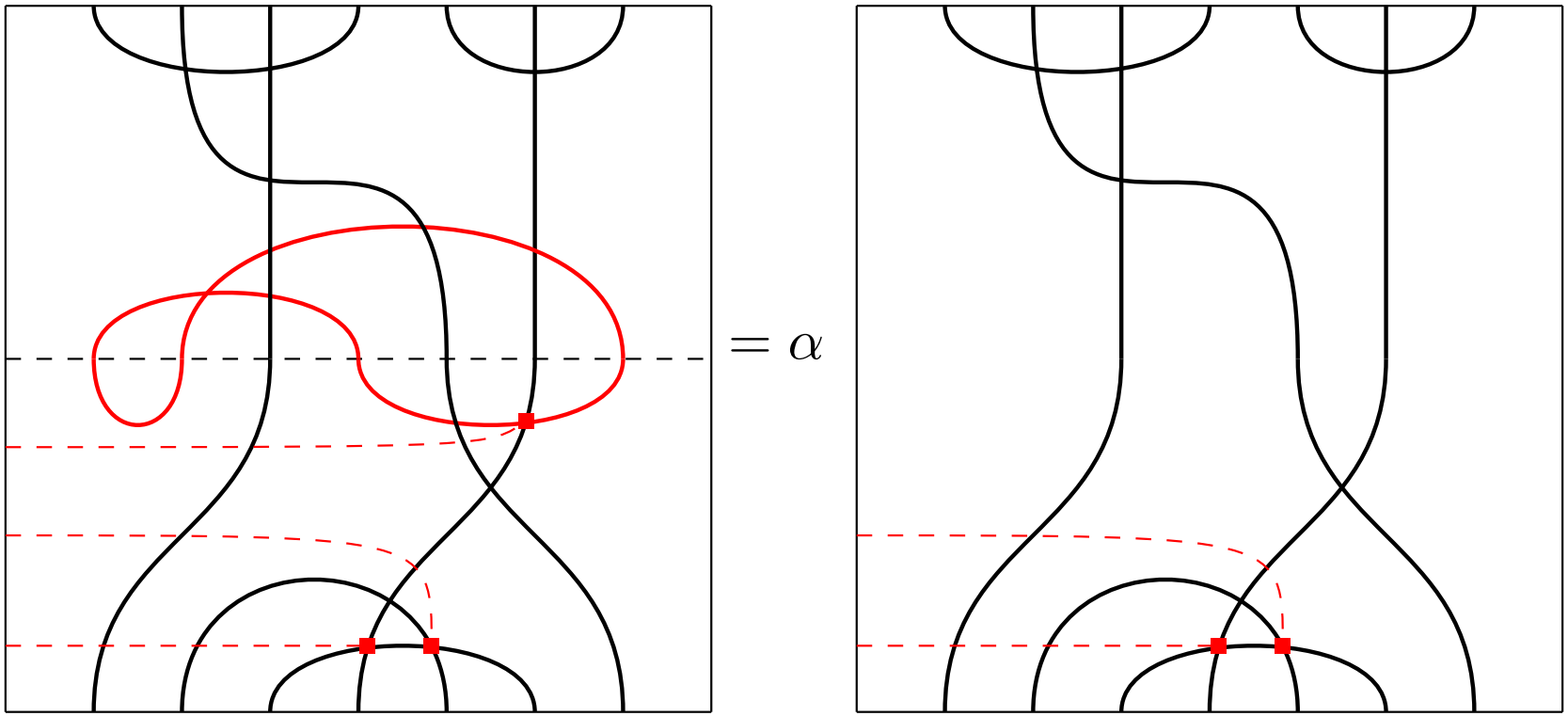
$$ht(p'\#p) \leq \max\{ht(p'), ht(p)\}$$

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Lemma (Height is non-increasing)

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Proof: $(p' \circ p = \alpha^{\#\text{loops}} p'\#p)$



Kadar-Martin-Yu Categories

Definition (Kadar, Martin, and Yu, 2014 ; KMY Category)

For $l = -1, 0, 1, \ldots, \infty$ define $J_l(\alpha) \subset Br(\alpha)$ to be the wide, $\mathbb{K}$-linear subcategory where now, $J_l(n, m)$ is spanned by pairings with height $\leq l$.

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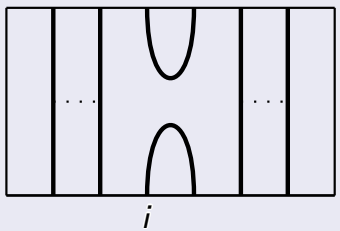
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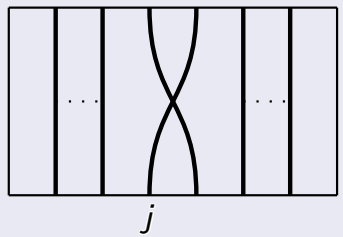
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Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}(\alpha)$ is generated by:

$e_i =$ 

, $s_j =$ 

,

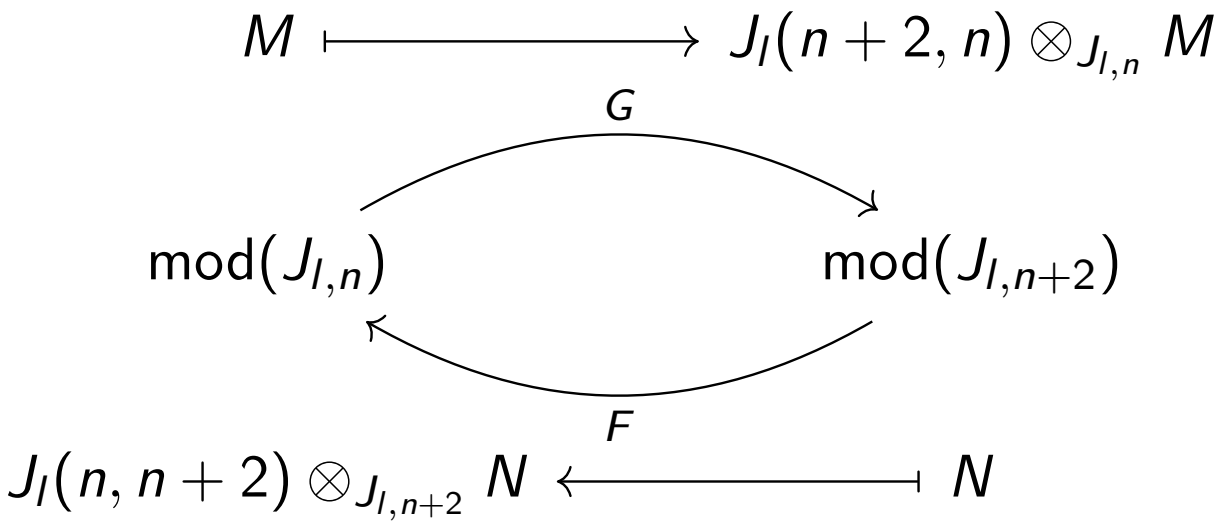
for $i = 1, \ldots, n-1$, and $j = 1, \ldots, \min(l+1, n-1)$. *i.e.* $\mathbb{K}S_n \subset Br_n$ whereas $\mathbb{K}S_{\min(l+2,n)} \subset J_{l,n}$.

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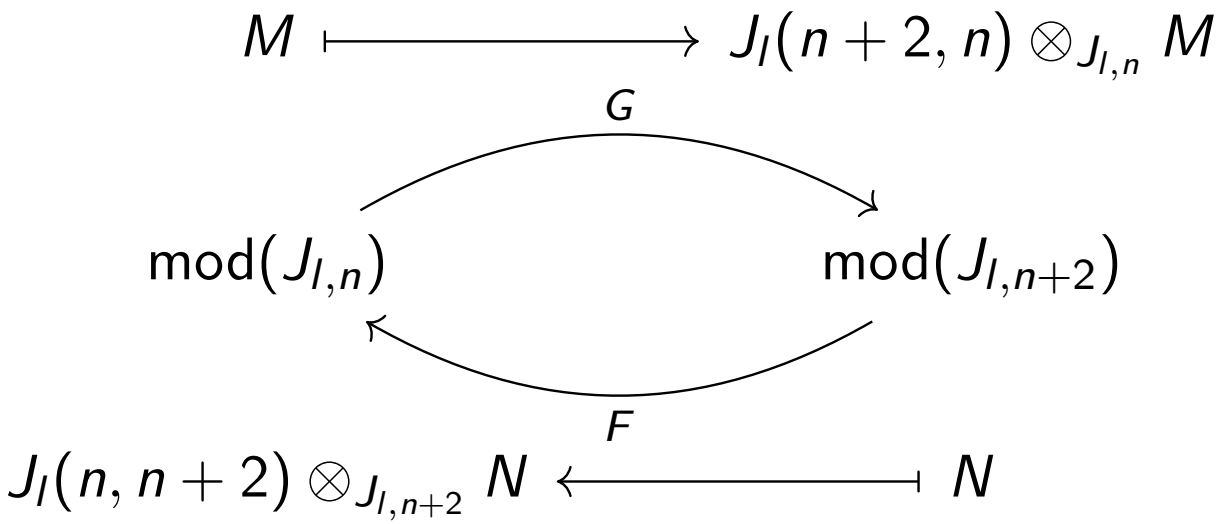
Representation Theory: Globalisation/Localisation

Use hom-spaces as bimodules:



Representation Theory: Globalisation/Localisation

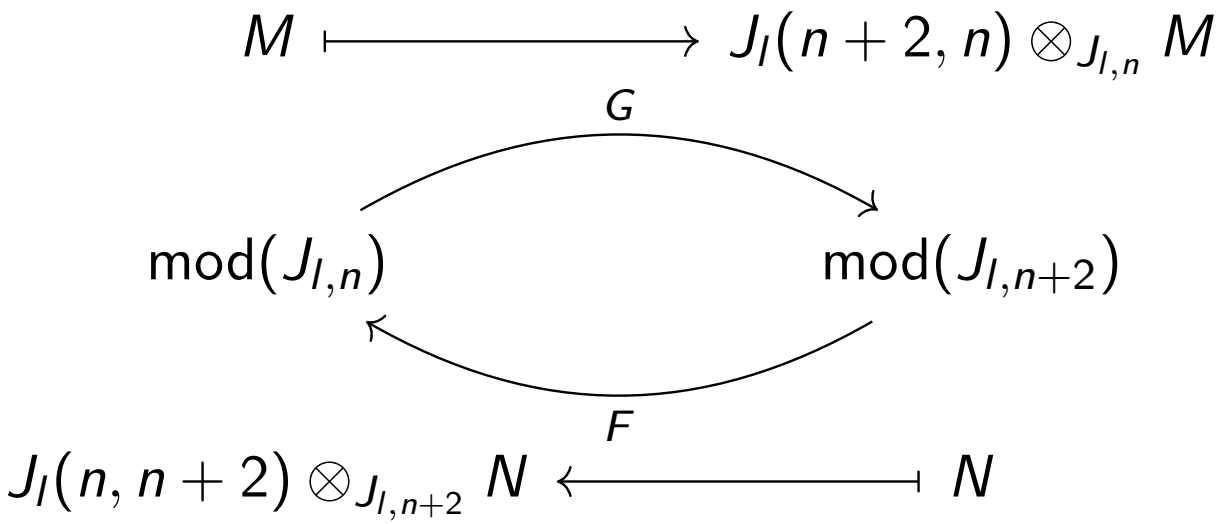
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Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$.

Representation Theory: Globalisation/Localisation

Use hom-spaces as bimodules:

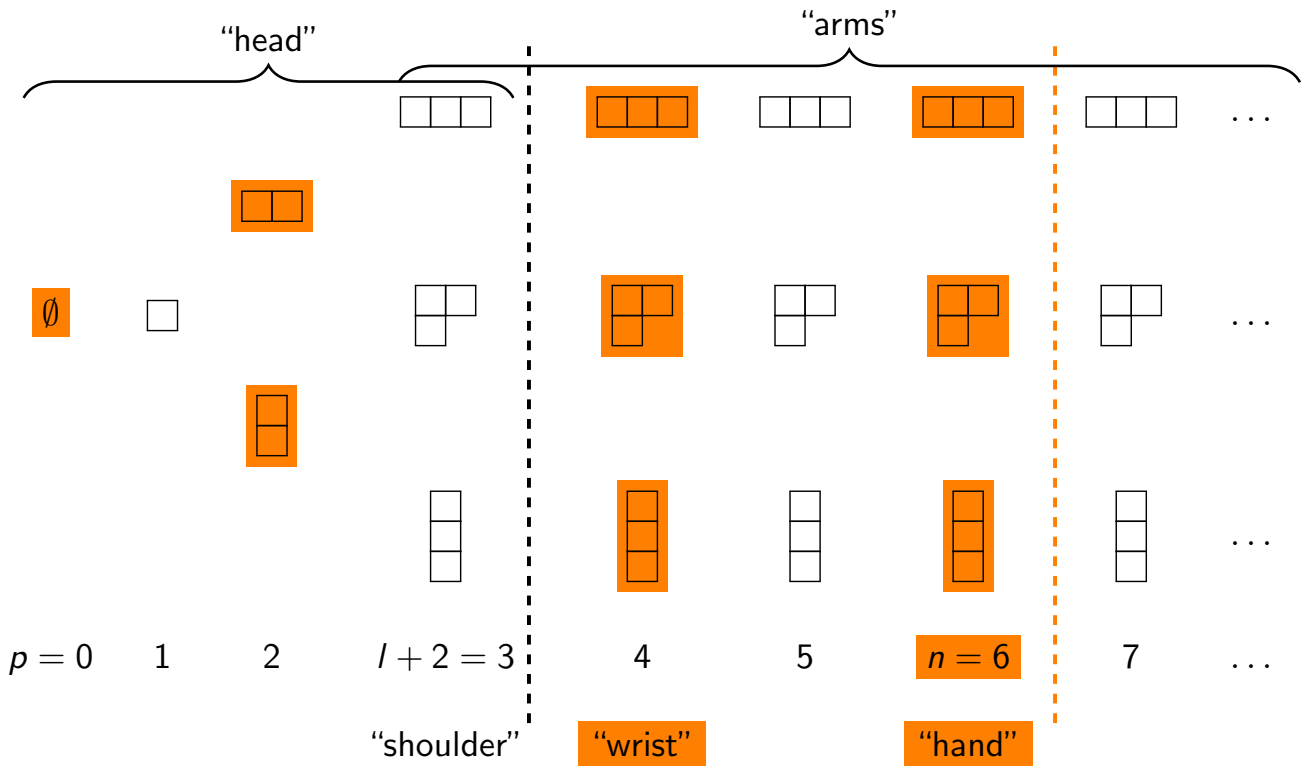


Then $F \circ G(M) = M$ but $G \circ F(N) = J_{l,n}^{\leq n-2} \otimes_{J_{l,n}} N$. Use to label simples inductively:

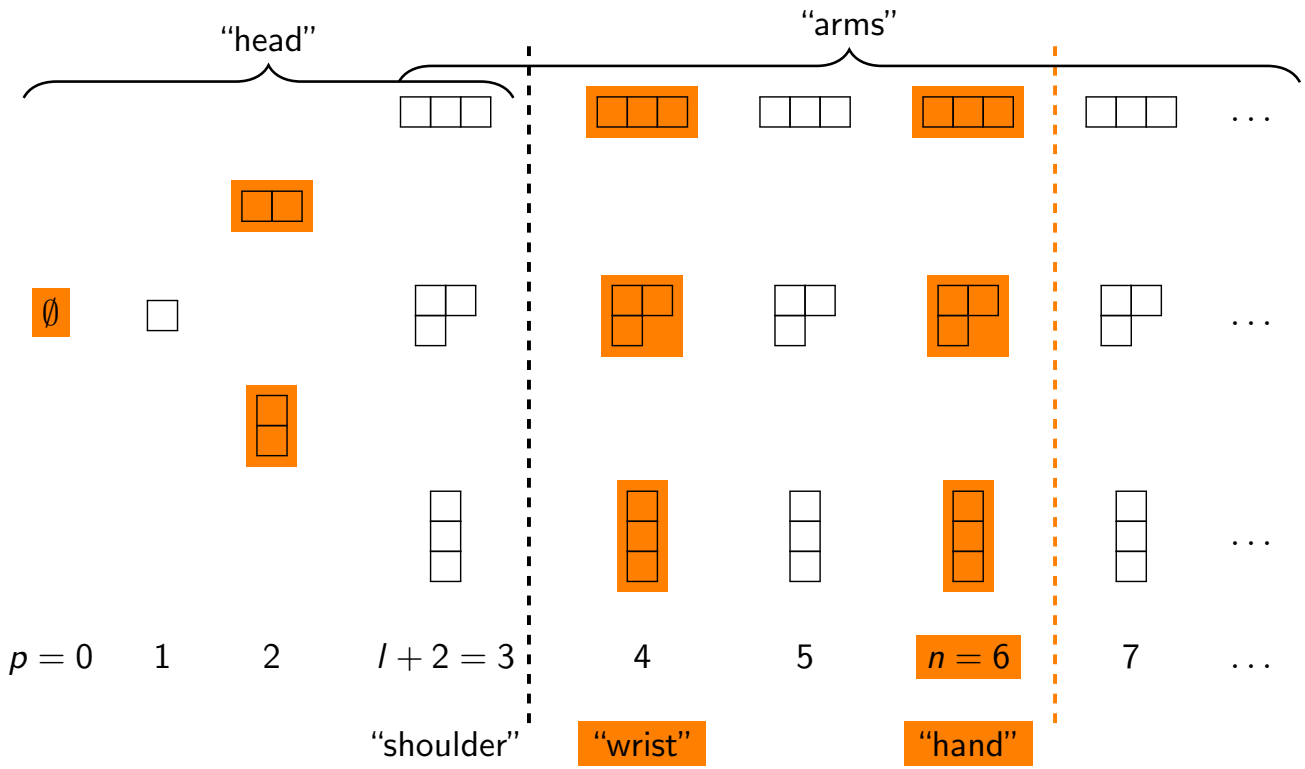
$$\begin{aligned} \Lambda_{l,n} &= \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_{l,n}^{\leq n-2}\right) \\ &= \Lambda_{l,n-2} \sqcup \Lambda(\mathbb{K}\, S_{\min(n,l+2)}) = \Lambda_{l,n-2} \sqcup \{\lambda \vdash \min(n, l+2)\}, \end{aligned}$$

$$\Rightarrow \qquad \Lambda_{l,n} = \{(p,\lambda) \mid p = n, n-2, \ldots, 0/1, \quad \lambda \vdash \min(p, l+2)\}$$

Representation Theory: Anatomy of mod($J_{l,-}$)



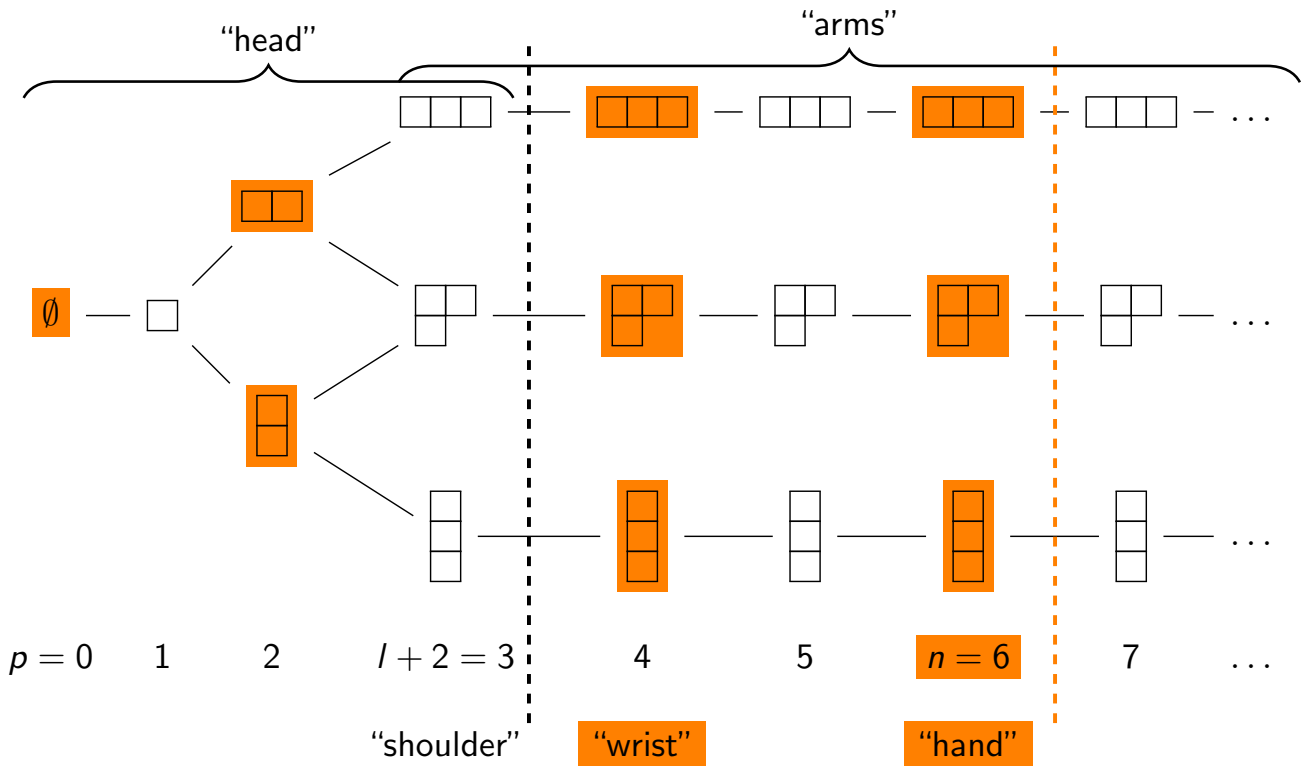
Representation Theory: Anatomy of mod($J_{l,-}$)



For each label have a **standard module**, constructed inductively:

$$\Delta^n_{(n,\lambda)} \simeq \mathcal{S}_\lambda, \qquad \Delta^n_{(n-2k,\lambda)} = G(\Delta^{n-2}_{(n-2k,\lambda)}).$$

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Encode induction/restriction rules on **Rollet diagram**, *e.g.* above.

Representation Theory: Standard Modules

Standard modules have simple heads $L_{(p,\lambda)}$ and determine $[L_{(p,\lambda)} : P_{(p,\lambda)}]$.

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Example: $n = 6$ and label $(4,(2,1))$. Write

$$\mathcal{S}_{(2,1)} = \mathbb{C}S_3 \cdot c_{(2,1)} = \mathbb{C}\{c_{(2,1)}, (2\,3)c_{(2,1)}\},$$

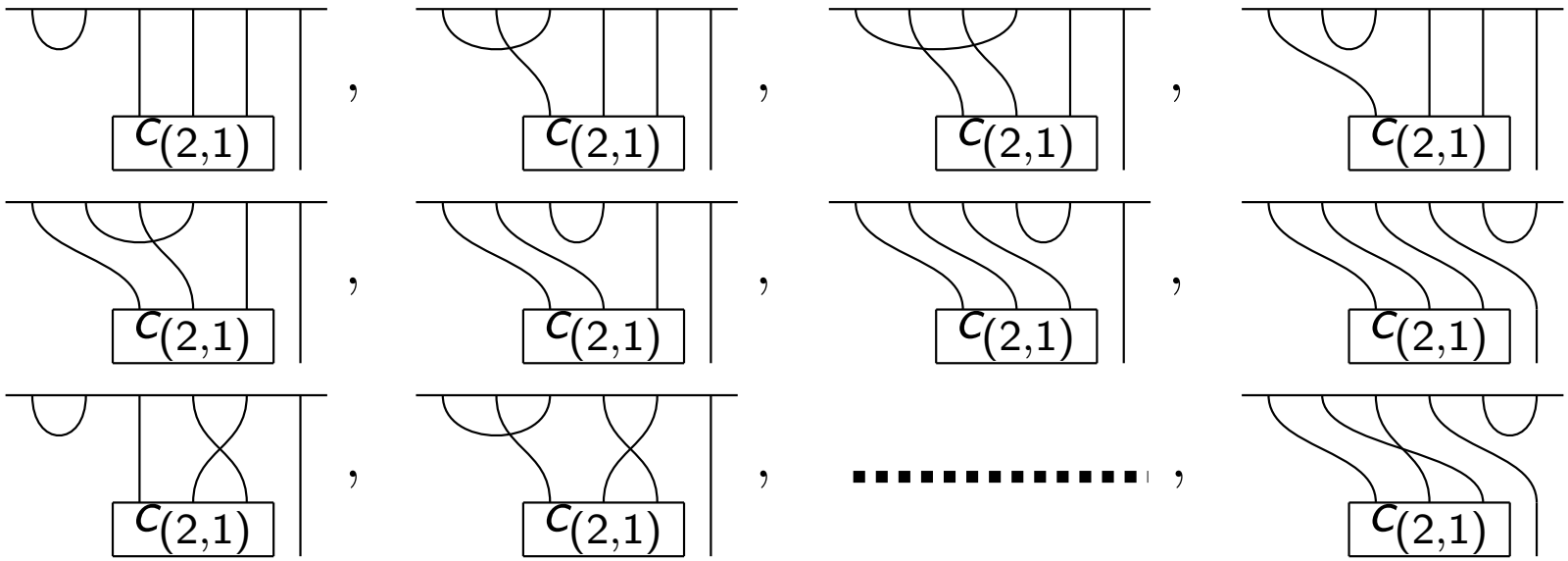
c_λ is a young symmetriser s.t.: $c_\lambda^2 = c_\lambda,\ c_\lambda^* = c_\lambda,\ c_\lambda \cdot \mathbb{C}S_{|\lambda|} \cdot c_\lambda = \mathbb{C}\, c_\lambda.$

Representation Theory: Standard Modules

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A basis for $\Delta^6_{(4,\lambda)}$ is:



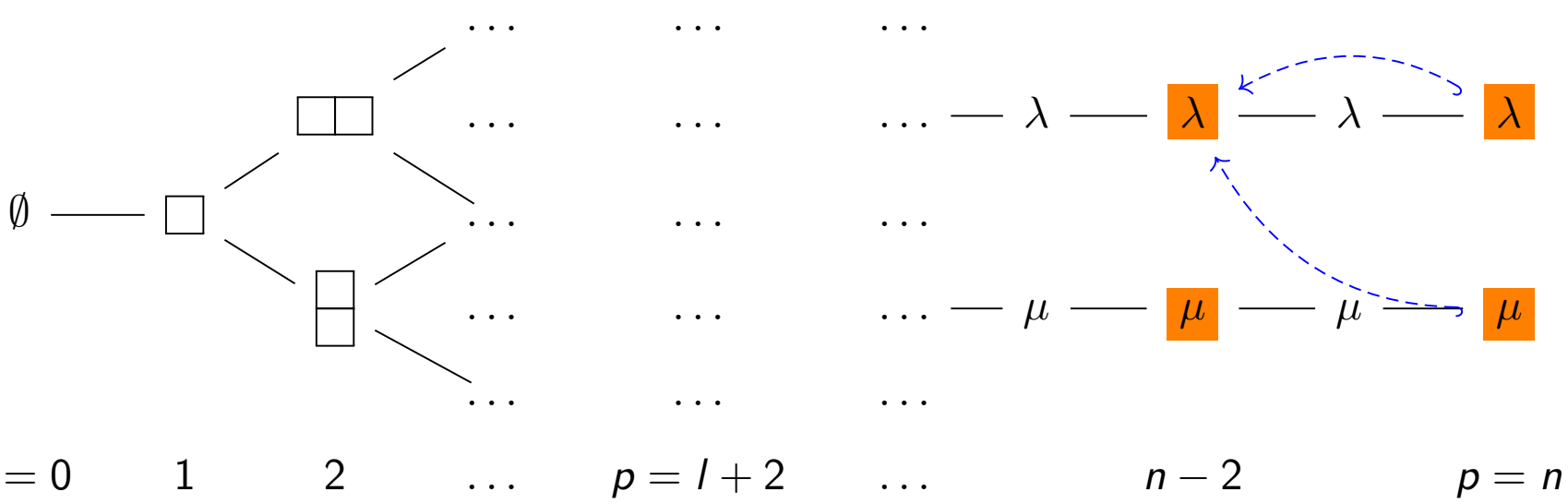
Representation Theory: Semisimplicity

Q: When is $J_{l,n}$ semisimple?

Representation Theory: Semisimplicity

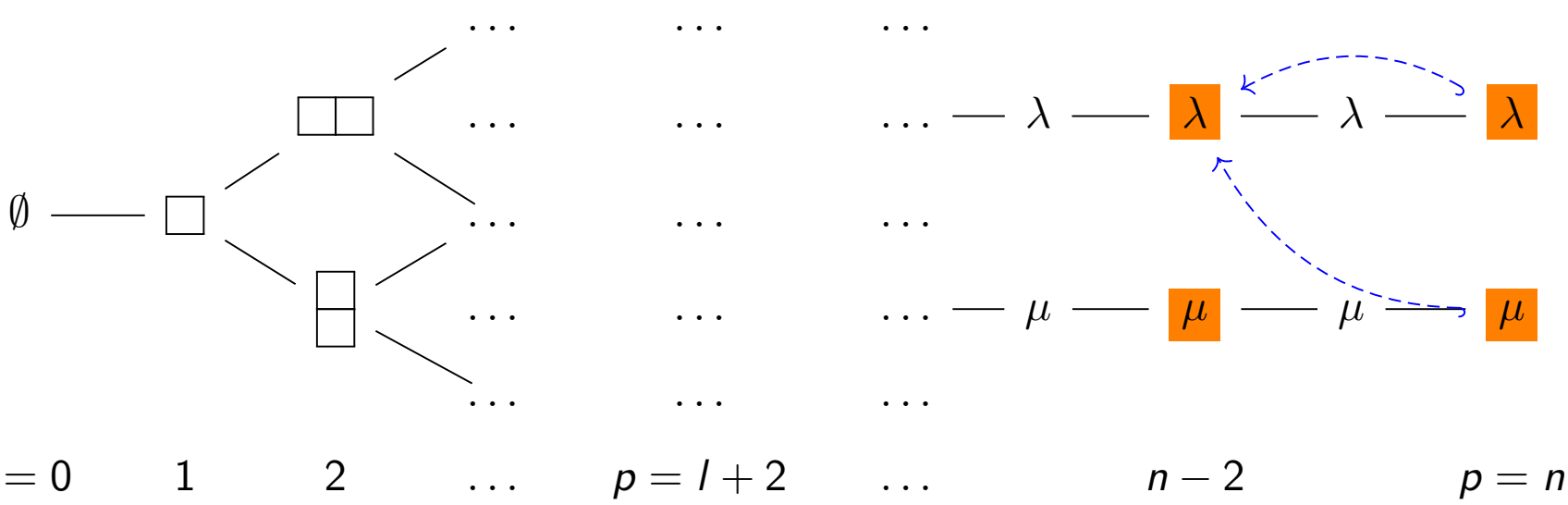
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A: When standards = simples. By tower of recollment frame work [Cox et al., 2004], enough to have no maps from "hand" to "wrist":



Representation Theory: Semisimplicity

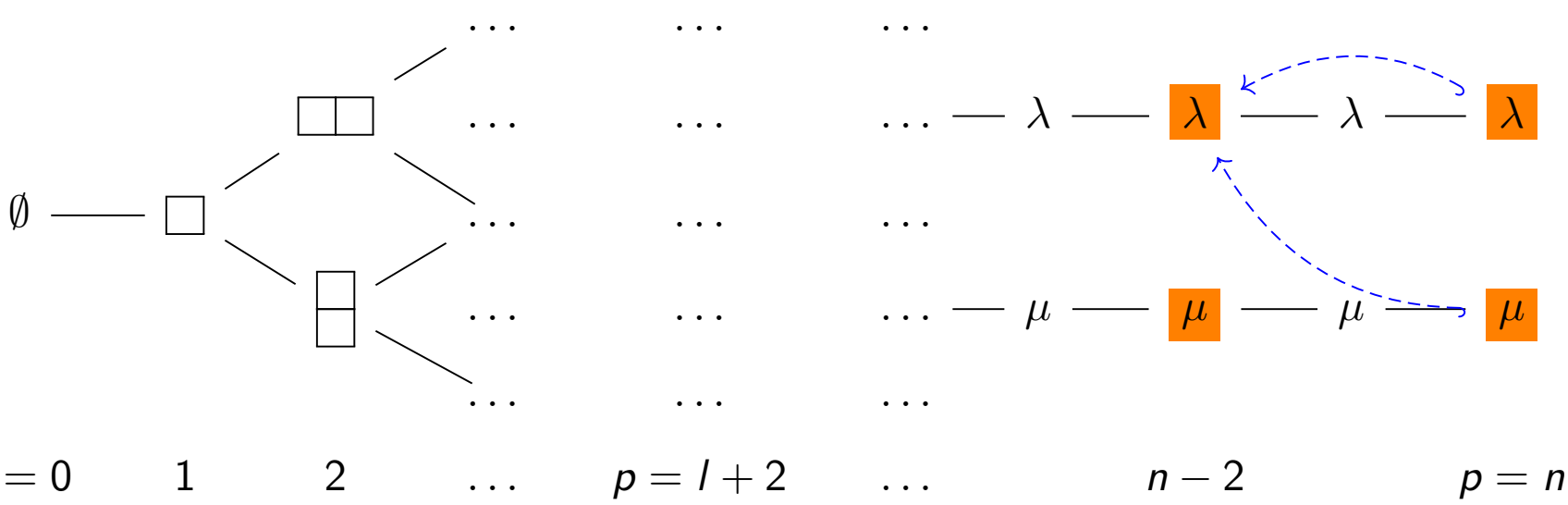
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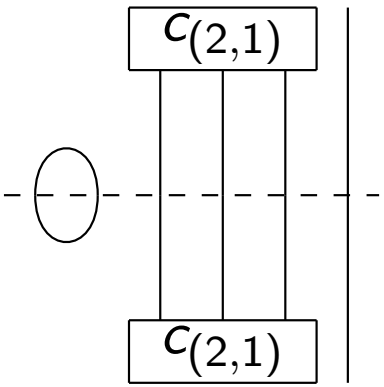


Reduce to looking for (Specht) submodules in $\Delta^n_{(n-2,\lambda)}$.
In turn this is reduced to studying non-degeneracy of a bilinear form

$$\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha].$$

Representation Theory: Bilinear Form

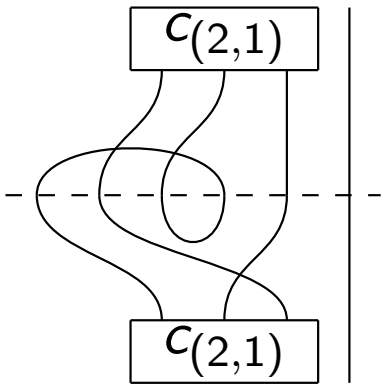
Q: When is $J_{l,n}$ (non-)semisimple?
A: ... study non-degeneracy of $\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha]$.



$$= \alpha\, c^{(4)}_{(2,1)}\, id_4\, c^{(4)}_{(2,1)}$$

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$$\langle _, _ \rangle = \alpha,$$



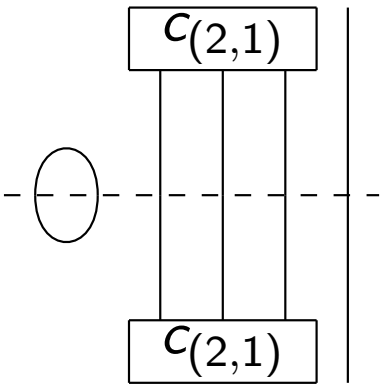
$$= c^{(4)}_{(2,1)}\, (1\ 2\ 3)\, c^{(4)}_{(2,1)}$$

$$= -\frac{1}{2}\, c^{(4)}_{(2,1)}$$

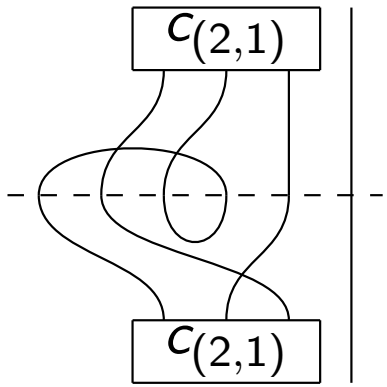
$$\langle _, _ \rangle = -\frac{1}{2}.$$

Representation Theory: Bilinear Form

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A: ... study non-degeneracy of $\Delta^n_{(n-2,\lambda)} \times \Delta^n_{(n-2,\lambda)} \to \mathbb{C}[\alpha]$.



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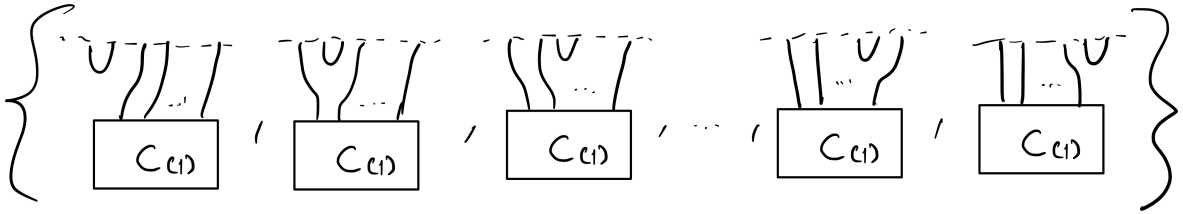
$$= c^{(4)}_{(2,1)}\, (1\ 2\ 3)\, c^{(4)}_{(2,1)}$$
$$= -\frac{1}{2}\, c^{(4)}_{(2,1)}$$

$$\langle _, _ \rangle = \alpha, \qquad \langle _, _ \rangle = -\frac{1}{2}.$$

Form is symmetric and contravariant with respect to $J_{l,n}$-action, *i.e.*

$$\langle \xi, y\zeta \rangle = \langle y^* \xi, \zeta \rangle.$$

Representation Theory: Bilinear Form - TL Case

$\mathcal{B} =$ 

Gram matrix of form:

$$\Gamma(\Delta_{(n-2,(1))}) = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 \\ 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow \begin{aligned} P_n^{(1)}(\alpha) &:= |\Gamma(\Delta_{(n-2,(1))})| \\ &= U_{n-1}(\alpha/2) \end{aligned}$$

Chebyshev recursion: $P_{n+1}^{(1)}(\alpha) = \alpha P_n^{(1)}(\alpha) - P_{n-1}^{(1)}(\alpha)$,

$$P_1^{(1)} = 1, \qquad P_2^{(1)} = \alpha$$

$$P_3^{(1)} = \alpha^2 - 1, \quad P_5^{(1)} = \alpha(\alpha^2 - 2), \quad P_4^{(1)} = \left(\alpha^2 - \alpha - 1\right)\left(\alpha^2 + \alpha - 1\right)$$

Roots: $P_n^{(1)}(\alpha) = 0$ at $\alpha = q + q^{-1}$ and $q^{2n} = 1$ (or $\alpha = 2\cos(k\pi/n)$).

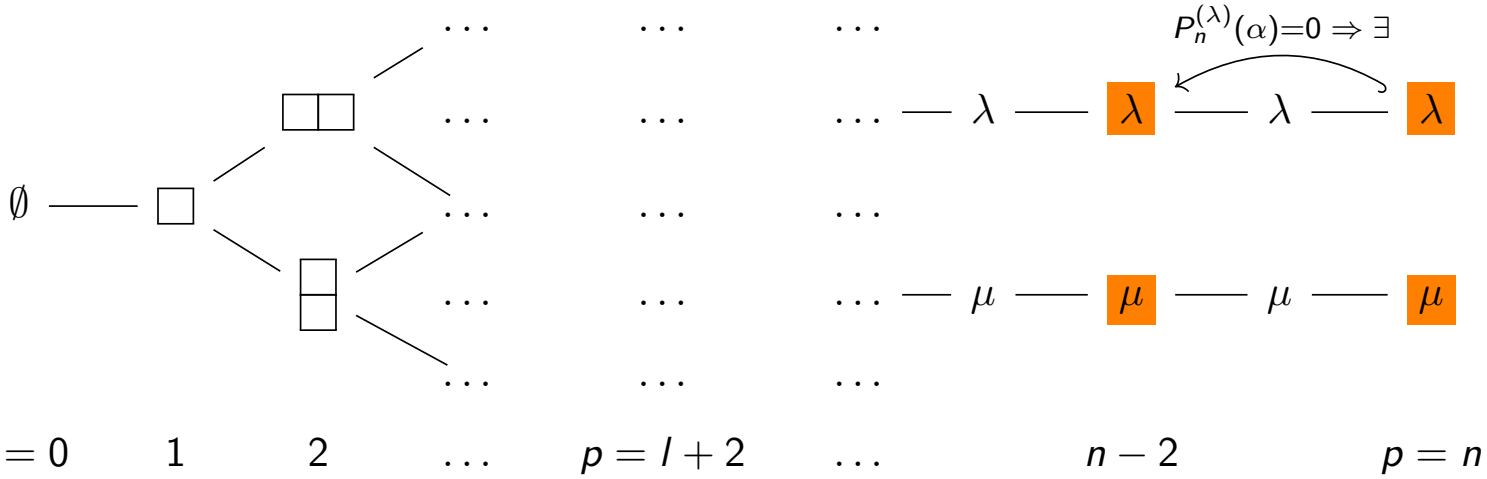
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Morris and Martin, 2025] - Generalised Roots of Unity

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**. Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.

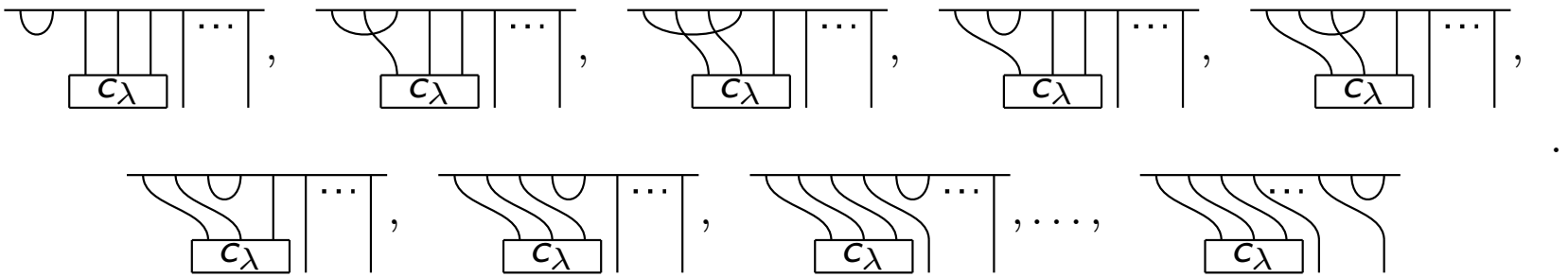


Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$,

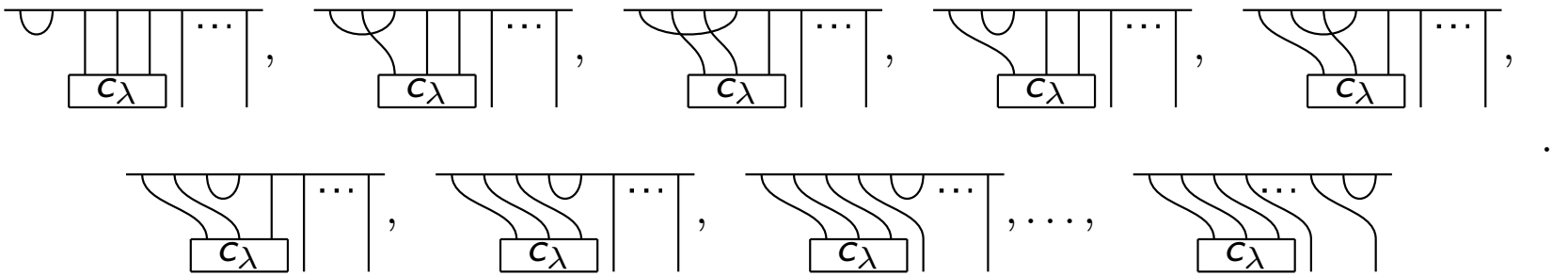
Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, *e.g.* $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



Representation Theory: Bilinear Form - Det. Factorisation

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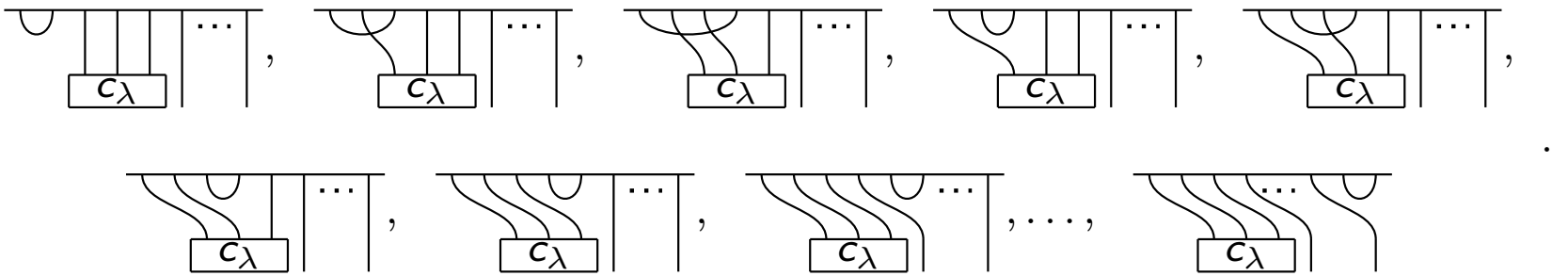
Need to compute determinant of:

$$\left(\begin{array}{ccccccc|cccccccc}
\alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l}
u_{12}b \\ u_{13}b \\ u_{14}b \\ u_{23}b \\ u_{24}b \\ u_{34}b \\ u_{45}b
\end{array}\Bigg\} \text{head}
\qquad
\begin{array}{l}
u_{56}b \\ u_{67}b \\ u_{78}b \\ u_{89}b \\ u_{9,10}b \\ u_{10,11}b \\ u_{11,12}b \\ u_{12,13}b
\end{array}\Bigg\} \text{tail}$$

,

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Warm up $d_\lambda = 1$, *e.g.* $l = 1$ and $\lambda = (3), (1^3)$ and $n = 13$. Basis:



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$$\left(\begin{array}{ccccccc|cccccccc}
\alpha & 1 & \epsilon & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 1 & \alpha & 0 & 1 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \epsilon & \epsilon & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
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u_{56}b \\ u_{67}b \\ u_{78}b \\ u_{89}b \\ u_{9,10}b \\ u_{10,11}b \\ u_{11,12}b \\ u_{12,13}b
\end{array}\Bigg\} \text{tail} \quad ,$$

By expanding along last row we see the determinant satisfies Chebyshev recursion in n. We therefore, obtain the desired form: $\det = C^{(\lambda)}(\alpha)P_n^{(\lambda)}(\alpha)$.

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$
\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)
\begin{array}{l} \\ \\ \\

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)$$

k_1 (rows 8–10), k_2 (rows 19–21)

This has two Chebyshev expansions: $\det = C^{(\lambda)}(\alpha)\, P^{(\lambda)}_{n_1}(\alpha) = D^{(\lambda)}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$.

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Now $d_\lambda = 2$, *e.g.* $l = 1$ and $\lambda = (2,1)$. Need to compute a determinant like:

$$\left(\begin{array}{ccccccc|ccc|ccccccc|ccc}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon & 0 & \epsilon & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & \delta & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 1 & \alpha & 0 & 1 & -1 & \epsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & -1 & 0 & 1 & 1 & \alpha & 1 & \epsilon' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \delta' & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & \epsilon' & 1 & \alpha & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha
\end{array}\right)$$

k_1 (rows 8–10) k_2 (rows 19–21)

This has two Chebyshev expansions: $\det = C^{(\lambda)}(\alpha)\, P^{(\lambda)}_{n_1}(\alpha) = D^{(\lambda)}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$.
Need to

1. Show $P^{(\lambda)}_n(\alpha) = Q^{(\lambda)}_n(\alpha)$. (Interesting)
2. Show $P^{(\lambda)}_{n_1}(\alpha)\, Q^{(\lambda)}_{n_2}(\alpha)$ divides determinant. (Boring)

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

1. Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
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To prove 1. we show both series have the same initial conditions by a row reduction procedure...

Representation Theory: Bilinear Form - Det. Factorisation

Proof Sketch: Need to

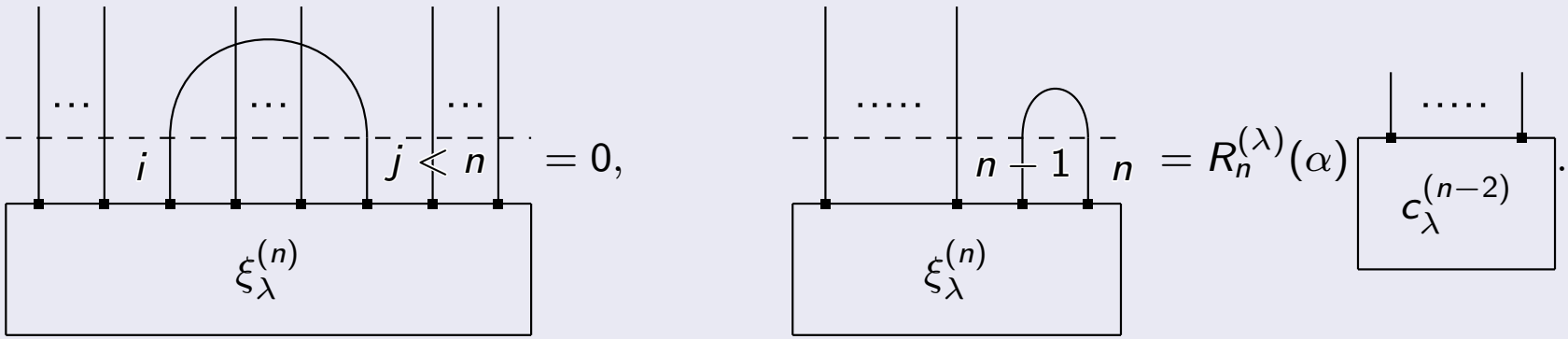
1. Show $P_n^{(\lambda)}(\alpha) = Q_n^{(\lambda)}(\alpha)$. (Interesting)
2. Show $P_{n_1}^{(\lambda)}(\alpha)\, Q_{n_2}^{(\lambda)}(\alpha)$ divides determinant. (Boring)

To prove 1. we show both series have the same initial conditions by a row reduction procedure... uses a nice family of elements $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$.

Lemma

For each $n \geq l+4$ there exists a unique $\xi_\lambda^{(n)} \in \Delta^n_{(n-2,\lambda)}$ (polynomial coefficients) such that:

1. *$\xi_\lambda^{(n)}$ is non-zero for all specialisations $\alpha \in \mathbb{C}$,*
2. *We have (in $\Delta^{n-2}_{(n-2,\lambda)}$)*



When $\alpha \in \mathbb{C}$ s.t. $R_n^{(\lambda)}(\alpha) = 0$, $\xi_\lambda^{(n)}$ generates a submodule $J_{l,n}\xi_\lambda^{(n)} \simeq \mathcal{S}_\lambda \subset \Delta^n_{(n-2,\lambda)}$. □

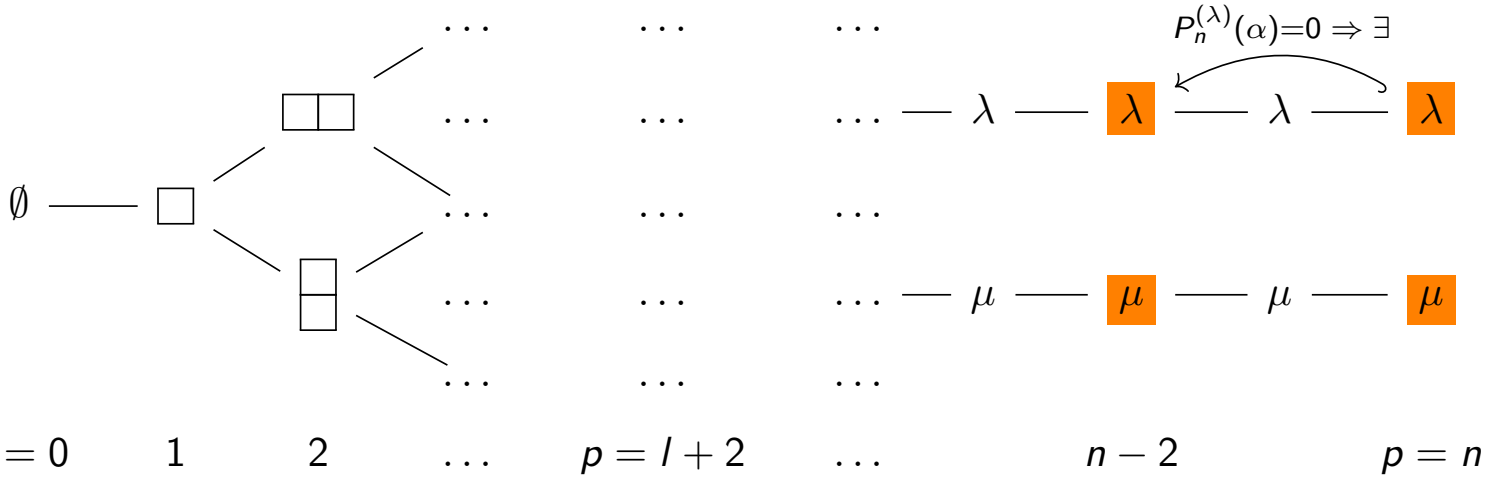
Representation Theory: Bilinear Form - Det. Factorisation

Theorem [Martin-M.,2025] - Generalised Root of Unity Criteria

Fix l, $n \geq l+4$, $\lambda \vdash l+2$. $\exists$ polynomials $C^{(\lambda)}(\alpha)$, $P_n^{(\lambda)}(\alpha) \in \mathbb{C}[\alpha]$ s.t.

$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda},$$

where $d_\lambda = \dim(\mathcal{S}^\lambda)$ and the $P_n^{(\lambda)}$ satisfies the **Chebyshev recursion**. Furthermore, when $\alpha \in \mathbb{C}$ is a root of $P_n^{(\lambda)}(\alpha)$ we have $\Delta^n_{(n,\lambda)} \hookrightarrow \Delta^n_{(n-2,\lambda)}$.



Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



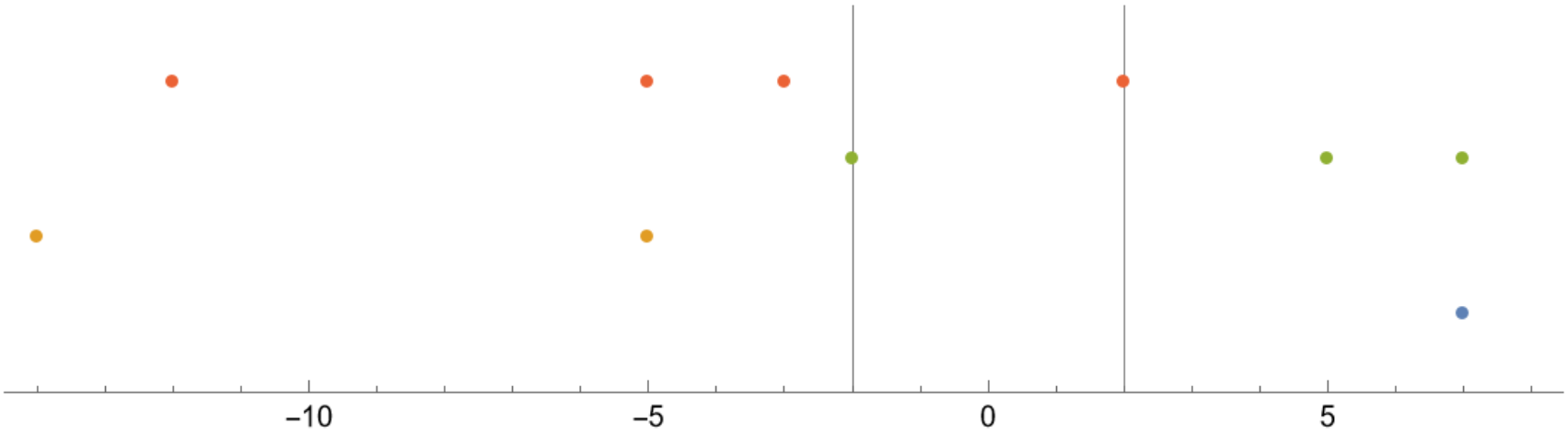


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = -1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



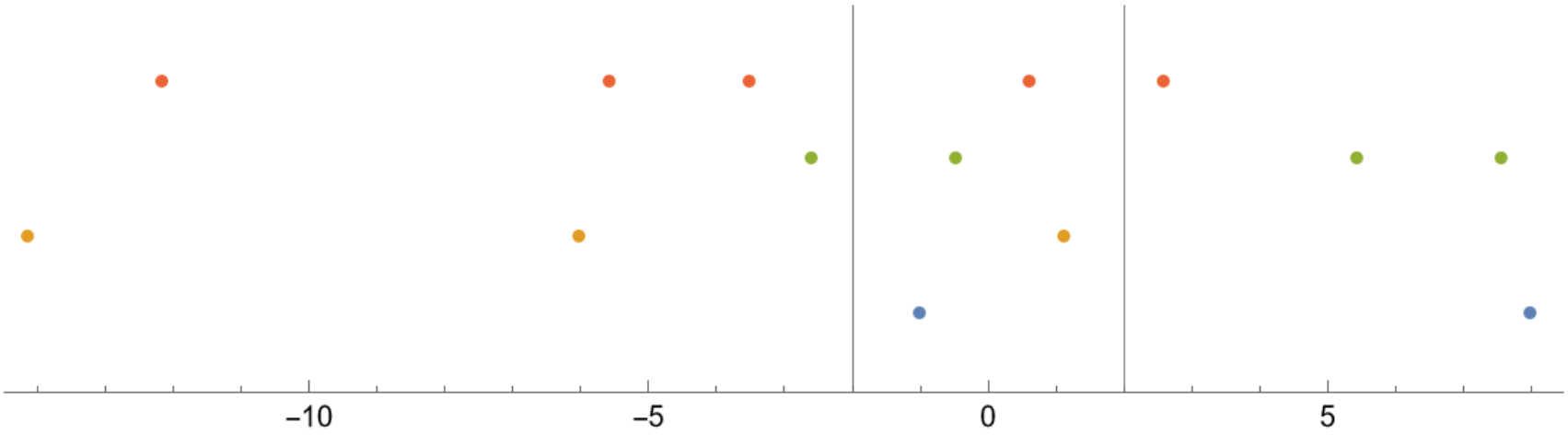


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 0$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



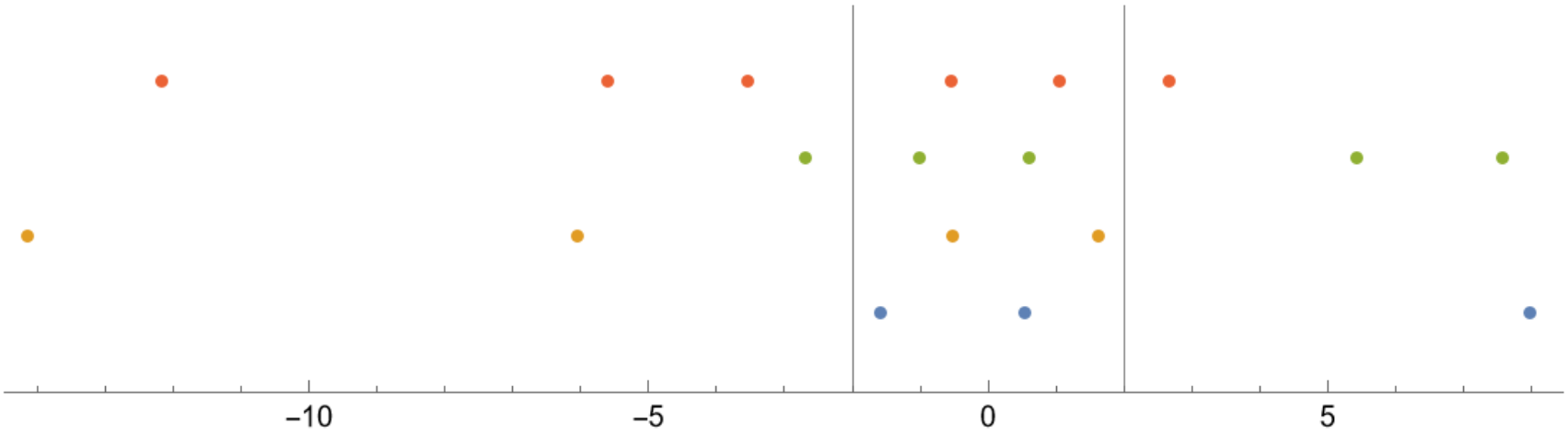


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 1$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



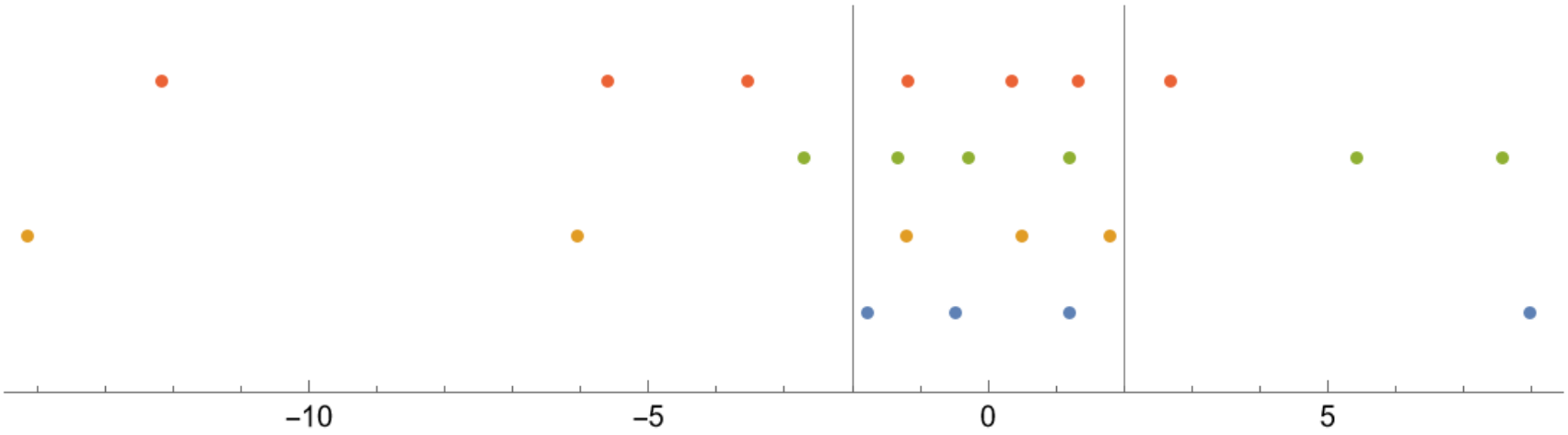


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 2$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



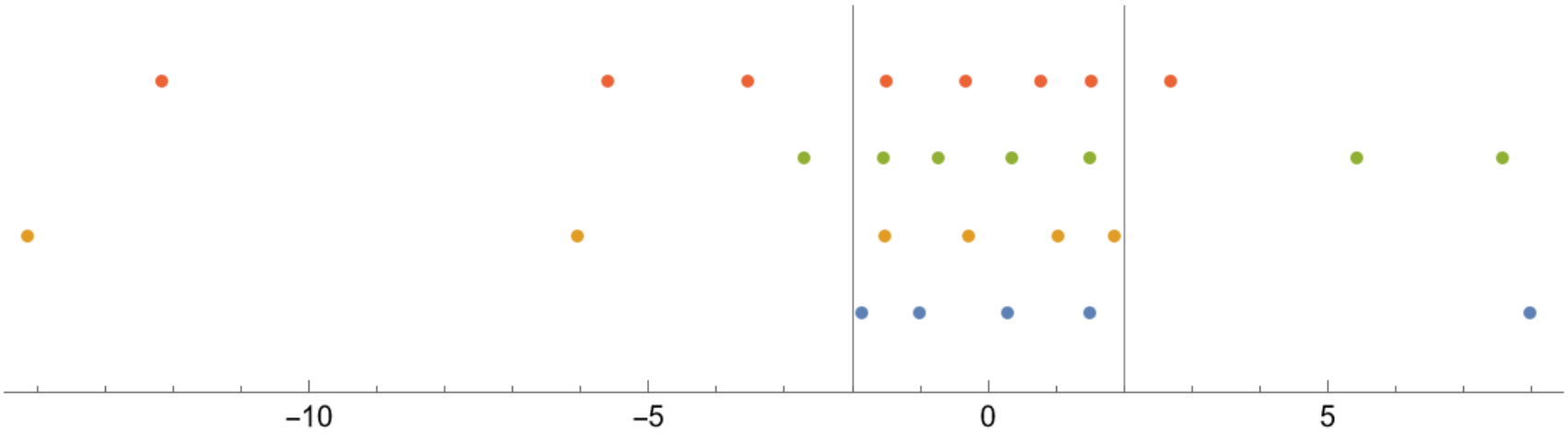


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 3$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



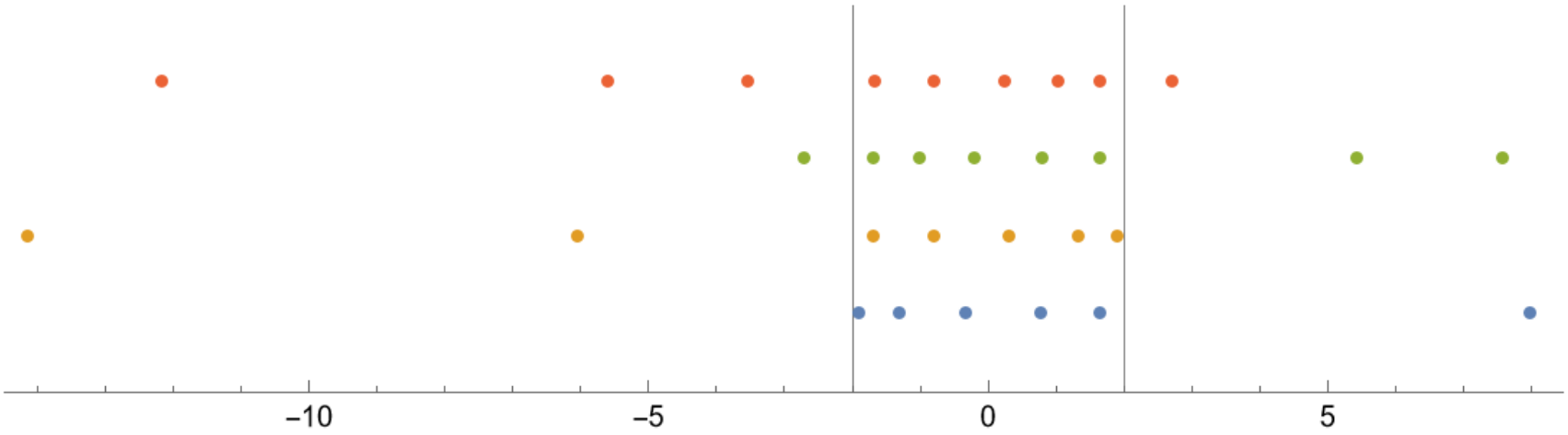


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 4$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



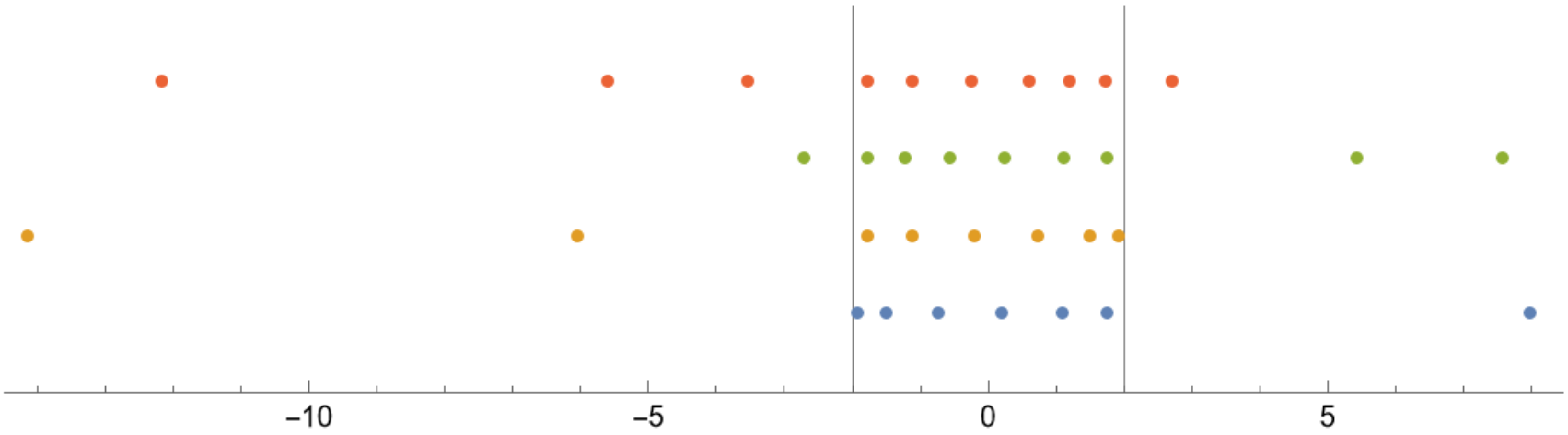


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 5$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



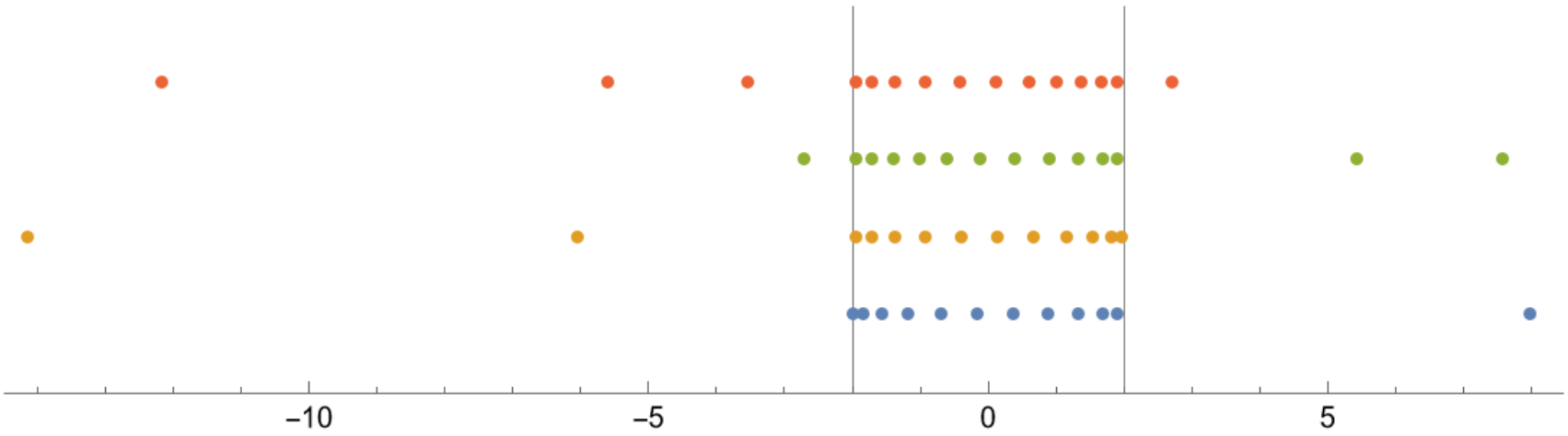


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 10$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



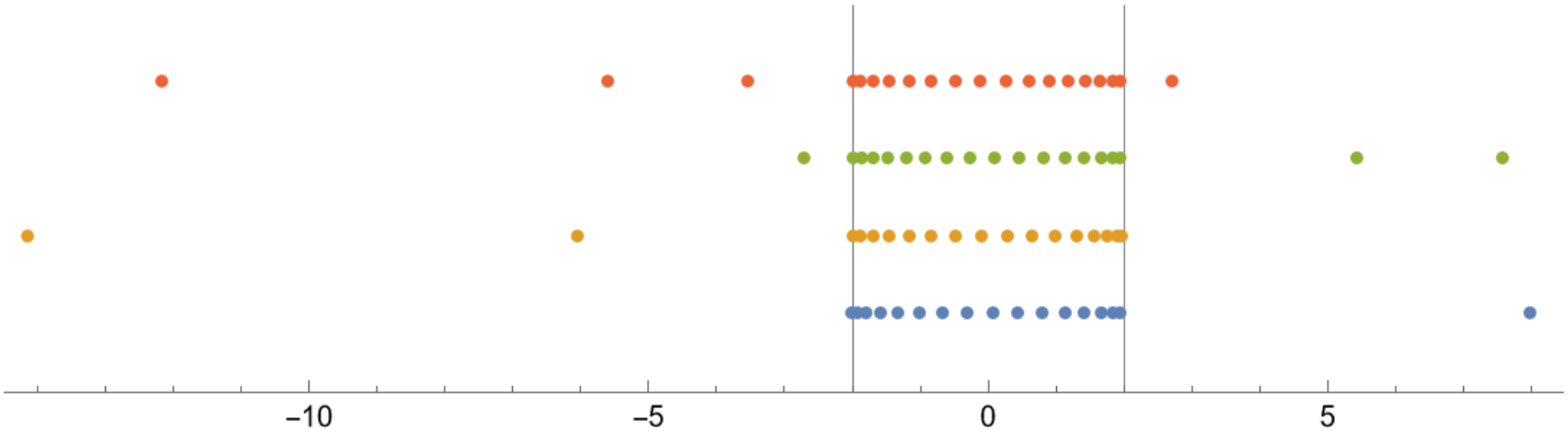


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 15$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



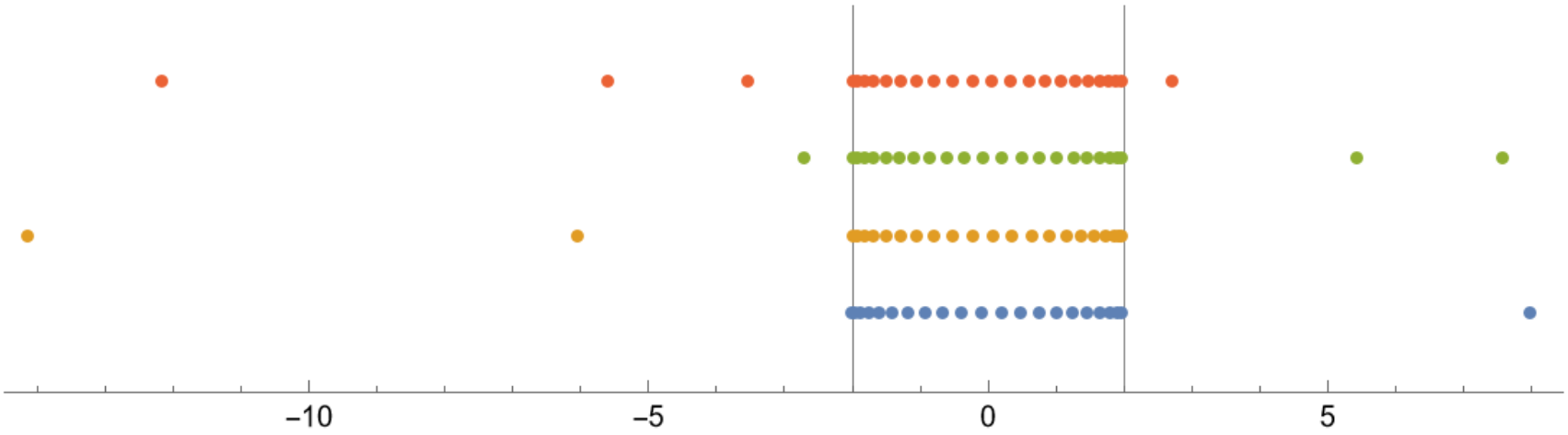


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 20$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



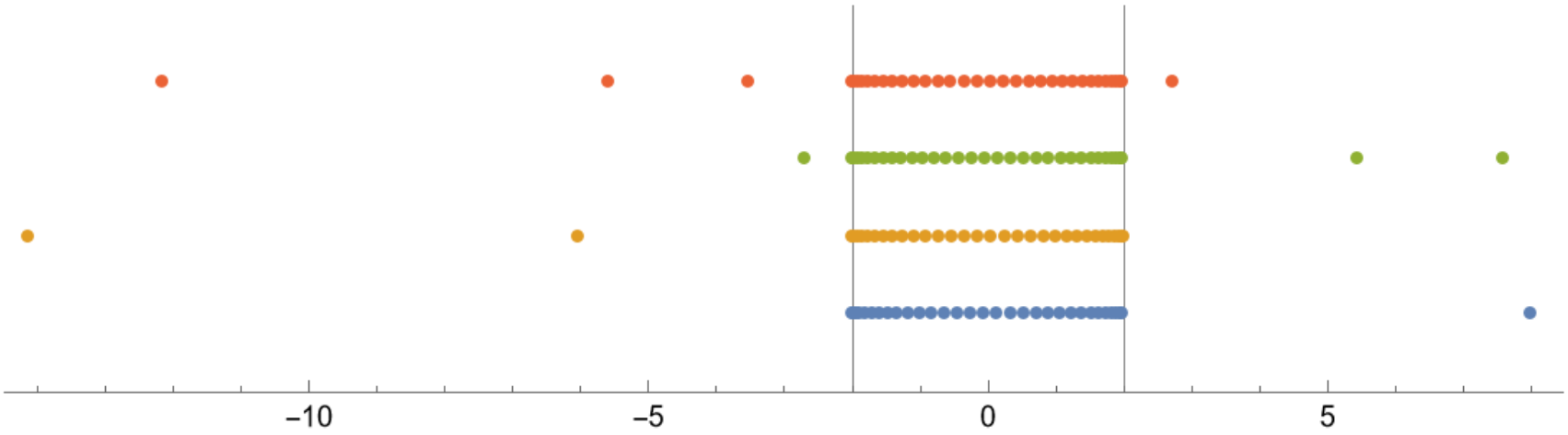


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 30$

Rep. Theory: Generalised roots of unity

We plot roots for $\lambda \vdash l + 2 = 8$ in the cases below:



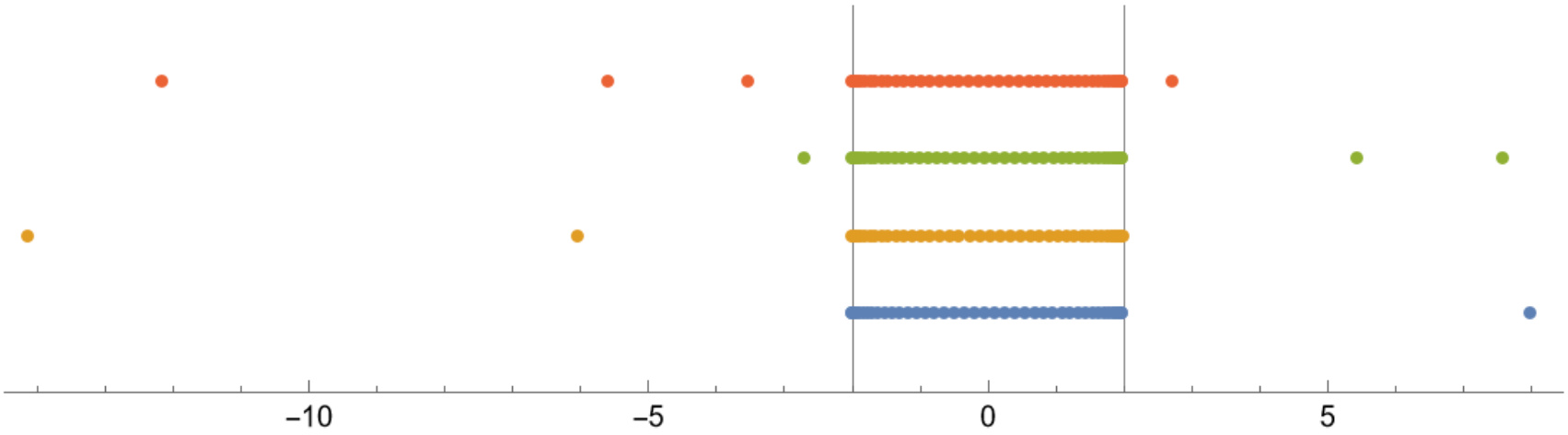


Figure: Roots of $P^{(\lambda)}_{l+4+k}$, with $k = 40$

Representation Theory: Bilinear Form - Det. Factorisation

Computed for all $\lambda \vdash l + 2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).

Representation Theory: Bilinear Form - Det. Factorisation

Computed for all $\lambda \vdash l + 2 < 6$ and some infinite families of λ (rows, columns and things that differ by one box).

Q: What about the constant piece of the factorisation?

Representation Theory: Bilinear Form - Det. Factorisation

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Q: What about the constant piece of the factorisation?

Example: For $l = 0$ we have $\lambda = (2), (1^2)$, $d_\lambda = 1$

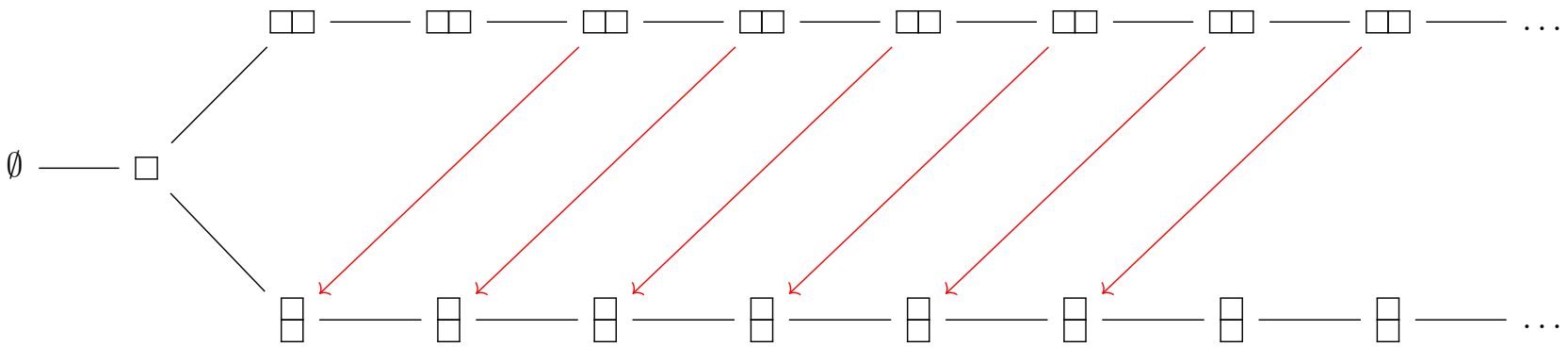
$$C^{(2)}(\alpha) = (\alpha - 1), \qquad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

Representation Theory: Bilinear Form - Det. Factorisation

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Q: What about the constant piece of the factorisation?
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$$C^{(2)}(\alpha) = (\alpha - 1), \qquad C^{(1^2)}(\alpha) = (\alpha - 1)(\alpha + 2).$$

Zeros correspond to maps "across arms" i.e. at $\alpha = -2$ we have $\Delta^n_{(n,(2))} \hookrightarrow \Delta^n_{(n-2,(1^2))}$.



Representation Theory: Bilinear Form - Det. Factorisation

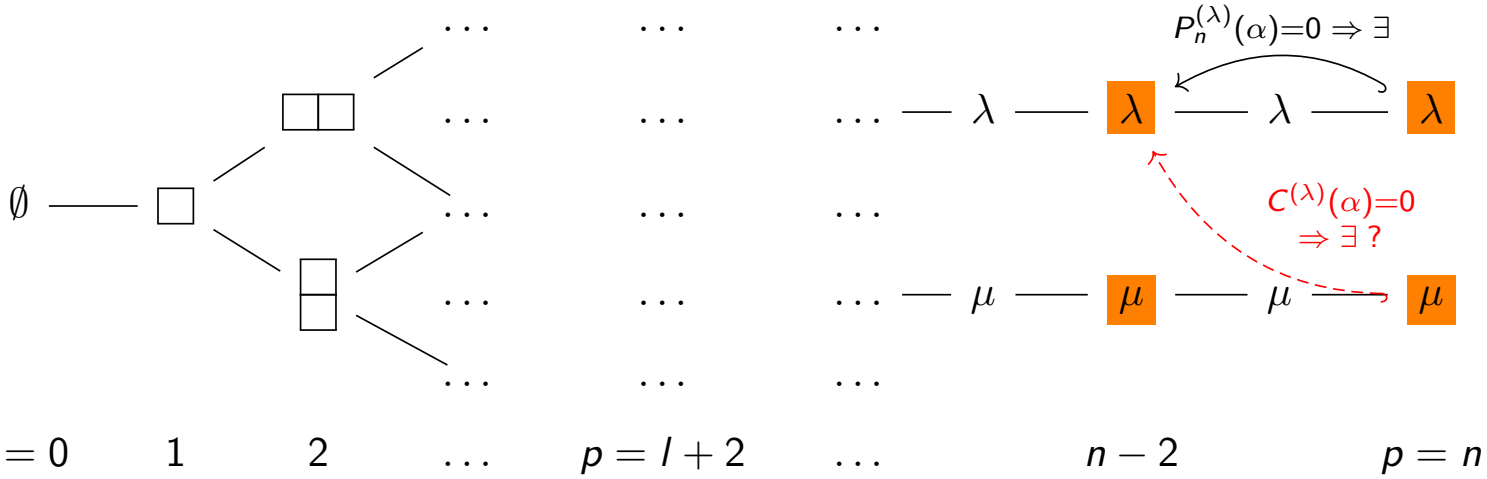
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$$\det(\Gamma(\Delta^n_{(n-2,\lambda)}))(\alpha) = C^{(\lambda)}(\alpha)\left(P_n^{(\lambda)}(\alpha)\right)^{d_\lambda} \dots$$

Conjecture

The $C^{(\lambda)}(\alpha)$ have integer roots which correspond to "non-TL behaviour".



Thank you!

Questions?

References:

- Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv 2407.07028. arXiv: 2407.07028 [math.RT]. URL: https://arxiv.org/abs/2407.07028.
- Cox, Anton et al. (2004). *Representation theory of towers of recollement: theory, notes, and examples*. arXiv 0411395. arXiv: math/0411395 [math.RT].
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- Morris, Benjamin and Paul P. Martin (2025). *On semisimplicity criteria and non-semisimple representation theory for the Kadar-Yu algebras*. arXiv: 2512.24535 [math.RT]. URL: https://arxiv.org/abs/2512.24535.