UNIVERSITY OF LEEDS

A Diagram Category for Non-Orientable Surfaces

Benjamin Morris[1]
Joint work with Dionne Ibarra[2], Gabriel Montoya-Vega[3], and Paul Martin[1] (supervisor)

[1]University of Leeds
[2]Monash University, Melbourne
[3]CUNY Graduate Center, NYC

Leeds School of Mathematics PGR Conference, July 2024

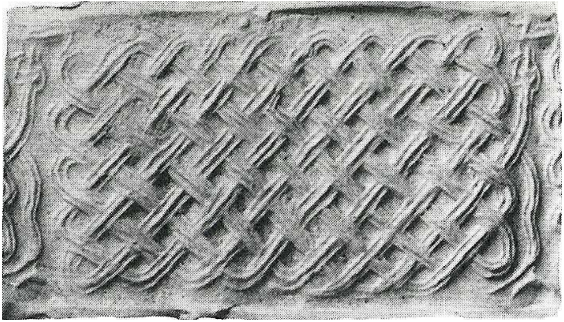
Motivation

[1] M.C. Heath, *Hesperia* 27 (1958), pl.22, no. S57

[2] D. Wolkstein, S.N. Kramer, Inanna, Harper Collins Publisher, 1983

Motivation

No. S57



Lerna, Greece,
Circa 2500-2200BC [1]

Ur, Mesopotamia,
Circa 2600-2500BC [2]

[1] M.C. Heath, *Hesperia* 27 (1958), pl.22, no. S57

[2] D. Wolkstein, S.N. Kramer, Inanna, Harper Collins Publisher, 1983

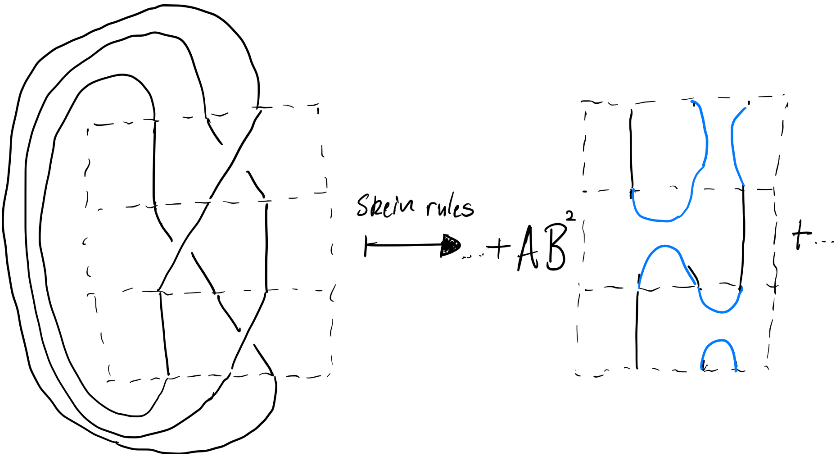
Motivation

Maths to organise drawings

UNIVERSITY OF LEEDS

Motivation

Maths to organise drawings



The Temperley-Lieb Category

The Temperley-Lieb Category

Fix a unital commutative ring R and suppose $\alpha \in R$.

The Temperley-Lieb Category

Fix a unital commutative ring R and suppose $\alpha \in R$. The category $TL(\alpha)$ is an R-linear category defined as follows:

The Temperley-Lieb Category

Fix a unital commutative ring R and suppose $\alpha \in R$. The category $TL(\alpha)$ is an R-linear category defined as follows:

- **Objects:** Non-negative integers

The Temperley-Lieb Category

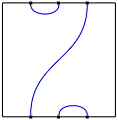
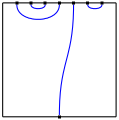
Fix a unital commutative ring R and suppose $\alpha \in R$. The category $TL(\alpha)$ is an R-linear category defined as follows:

- **Objects:** Non-negative integers
- **Morphisms:** $\mathrm{Hom}(n, m)$ is $\{0\}$ if $n + m = 1 \mod 2$, otherwise R-linear combinations of type n, m "TL-diagrams",

The Temperley-Lieb Category

Fix a unital commutative ring R and suppose $\alpha \in R$. The category $TL(\alpha)$ is an R-linear category defined as follows:

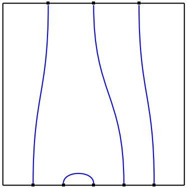
- **Objects:** Non-negative integers
- **Morphisms:** $\text{Hom}(n, m)$ is $\{0\}$ if $n + m = 1 \mod 2$, otherwise R-linear combinations of type n, m "TL-diagrams", *e.g.*

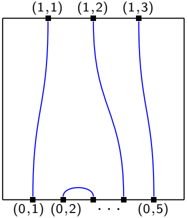
 $\in \text{Hom}(3, 3)$,  $\in \text{Hom}(1, 7)$.

Temperley-Lieb Diagrams

Temperley-Lieb Diagrams

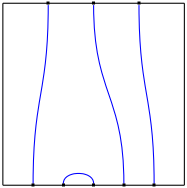
What we see:

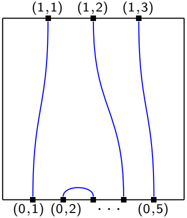




Temperley-Lieb Diagrams

What we see:



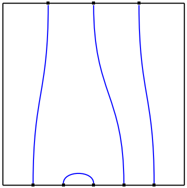


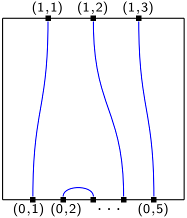
What we mean:

$$\Big\{ \{(0,1),(1,1)\},\{(0,2),(0,3)\},\{(0,4),(1,2)\},\{(0,5),(1,3)\} \Big\}$$

Temperley-Lieb Diagrams

What we see:





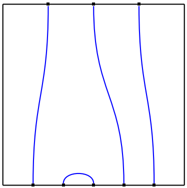
What we mean:

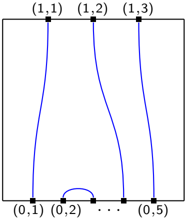
$$\Big\{ \{(0,1),(1,1)\}, \{(0,2),(0,3)\}, \{(0,4),(1,2)\}, \{(0,5),(1,3)\} \Big\}$$

What is a crossing?

Temperley-Lieb Diagrams

What we see:





What we mean:

$$\Big\{ \{(0,1),(1,1)\},\{(0,2),(0,3)\},\{(0,4),(1,2)\},\{(0,5),(1,3)\} \Big\}$$

What is a crossing? Order the vertices AC starting from $(0,1)$ as the minimum. Then $\{v,v'\}$ crosses $\{u,u'\}$ if $v \prec u \prec v' \prec u'$.

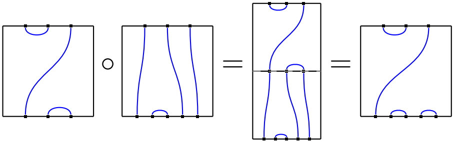
Temperley-Lieb Diagrams: Composition

Temperley-Lieb Diagrams: Composition

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \rightarrow \mathrm{Hom}(n, l)$ is defined on diagrams by vertically "stacking" $((\phi, \psi) \mapsto \psi \circ \phi)$:

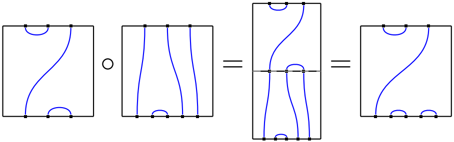
Temperley-Lieb Diagrams: Composition

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is defined on diagrams by vertically "stacking" $((\phi, \psi) \mapsto \psi \circ \phi)$:

$D_2 \circ D_1 =$  $= D_2 \# D_1.$

Temperley-Lieb Diagrams: Composition

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is defined on diagrams by vertically "stacking" $((\phi, \psi) \mapsto \psi \circ \phi)$:

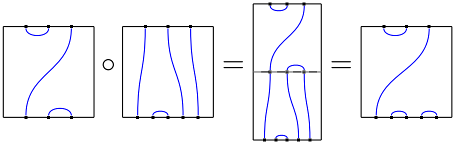
$D_2 \circ D_1 =$  $= D_2 \# D_1$.

What about

Temperley-Lieb Diagrams: Composition

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is defined on diagrams by vertically "stacking" $((\phi, \psi) \mapsto \psi \circ \phi)$:

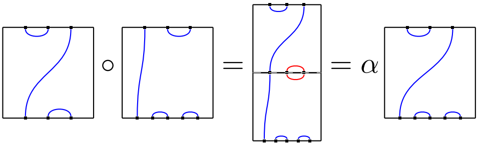
$D_2 \circ D_1 =$



$= D_2 \# D_1.$

What about

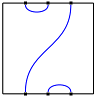
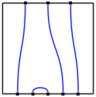
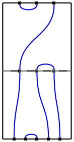
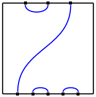
$D_2 \circ D_1' =$



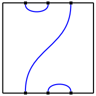
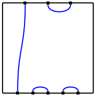
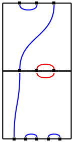
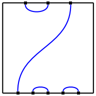
$= \alpha\, D_2 \# D_1'$

Temperley-Lieb Diagrams: Composition

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is defined on diagrams by vertically "stacking" $((\phi, \psi) \mapsto \psi \circ \phi)$:

$D_2 \circ D_1 =$  $\circ$  $=$  $=$  $= D_2 \# D_1.$

What about

$D_2 \circ D_1' =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1'$

Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)}\, D_2 \# D_1.$

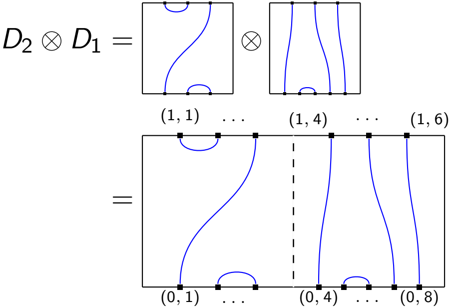
Temperley-Lieb Diagrams: Tensor Product

Temperley-Lieb Diagrams: Tensor Product

$\otimes : \mathrm{Hom}(n_1, m_1) \times \mathrm{Hom}(n_2, m_2) \to \mathrm{Hom}(n_1 \otimes n_2, m_1 \otimes m_2)$ is defined on diagrams by **horizontally** “stacking”:

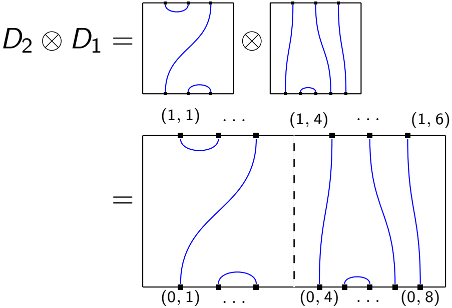
Temperley-Lieb Diagrams: Tensor Product

$\otimes : \text{Hom}(n_1, m_1) \times \text{Hom}(n_2, m_2) \to \text{Hom}(n_1 \otimes n_2, m_1 \otimes m_2)$ is defined on diagrams by **horizontally** "stacking":



Temperley-Lieb Diagrams: Tensor Product

$\otimes : \mathrm{Hom}(n_1, m_1) \times \mathrm{Hom}(n_2, m_2) \to \mathrm{Hom}(n_1 \otimes n_2, m_1 \otimes m_2)$ is defined on diagrams by **horizontally** "stacking":

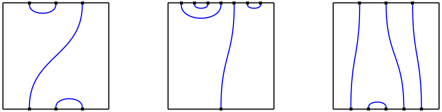


where $n_1 \otimes n_2 = n_1 + n_2$.

How to draw TL diagrams on surfaces?

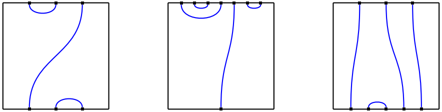
How to draw TL diagrams on surfaces?

TL-diagrams can be drawn on a square frame



How to draw TL diagrams on surfaces?

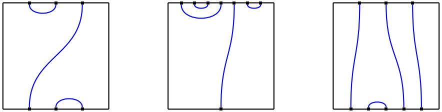
TL-diagrams can be drawn on a square frame



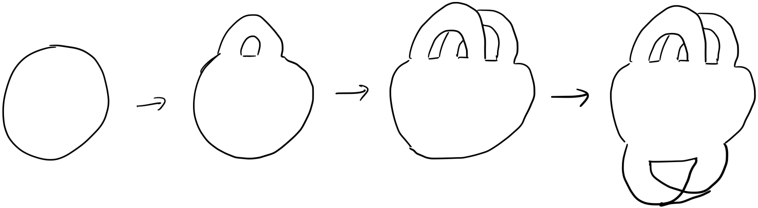
We would like to draw diagrams on different surfaces.

How to draw TL diagrams on surfaces?

TL-diagrams can be drawn on a square frame



We would like to draw diagrams on different surfaces. Use a "disc with bands" model for surfaces:



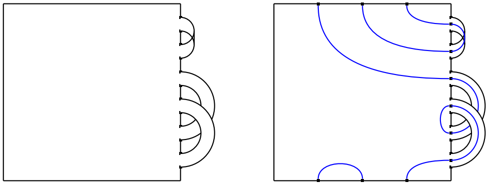
How to draw TL diagrams on surfaces?

How to draw TL diagrams on surfaces?

Marry square frame with this model - "square with bands" (SWB) diagrams:

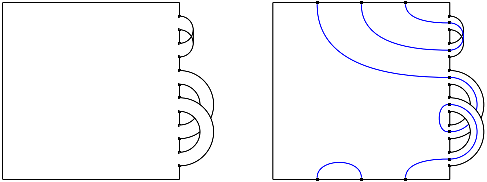
How to draw TL diagrams on surfaces?

Marry square frame with this model - "square with bands" (SWB) diagrams:



How to draw TL diagrams on surfaces?

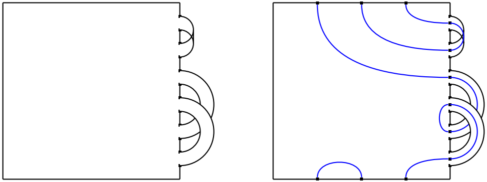
Marry square frame with this model - "square with bands" (SWB) diagrams:



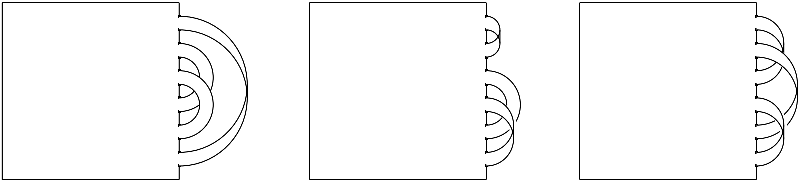
Trade-off: Non-unique way of representing surfaces!

How to draw TL diagrams on surfaces?

Marry square frame with this model - "square with bands" (SWB) diagrams:



Trade-off: Non-unique way of representing surfaces!



Twisted Chord Diagrams

Twisted Chord Diagrams

A **twisted chord diagram** (TCD) of rank N is a pair (P, s):

Twisted Chord Diagrams

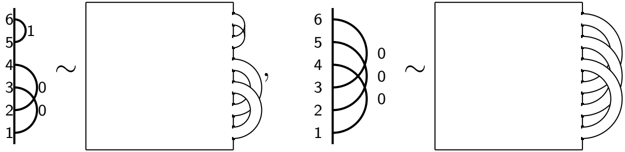
A **twisted chord diagram** (TCD) of rank N is a pair (P, s):

- P is a pair partition of the set $\{1, 2, \ldots, 2N\}$,

Twisted Chord Diagrams

A **twisted chord diagram** (TCD) of rank N is a pair (P, s):

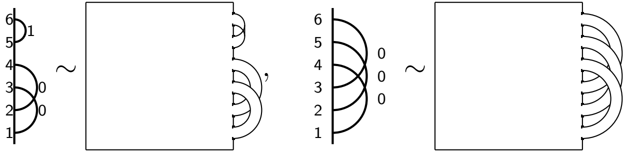
- P is a pair partition of the set $\{1, 2, \ldots, 2N\}$,
- s is a $\mathbb{Z}_2$ labelling $s : P \to \mathbb{Z}_2$.



Twisted Chord Diagrams

A **twisted chord diagram** (TCD) of rank N is a pair (P, s):

- P is a pair partition of the set $\{1, 2, \ldots, 2N\}$,
- s is a $\mathbb{Z}_2$ labelling $s : P \to \mathbb{Z}_2$.



A TCD is **orientable** if $s(P) = \{0\}$.

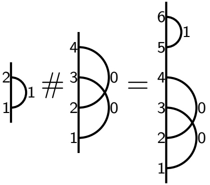
Twisted Chord Diagrmas

Twisted Chord Diagrmas

For two TCD $(P_1, s_1) \in \mathcal{TC}_{N_1}$ and $(P_2, s_2) \in \mathcal{TC}_{N_2}$, their vertical juxtaposition is a twisted chord diagram of rank $N_1 + N_2$:

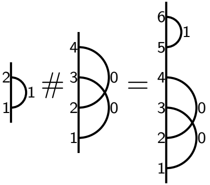
Twisted Chord Diagrmas

For two TCD $(P_1, s_1) \in \mathcal{TC}_{N_1}$ and $(P_2, s_2) \in \mathcal{TC}_{N_2}$, their vertical juxtaposition is a twisted chord diagram of rank $N_1 + N_2$:



Twisted Chord Diagrmas

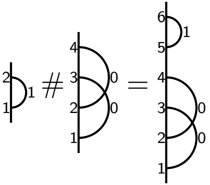
For two TCD $(P_1, s_1) \in \mathcal{TC}_{N_1}$ and $(P_2, s_2) \in \mathcal{TC}_{N_2}$, their vertical juxtaposition is a twisted chord diagram of rank $N_1 + N_2$:



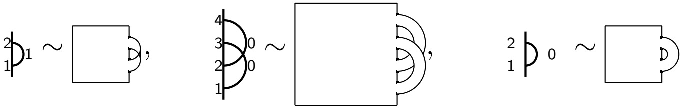
Small rank "prime" diagrams:

Twisted Chord Diagrmas

For two TCD $(P_1, s_1) \in \mathcal{TC}_{N_1}$ and $(P_2, s_2) \in \mathcal{TC}_{N_2}$, their vertical juxtaposition is a twisted chord diagram of rank $N_1 + N_2$:



Small rank "prime" diagrams:



Chordsliding

Chordsliding

SWB diagrams give non-unique realisation of surfaces...

Chordsliding

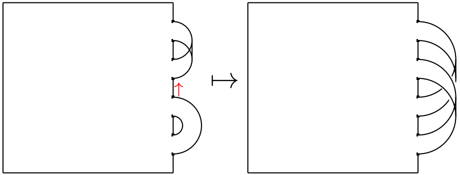
SWB diagrams give non-unique realisation of surfaces...

Idea: "Slide one end of a band"

Chordsliding

SWB diagrams give non-unique realisation of surfaces...

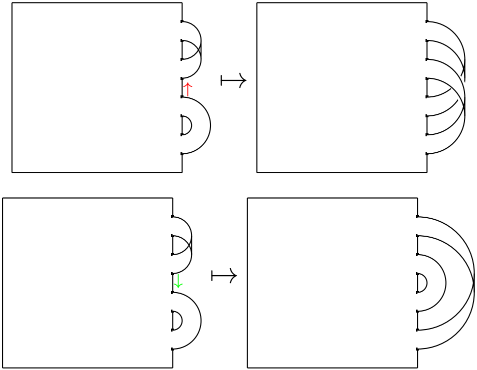
Idea: "Slide one end of a band"



Chordsliding

SWB diagrams give non-unique realisation of surfaces...

Idea: "Slide one end of a band"

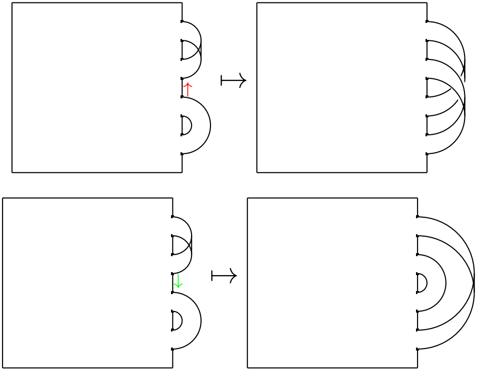


UNIVERSITY OF LEEDS

Chordsliding

SWB diagrams give non-unique realisation of surfaces...

Idea: "Slide one end of a band"



View this as a map $h_{(i,\pm 1)} : \mathcal{TC}_N \to \mathcal{TC}_N$,
$h_{(i,\pm 1)} : (P, s) \mapsto \left(\sigma(P), s' \circ \sigma^{-1}\right)$, $\sigma = \sigma_{(i \pm 1, P, s)} \in \mathrm{Sym}_{2N}$.

Chordsliding

Chordsliding

Can define an equivalence on $\mathcal{TC}$ by $(P, s) \sim (P', s')$ if (P', s') is obtained from (P, s) by a finite sequence of chordslides.

Chordsliding

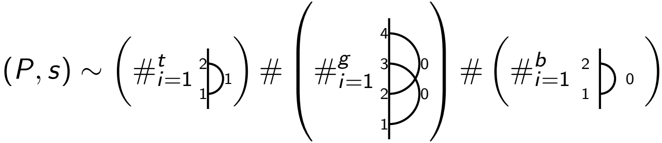
Can define an equivalence on $\mathcal{TC}$ by $(P, s) \sim (P', s')$ if (P', s') is obtained from (P, s) by a finite sequence of chordslides.

Fact: For any $(P, s) \in \mathcal{TC}_N$, there exist unique integers $g, b \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, 2\}$ such that

Chordsliding

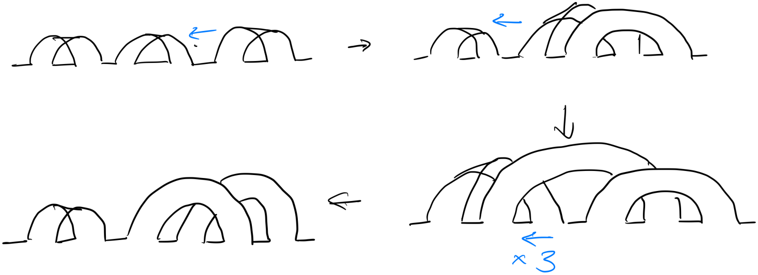
Can define an equivalence on $\mathcal{TC}$ by $(P, s) \sim (P', s')$ if (P', s') is obtained from (P, s) by a finite sequence of chordslides.

Fact: For any $(P, s) \in \mathcal{TC}_N$, there exist unique integers $g, b \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, 2\}$ such that



Three Twisted Bands to One

Three Twisted Bands to One

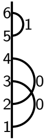


Intersection Matrix

Fact: For any $(P, s) \in \mathcal{TC}_N$, there exist unique integers $g, b \in \mathbb{Z}_{\geq 0}$ and $t \in \{0, 1, 2\}$ such that

$$(P, s) \sim (\#_{i=1}^{t}\text{Möb})\#(\#_{i=1}^{g}\text{Tor})\#(\#_{i=1}^{b}\text{Ann})$$

Uniqueness? **intersection matrix** $T(P, s) \in M_{N\times N}(\mathbb{Z}_2)$:



$$\mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $(P, s) \sim (P, s')$ then $T(P, s)$ and $T(P', s')$ are related by elementary RC op.s $\Rightarrow b = \text{Null}(T(P, s))$.
Let $\mathcal{TC}_N^* = \{(P, s) \in \mathcal{TC}_N \mid T(P, s) \text{ non-singular}\}$. Then $\# : \mathcal{TC}_{N_1}^* \times \mathcal{TC}_{N_2}^* \to \mathcal{TC}_{N_1+N_2}^*$

SWB diagrams

SWB diagrams

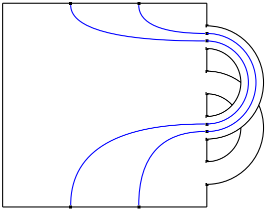
We have the "frames" for our diagrams: $(P, s) \in \mathcal{TC}_N^*$.

SWB diagrams

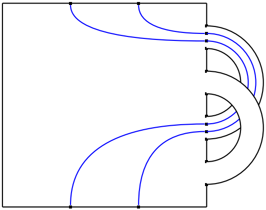
We have the "frames" for our diagrams: $(P, s) \in \mathcal{TC}_N^*$. How to **unambiguously** encode a diagram on this frame?

SWB diagrams

We have the "frames" for our diagrams: $(P, s) \in \mathcal{TC}_N^*$. How to **unambiguously** encode a diagram on this frame?

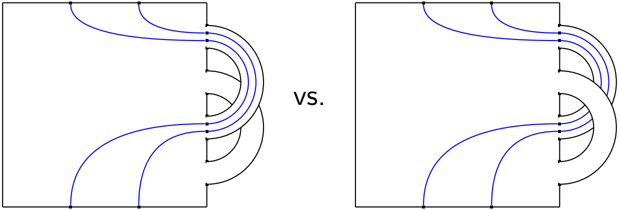


vs.



SWB diagrams

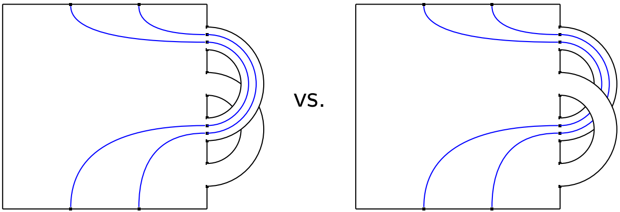
We have the “frames” for our diagrams: $(P, s) \in \mathcal{TC}^*_N$. How to **unambiguously** encode a diagram on this frame?



We record (P, s) for the surface,

SWB diagrams

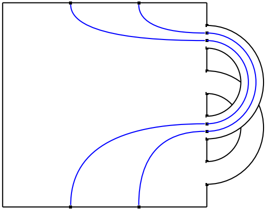
We have the "frames" for our diagrams: $(P, s) \in \mathcal{TC}_N^*$. How to **unambiguously** encode a diagram on this frame?



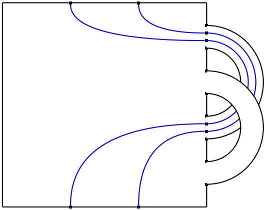
We record (P, s) for the surface, and $f : P \to \mathbb{Z}_{\geq 0}$ through lines for each band.

SWB diagrams

We have the "frames" for our diagrams: $(P, s) \in \mathcal{TC}_N^*$. How to **unambiguously** encode a diagram on this frame?



vs.



We record (P, s) for the surface, and $f : P \to \mathbb{Z}_{\geq 0}$ through lines for each band. Then we just need to record the crossingless pairing inside the square, E.

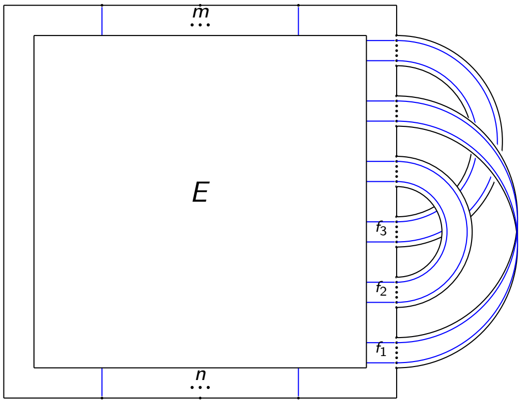
SWB diagrams

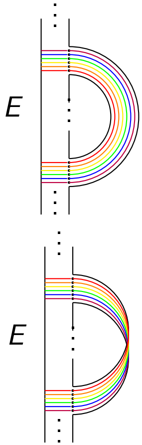
SWB diagrams

SWB diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$,

SWB diagrams

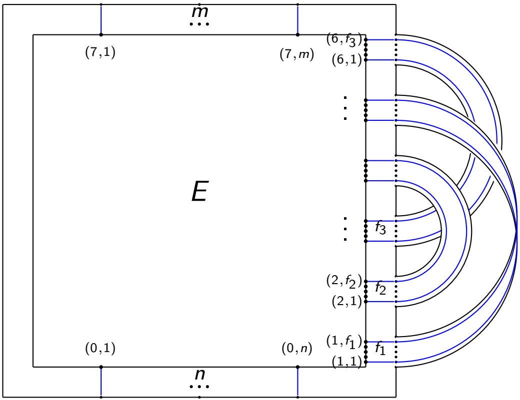
SWB diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$,

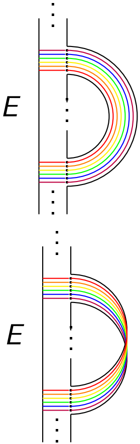




SWB diagrams

SWB diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$,





Can organise this as a graph $G(\Theta)$.

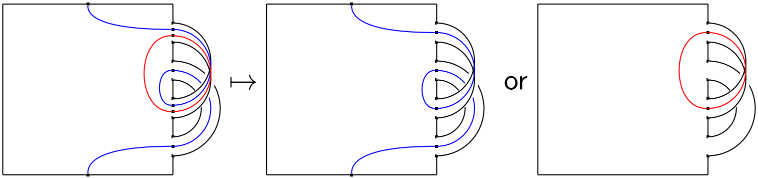
SWB diagrams

Organising SWB diagrams Θ as a graph $G(\Theta)$ means we can describe operations on Θ by its effect on $G(\Theta)$!

SWB diagrams

Organising SWB diagrams Θ as a graph $G(\Theta)$ means we can describe operations on Θ by its effect on $G(\Theta)$!

Example: We can "delete components" of $G(\Theta)$



SWB diagrams

SWB diagrams

Given a diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$ and some connected component $\Gamma \subset G(\Theta)$,

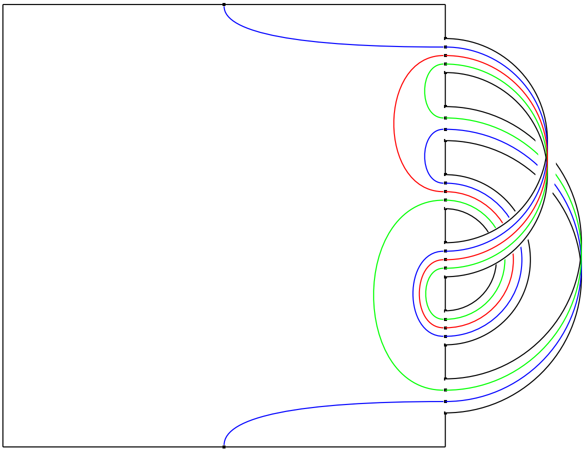
UNIVERSITY OF LEEDS

SWB diagrams

Given a diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$ and some connected component $\Gamma \subset G(\Theta)$, define the **twist** $\tau_\Gamma \in \mathbb{Z}_2$:

SWB diagrams

Given a diagram $\Theta = (P, s, f, E) \in Sq_N(n, m)$ and some connected component $\Gamma \subset G(\Theta)$, define the **twist** $\tau_\Gamma \in \mathbb{Z}_2$:



$\tau = 1,$
$\tau = 0,$
$\tau = 0.$

SWB diagrams - Vertical Juxtaposition

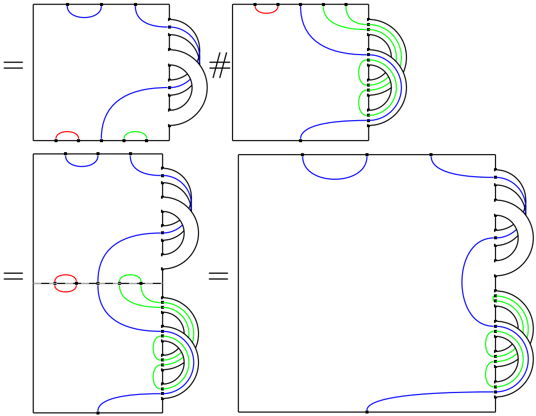
SWB diagrams - Vertical Juxtaposition

We want to vertically stack our diagrams:

SWB diagrams - Vertical Juxtaposition

We want to vertically stack our diagrams:

$$\Theta_2 \# \Theta_1 =$$



$L(\Theta_1, \Theta_2) = 1.$

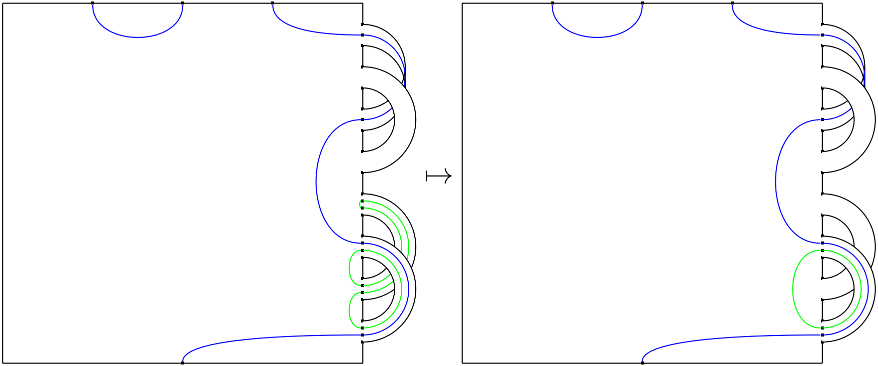
SWB diagrams - Isotopy

SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move:

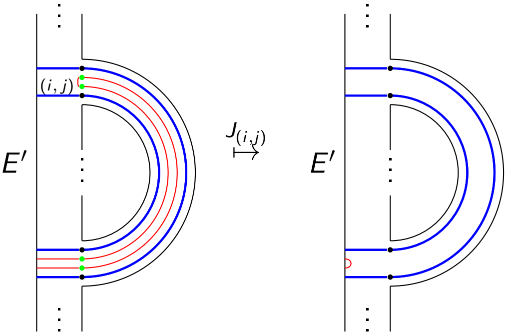
SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move:



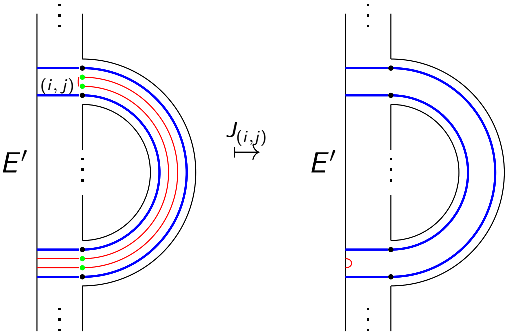
SWB diagrams - Isotopy

Generically, we can remove "turnbacks" by "pull-throughs"



SWB diagrams - Isotopy

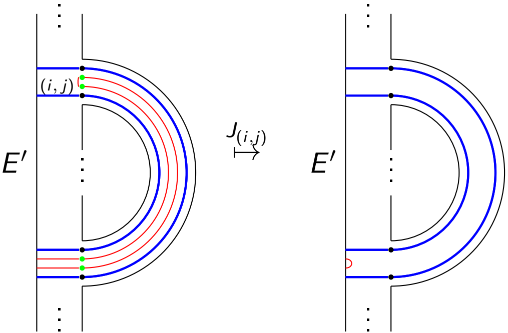
Generically, we can remove "turnbacks" by "pull-throughs"



$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', E'')$$

SWB diagrams - Isotopy

Generically, we can remove "turnbacks" by "pull-throughs"

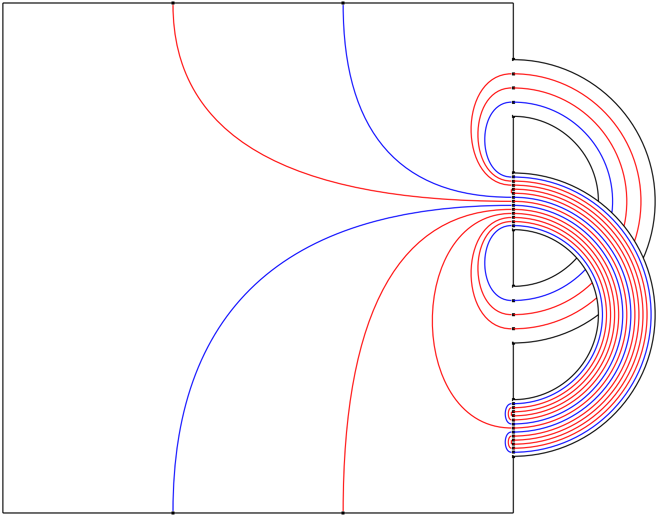


$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', E'')$$

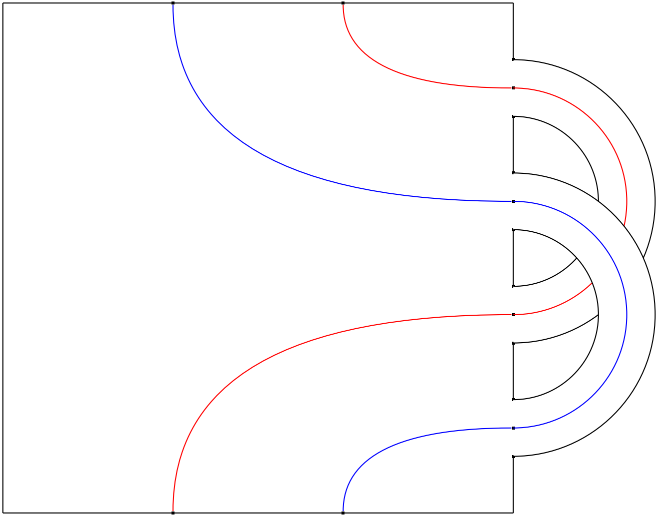
Can generate an equivalence relation with this move.

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks!

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks!



Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks!

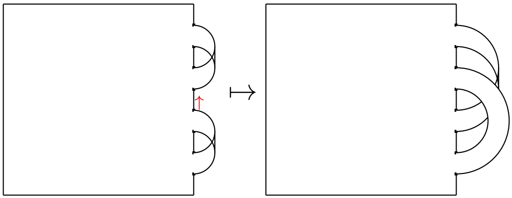


SWB diagrams - Handlesliding

We have the “chordsliding” equivalence move on our surfaces:

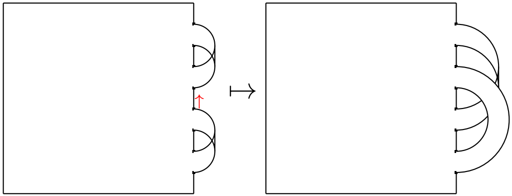
SWB diagrams - Handlesliding

We have the “chordsliding” equivalence move on our surfaces:



SWB diagrams - Handlesliding

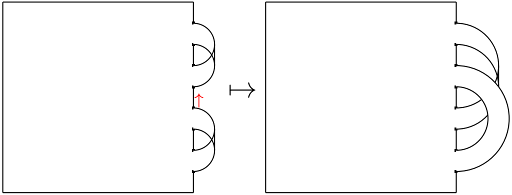
We have the “chordsliding” equivalence move on our surfaces:



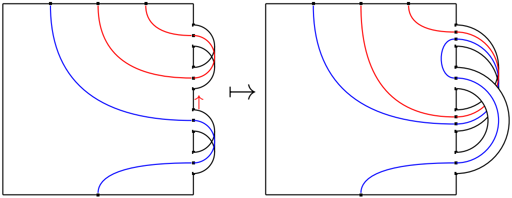
Now lets extend this to moves on our diagrams

SWB diagrams - Handlesliding

We have the "chordsliding" equivalence move on our surfaces:



Now lets extend this to moves on our diagrams

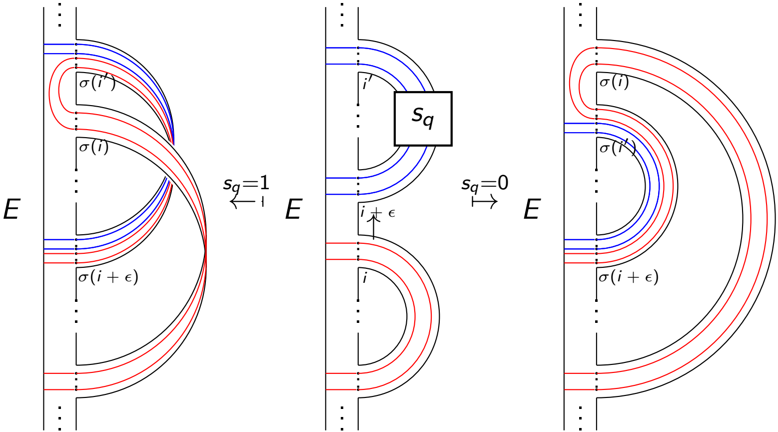


SWB diagrams - Handlesliding

Generically: “Two bands involved”

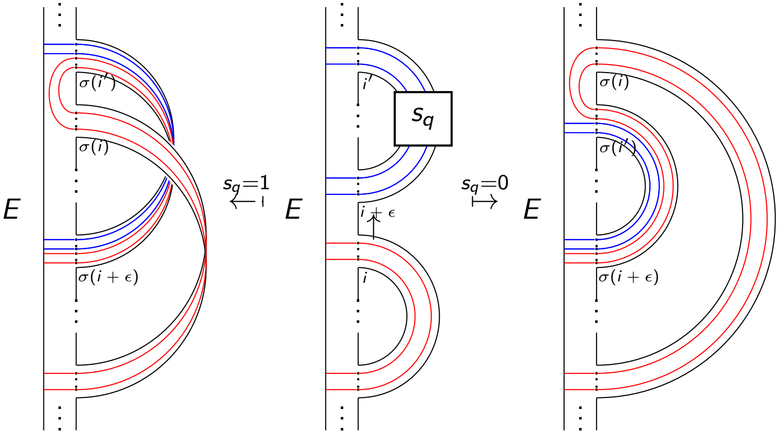
SWB diagrams - Handlesliding

Generically: "Two bands involved"



SWB diagrams - Handlesliding

Generically: "Two bands involved"



$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, E \cup \{\text{"new red arcs"}\})$$

SWB diagrams - Handleslide Equivalence

SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P', s') by a finite sequence of chordslides.

SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P', s') by a finite sequence of chordslides.

What about on our diagrams? Suppose we define a relation by $\Theta \sim \Theta'$ if Θ' can be obtained from Θ by a finite sequence of handleslides;

SWB diagrams - Handleslide Equivalence

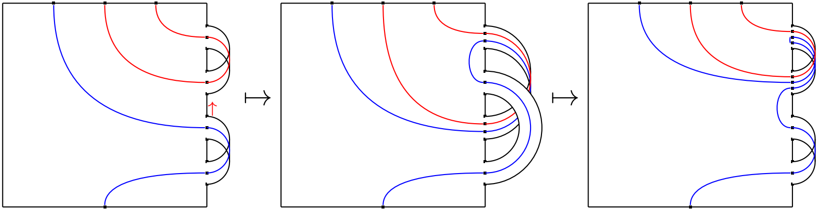
On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P', s') by a finite sequence of chordslides.

What about on our diagrams? Suppose we define a relation by $\Theta \sim \Theta'$ if Θ' can be obtained from Θ by a finite sequence of handleslides; this only defines an equivalence relation on **isotopy classes**!! - call this **Handleslide (HS) Equivalence**.

SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P', s') by a finite sequence of chordslides.

What about on our diagrams? Suppose we define a relation by $\Theta \sim \Theta'$ if Θ' can be obtained from Θ by a finite sequence of handleslides; this only defines an equivalence relation on **isotopy classes**!! - call this **Handleslide (HS) Equivalence**.



The Category $\mathcal{SQ}$

Let R be a unital commutative ring with $\alpha, \beta \in R$.

The Category $\mathcal{SQ}$

Let R be a unital commutative ring with $\alpha, \beta \in R$. The category $\mathcal{SQ}(\alpha, \beta)$ is defined as the R-linear category with:

The Category $\mathcal{SQ}$

Let R be a unital commutative ring with $\alpha, \beta \in R$. The category $\mathcal{SQ}(\alpha, \beta)$ is defined as the R-linear category with:

- Objects: Non-negative integers

The Category $\mathcal{SQ}$

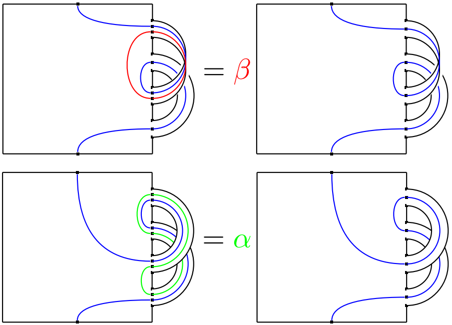
Let R be a unital commutative ring with $\alpha, \beta \in R$. The category $\mathcal{SQ}(\alpha, \beta)$ is defined as the R-linear category with:

- Objects: Non-negative integers
- Morphisms: $\mathrm{Hom}(n, m) = \{0\}$ if $n + m = 1 \mod 2$, and otherwise it is R-linear combinations of HS equivalence classes of SWB diagrams, $[\Theta]_{\mathrm{HS}}$,

The Category $\mathcal{SQ}$

Let R be a unital commutative ring with $\alpha, \beta \in R$. The category $\mathcal{SQ}(\alpha, \beta)$ is defined as the R-linear category with:

- Objects: Non-negative integers
- Morphisms: $\mathrm{Hom}(n, m) = \{0\}$ if $n + m = 1 \mod 2$, and otherwise it is R-linear combinations of HS equivalence classes of SWB diagrams, $[\Theta]_{\mathrm{HS}}$, **Modulo** the "delooping" relations:



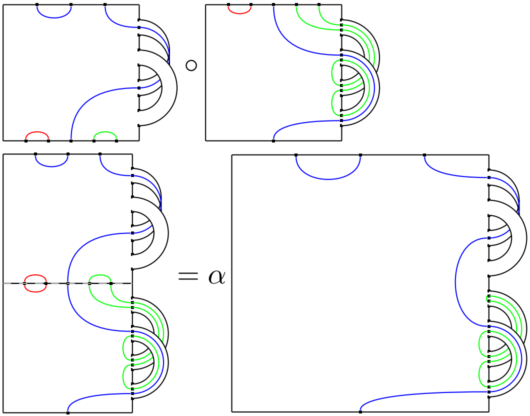
The Category $\mathcal{SQ}$

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is given by $\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \, \overline{\Theta_2 \# \Theta_1}$:

The Category $\mathcal{SQ}$

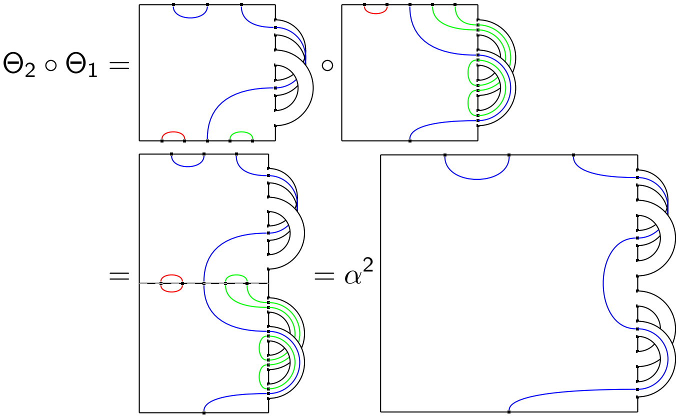
Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is given by $\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)}\, \overline{\Theta_2 \# \Theta_1}$:

$\overline{\Theta_2} \circ \overline{\Theta_1} = \ \circ \ = \ = \alpha$



The Category $\mathcal{SQ}$

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is given by $\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}$:



The Category $\mathcal{SQ}$ - Basic Facts

The Category $\mathcal{SQ}$ - Basic Facts

Fact 1: For any $\Theta \in Sq(n,m)$, there exist **unique** integers l_u and l_t such that:

$$\overline{\Theta} = \alpha^{l_u}\beta^{l_t}\,\overline{\Theta'} \in \mathrm{Hom}(n,m),$$

where $\Theta' \in Sq(n,m)$ has no loops.

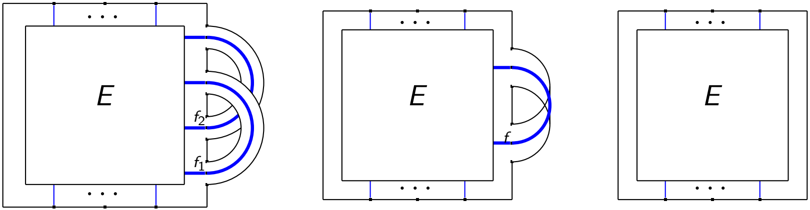
The Category $\mathcal{SQ}$ - Basic Facts

Fact 1: For any $\Theta \in Sq(n,m)$, there exist **unique** integers l_u and l_t such that:

$$\overline{\Theta} = \alpha^{l_u}\beta^{l_t}\,\overline{\Theta'} \in \mathrm{Hom}(n,m),$$

where $\Theta' \in Sq(n,m)$ has no loops.

Fact 2: Any morphism $\overline{\Theta} \in \mathrm{Hom}(n,m)$ has a factorisation in terms of diagrams of the following form

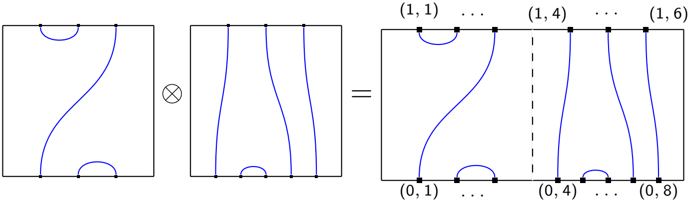


The Category $\mathcal{SQ}$ - Tensor Product

Recall: In the TL case we had a tensor product given by $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:

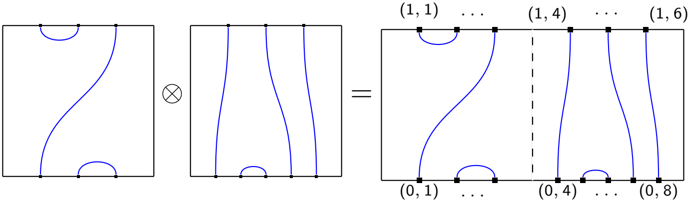
The Category $\mathcal{SQ}$ - Tensor Product

Recall: In the TL case we had a tensor product given by $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:



The Category $\mathcal{SQ}$ - Tensor Product

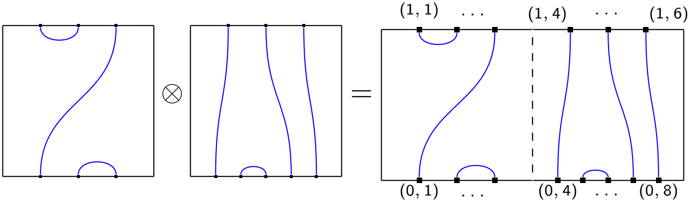
Recall: In the TL case we had a tensor product given by $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:



Can we extend this to a tensor product on $\mathcal{SQ}$:

The Category $\mathcal{SQ}$ - Tensor Product

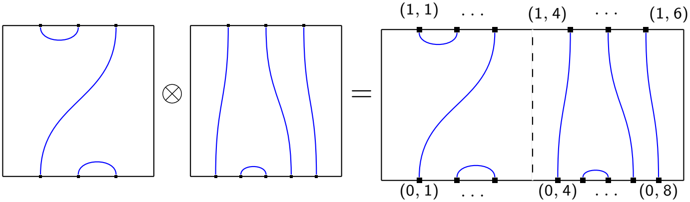
Recall: In the TL case we had a tensor product given by $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:



Can we extend this to a tensor product on $\mathcal{SQ}$: On objects $n_1 \otimes n_2 = n_1 + n_2$.

The Category $\mathcal{SQ}$ - Tensor Product

Recall: In the TL case we had a tensor product given by $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:



Can we extend this to a tensor product on $\mathcal{SQ}$: On objects $n_1 \otimes n_2 = n_1 + n_2$. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

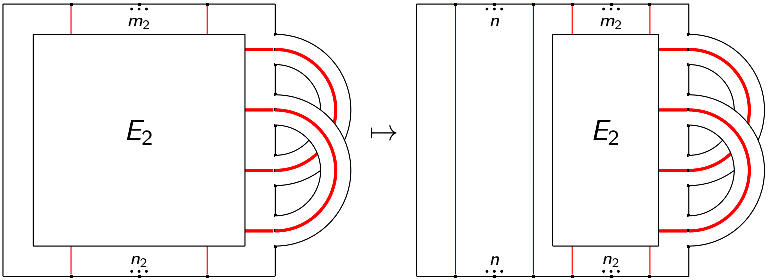
The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:

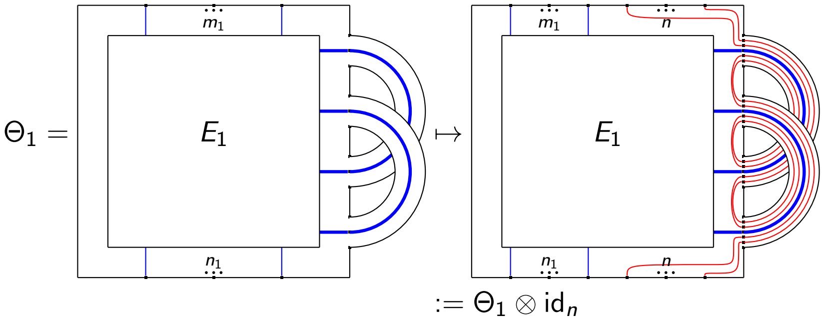
$$\Theta_2 = \quad \mapsto$$



$$:= \mathrm{id}_n \otimes \Theta_2$$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:



The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:

The Category $\mathcal{SQ}$ - Tensor Product

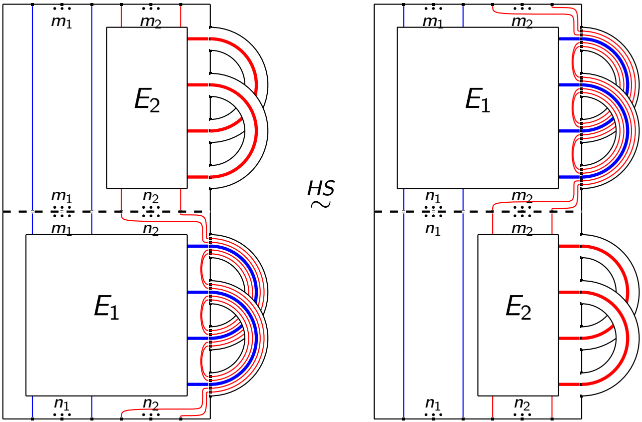
Indirect answer: Step 3 - Insist upon functoriality:

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\mathrm{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \mathrm{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \mathrm{id}_{m_2})} \circ \overline{(\mathrm{id}_{n_1} \otimes \Theta_2)}$$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\mathrm{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \mathrm{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \mathrm{id}_{m_2})} \circ \overline{(\mathrm{id}_{n_1} \otimes \Theta_2)}$$



Outlook

Some directions for future work:

Outlook

Some directions for future work:

- Construct “finite-dimensional” quotients via the finitising scheme

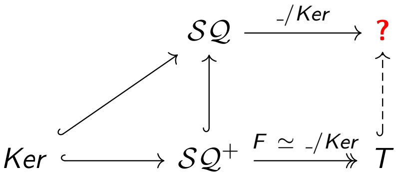
$$\begin{array}{ccccc} & & \mathcal{SQ} & \xrightarrow{\_/Ker} & \color{red}{\textbf{?}} \\ & \nearrow & \uparrow & & \uparrow \\ Ker & \hookrightarrow & \mathcal{SQ}^{+} & \xrightarrow{F \,\simeq\, \_/Ker} & T \end{array}$$

… Topological invariants?

Outlook

Some directions for future work:

- Construct "finite-dimensional" quotients via the finitising scheme



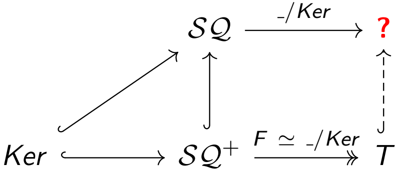
... Topological invariants?

- Computational questions: algorithms/presentations?

Outlook

Some directions for future work:

- Construct "finite-dimensional" quotients via the finitising scheme



... Topological invariants?

- Computational questions: algorithms/presentations?
- Compare with a more abstract/geometric construction: connect handlesliding with known presentation of mapping class groups.

THANK YOU

Questions?