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Generalised roots of unity: interpolating between Temperley-Lieb and Brauer representation theory

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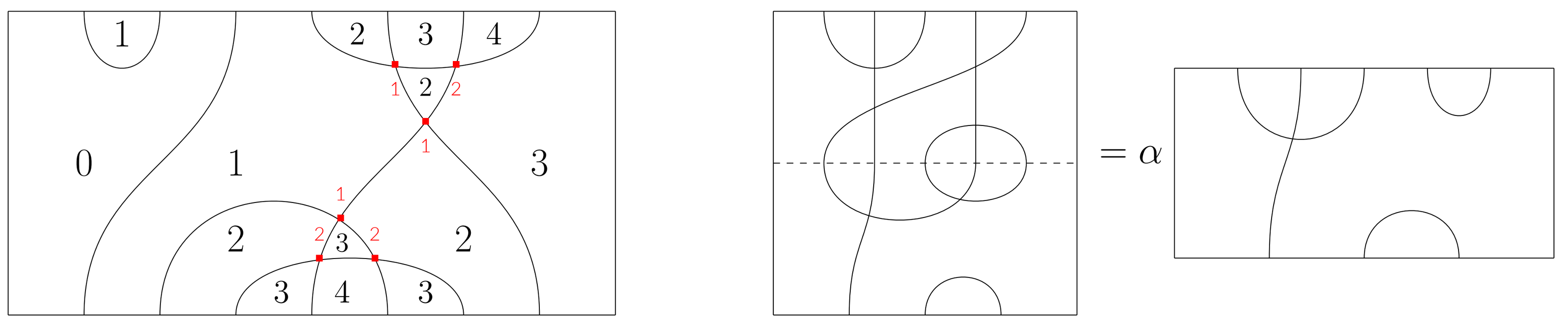
The Kadar-Yu Categories/Algebras

In [3], Kadar, Martin, and Yu, defined the notion of (left) **height** for a Brauer "diagram" as follows: we label the alcoves of a diagram with a height as below, and the height of the diagram is the maximum over all crossings (or -1 is no crossings), *e.g.* the diagram below left has height 2.



Note that height cannot increase under the usual Brauer composition, thus we define:

<u>Definition:</u> ($\alpha \in \mathbb{K}$) For each $l \in \{-1, 0, 1, \dots\}$ The **Kadar-Martin-Yu** (KMY) category, $J_l = J_l(\alpha)$, is the wide subcategory of $Br(\alpha)$ with hom spaces spanned by

$$J_l(n,m) = \{\, p \ \text{ a Brauer "diagram" of type } (n,m) \mid ht(p) \le l \,\}. \qquad \square$$

In particular, we study the KMY-algebras $J_{l,n} = J_l(n,n)$. These form a family of nested subalgebras **interpolating between the Temperley-Lieb (TL) and Brauer algebras**:

$$TL_n = J_{-1,n} \subset J_{0,n} \subset J_{1,n} \subset \cdots \subset J_{n-3,n} \subset J_{n-2,n} = J_{\infty,n} = Br_n.$$

A useful characterisation of $J_{l,n}$ from [1] is as follows:

<u>Proposition:</u> The algebra $J_{l,n}$ is generated by the following TL and symmetric group generators

(cup-cap) e_i for $i = 1, \dots, n-1$, (transposition) s_j for $j = 1, \dots, \min(n-1, l+1)$. $\square$

Simple/Standard Modules

We can study the using the Tower of Recollment (ToR) framework [2]. This leverages two adjunctions:

$$M \longmapsto J_l(n+2,n) \otimes_{J_{l,n}} M$$

$$\begin{tikzcd} \mathrm{mod}(J_{l,n}) \arrow[r, bend left, "G"] & \mathrm{mod}(J_{l,n+2}) \arrow[l, bend left, "F"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{mod}(J_{l,n}) \arrow[r, bend left, "\mathrm{Ind}"] & \mathrm{mod}(J_{l,n+1}) \arrow[l, bend left, "\mathrm{Res}"] \end{tikzcd}$$

$$J_l(n, n+2) \otimes_{J_{l,n+2}} N \longleftarrow N$$

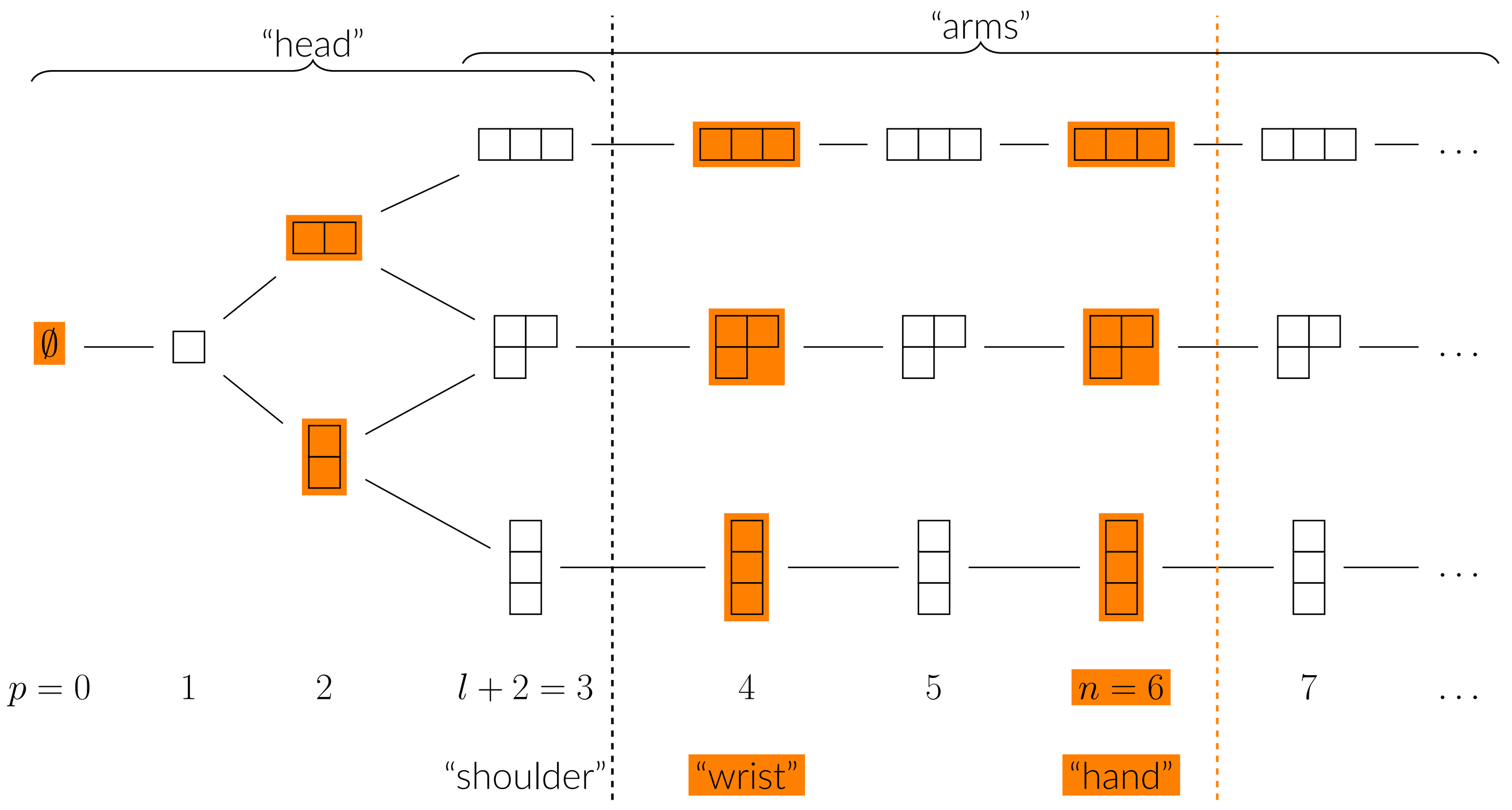
Since G is an embedding, we can compute a labelling set for simple modules inductively

$$\begin{aligned} \Lambda_{l,n} &= \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_l(n,n-2) \circ J_l(n-2,n)\right) \\ &= \Lambda_{l,n-2} \sqcup \Lambda(\mathbb{K}\, S_{\min(n,l+2)}) = \Lambda_{l,n-2} \sqcup \{\lambda \vdash \min(n, l+2)\}, \\ \Rightarrow \quad \Lambda_{l,n} &= \{(p,\lambda) \mid p = n, n-2, \dots, 0/1, \quad \lambda \vdash \min(p, l+2)\} \end{aligned}$$

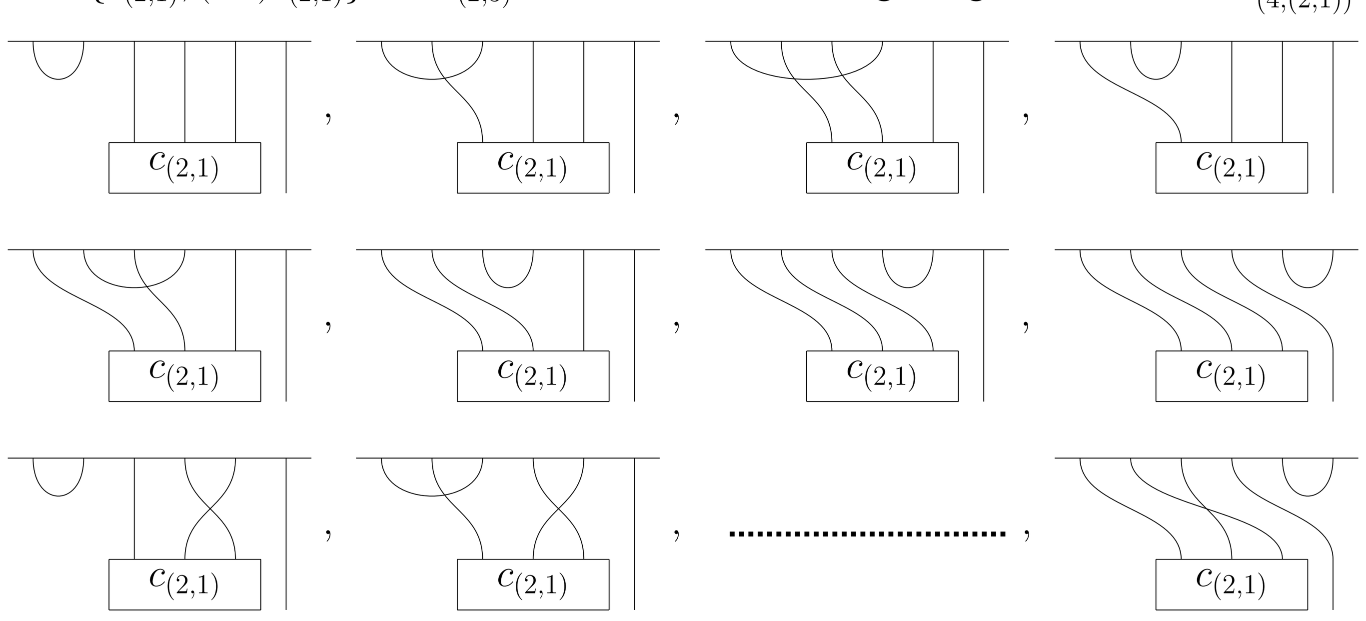
For each label $(p,\lambda) \in \Lambda_{l,n}$ we construct a **standard module** $\Delta^{(n)}_{(p,\lambda)}$ from the Specht module $\Delta^{(n=p)}_{(p,\lambda)} \simeq \mathcal{S}_\lambda$ using Globalisation

$$\Delta^{(n)}_{(p,\lambda)} := G^{(n-p)/2}\left(\Delta^{(p)}_{(p,\lambda)}\right).$$

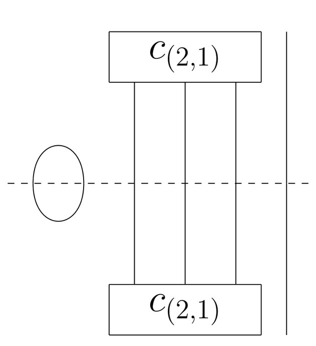
The family of module categories mod($J_{l,n}$) can be encoded in a **Rollet** diagram (projection of a Bratelli diagram) which encodes branching rules. *E.g.* for $l = 1$ (simple labels for $J_{1,6}$ highlighted in orange):



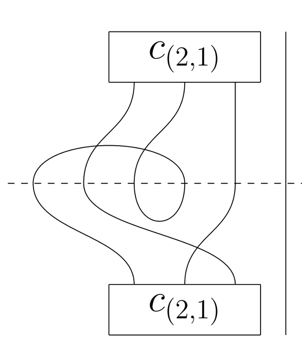
We can give a basis for each $\Delta^{(n)}_{(p,\lambda)}$ by gluing together a basis for $\mathcal{S}_\lambda$ and "half-diagrams" in $J_l(n,p)$, *e.g.* using the basis $\{c_{(2,1)}, (2\,3)c_{(2,1)}\}$ for $\mathcal{S}_{(2,3)}$ we have the following "diagram basis" for $\Delta^{(6)}_{(4,(2,1))}$:



We can determine the Cartan invariants $[L_{(p,\lambda)} : P_{(p,\lambda)}]$ if we know all maps between standards. In principal, the ToR framework reduces this to computing maps from the "hand" to "wrist". The existence of such maps is indicated by the degeneracy of a certain "diagrammatic" contravariant form on $\Delta^{(n)}_{(n-2,\lambda)}$:



$$= \alpha\, c^{(4)}_{(2,1)}\, id_4\, c^{(4)}_{(2,1)} \qquad = \alpha\, c^{(4)}_{(2,1)}$$



$$= c^{(4)}_{(2,1)}\,(1\;2\;3)\, c^{(4)}_{(2,1)} \qquad = -\frac{1}{2}\, c^{(4)}_{(2,1)}$$

Semisimplicity Criteria

Now fixing $\mathbb{K} = \mathbb{C}$, we determine (some) values of α for which one has hand to wrist maps; in particular, for such α the KMY algebras are not semisimple. (It has been shown $\alpha \in \mathbb{R}$ is necessary [1, 3]).

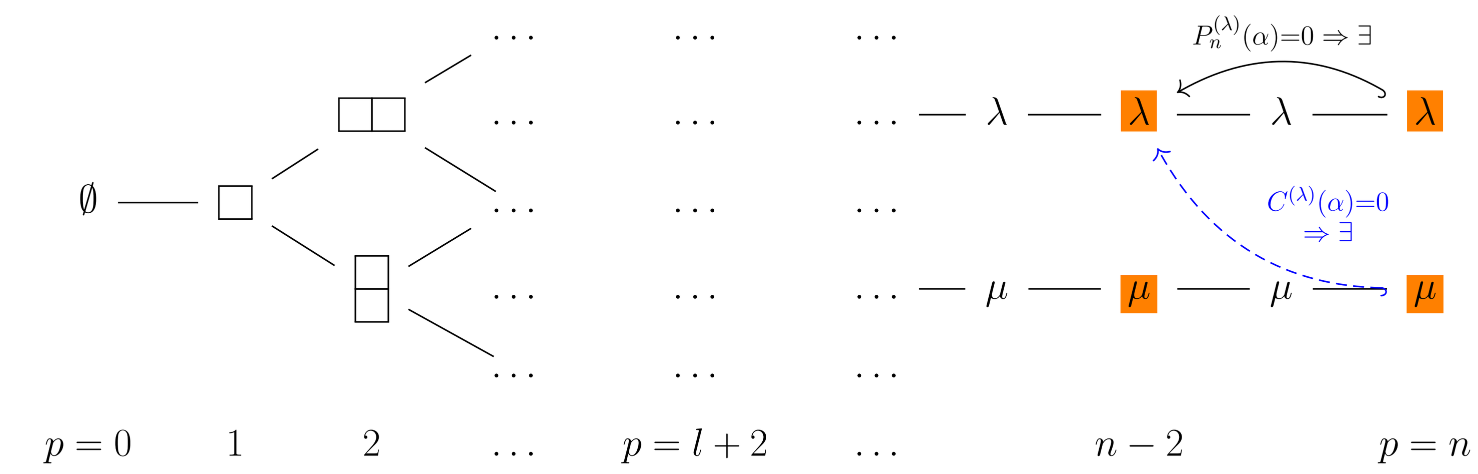
<u>Theorem [Martin-M., 25]:</u> For each $\lambda \vdash l+2$ and $n \ge l+4$ (*i.e.* in the arms of the Rollet diagram), there exists a polynomial $C^{(\lambda)} \in \mathbb{C}[\alpha]$, and a series of polynomials $P^{(\lambda)}_n \in \mathbb{C}[\alpha]$ satisfying the **Chebyshev recursion:**

$$P^{(\lambda)}_{n+1}(\alpha) = \alpha P^{(\lambda)}_n(\alpha) - P^{(\lambda)}_{n-1},$$

such that the determinant of the contravariant form on $\Delta^{(n)}_{(n-2,\lambda)}$ has the form:

$$\det\left(\Gamma(\Delta^{(n)}_{(n-2,\lambda)})\right) = C^{(\lambda)}(\alpha)\left(P^{(\lambda)}_n(\alpha)\right)^{d_\lambda}, \quad \text{where} \quad d_\lambda := \dim(\mathcal{S}_\lambda).$$

Furthermore, when $P^{(\lambda)}_n(\alpha) = 0$, there exists a map $\Delta^{(n)}_{(n-2,\lambda)} \hookrightarrow \Delta^{(n)}_{(n,\lambda)}$. $\square$

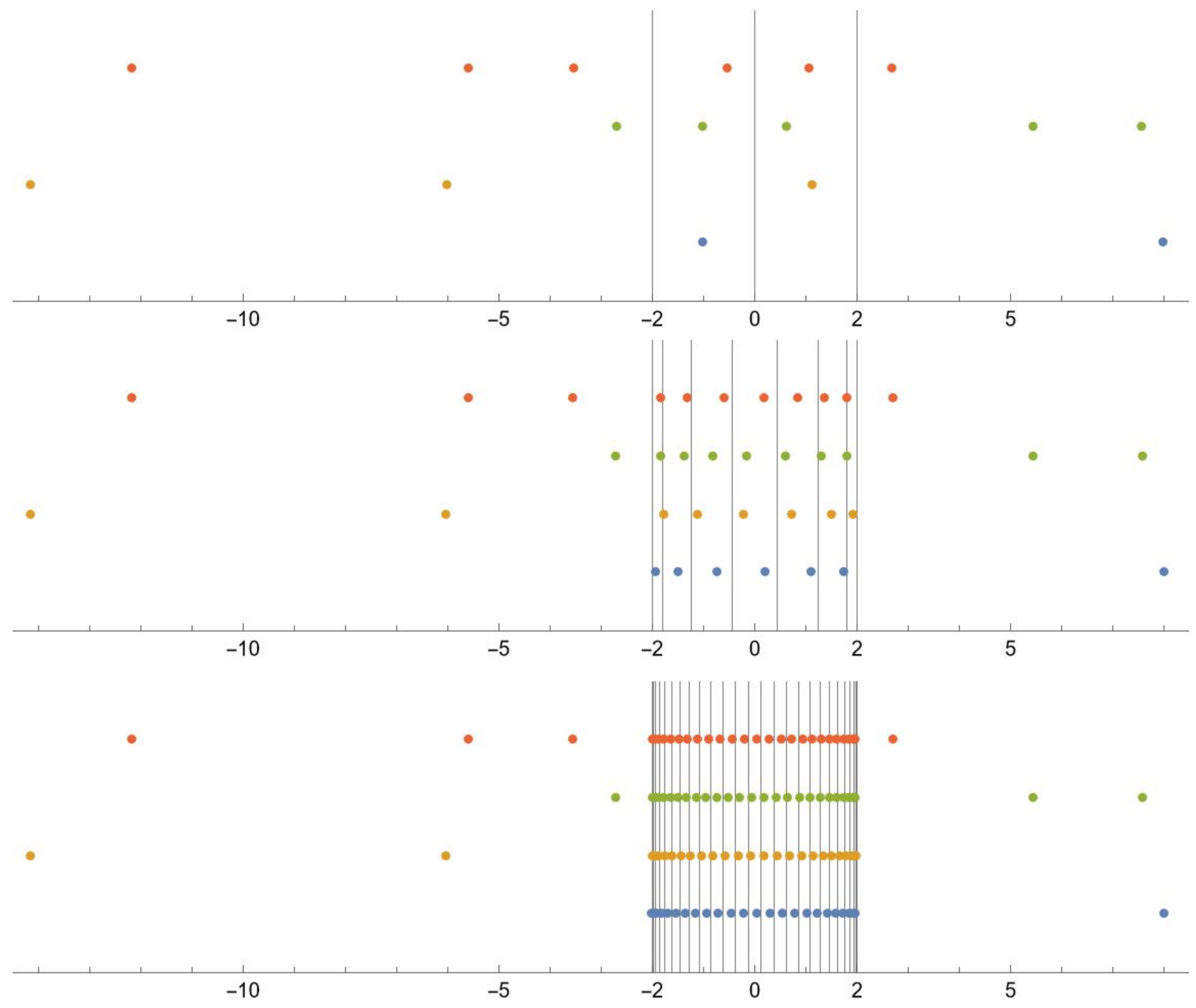


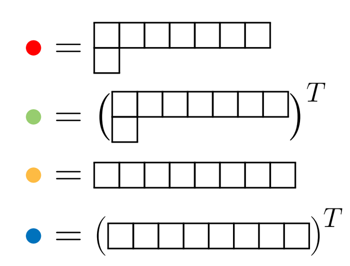
<u>Example:</u> When $l = -1$, we recover the classical "root of unity criteria" for the TL algebras. When $l = -1$ we have $J_{-1,n} = TL_n$, $\lambda = \square$, $d_\lambda = 1$, and $C^{(\lambda)}(\alpha) = 1$. Furthermore, $P^{(\lambda)}_n(\alpha)$ is the classical Chebyshev series:

$$P^{(\lambda)}_3(\alpha) = (\alpha-1)(\alpha+1), \quad P^{(\lambda)}_4(\alpha) = \alpha\left(\alpha^2-2\right), \quad P^{(\lambda)}_5(\alpha) = \left(\alpha^2-\alpha-1\right)\left(\alpha^2+\alpha-1\right)$$

with roots $\alpha = 2\cos(k\pi/n)$ for $k = 1, \dots, n-1$ (*i.e.* $q^n = 1$ when we parameterise as $\alpha = q + q^{-1}$).

<u>Example:</u> Some plots of the roots of the polynomials $P^{(\lambda)}_n$ (varying n) are given below in height $l = 6$. As n grows large, most roots "resemble" the classical values (indicated by the grey lines).



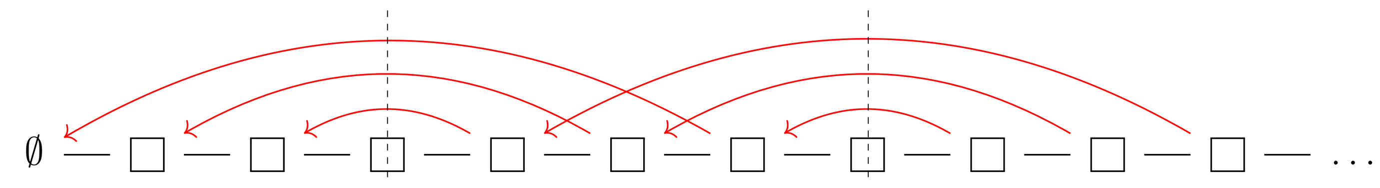


Scan for animation!

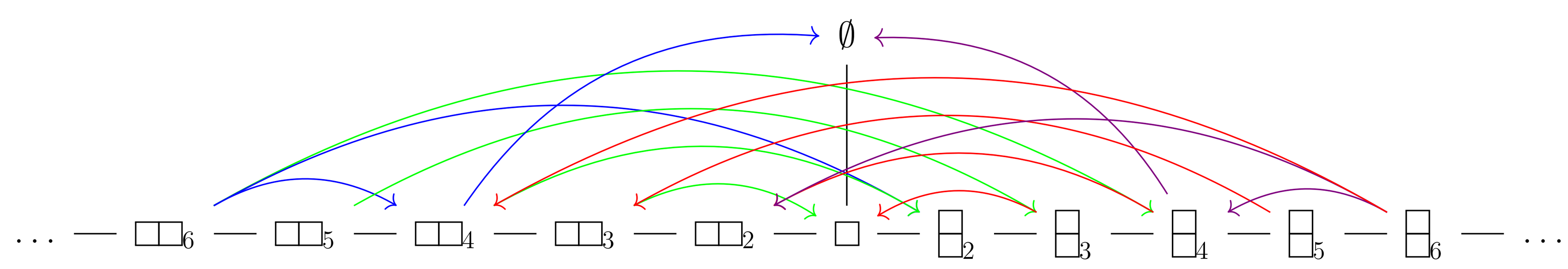
<u>Conjecture:</u> All roots of $C^{(\lambda)} \in \mathbb{C}[\alpha]$ are integers and correspond to maps of the form $\Delta^{(n)}_{(n,\mu)} \hookrightarrow \Delta^{(n)}_{(n-2,\lambda)}$.

Alcove Geometry?

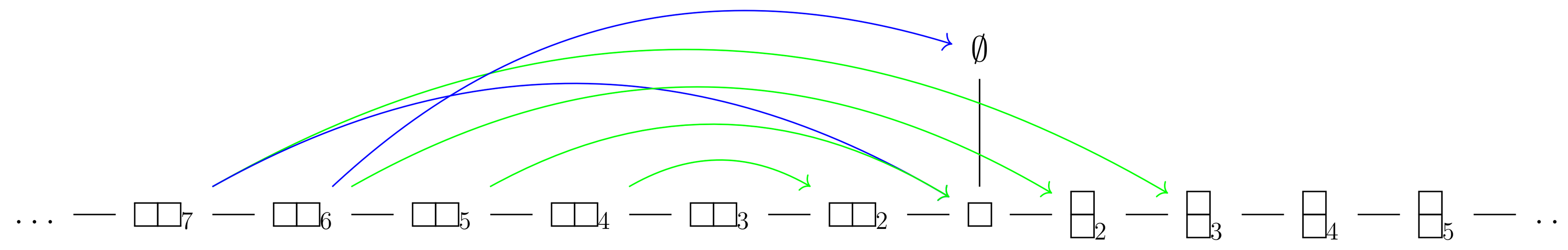
For TL and Brauer algebras, maps between standard modules can be understood by affine reflections, *e.g.* for TL when $P^{(1)}_4(\alpha) = 0$ maps between standards can be encoded on the Rollet diagram as:



For $l = 0$, when $\alpha = 1$ we have $C^{(1^2)}(\alpha) = C^{(2)}(\alpha) = 0 = P^{(1^2)}_{3k}(\alpha) = P^{(2)}_{3k}(\alpha)$ for $k \ge 1$. This primes the ToR with certain "hand to wrist" maps. We can then compute further maps using branching rules, *e.g.*:



Similarly, when $\alpha^2 + \alpha - 4 = 0$ we have $P^{(2)}_4(\alpha) = 0$, which primes the ToR with a map $\Delta^{(4)}_{(4,(2))} \hookrightarrow \Delta^{(4)}_{(2,(2))}$.



References

[1] Nouf Alraddadi and Alison Parker. *On semi-simplicity of KMY algebras*. 2024. arXiv: 2407.07028 [math.RT].

[2] Anton Cox et al. "Representation theory of towers of recollement: Theory, notes, and examples". In: *Journal of Algebra* 302.1 (Aug. 2006), pp. 340–360.

[3] Z. Kádár, P. P. Martin, and S. Yu. "On geometrically defined extensions of the Temperley–Lieb category in the Brauer category". In: *Math. Z.* 293.3-4 (Mar. 2019), pp. 1247–1276.