Q-Operators in Integrable Systems

Benjamin Morris

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Abstract

In this report we study the six-vertex model and its higher spin generalisations. We show how Baxter's Q-operator approach allows for the spectrum of the transfer matrix to be found for the standard case and for the higher spin generalisation. We examine two particular constructions of such Q-operators from [7] and [8] as well as the relationship between the two constructions. We also introduce a method, based on this relationship, of constructing the operator from [7] using a *composite* local L-operator in the hope of finding a particular limit of this operator, although this approach has not been fully developed yet.

1 Introduction

In this report we study the basics of integrable systems with a particular focus on the six-vertex model and its higher spin generalisations. Although a precise definition of integrability in a system does not exist Hitchin suggests three common features of integrable systems are the existence of a large number of conserved quantities, explicitly forms for solutions, and the presence of algebraic geometry [1].

The six vertex model is a famous lattice model introduced as an analogue for hydrogen bonding in ice by Pauling in 1935 [2]. It was solved for three specific cases by Lieb in 1967 using a Bethe Ansatz and then more generally by Sutherland in the same year [3,4]. The model is a square lattice model with M rows and N columns. It has toroidal boundary conditions meaning that the $(N+k)_{\text{th}}$ site on any row is the same as the k_{th} site and the $(M+l)_{\text{th}}$ site on any column is the same as the l_{th} site. It is obviously then most convenient to take $1 \leq k \leq N$ and $1 \leq l \leq M$.

At each vertex we imagine an oxygen atom and between any two adjacent atoms is a hydrogen ion which is closer to one of the atoms. This can be neatly represented by arrows; an arrow pointing towards a vertex represents a ion closer to that atom. Slater proposed that ions should satisfy the *ice rule* meaning each atom has exactly two ions closer to it than to adjacent atoms and two ions closer to adjacent atoms [5]. This gives rise to six possible arrangements at each vertex which are shown below in arrow form.

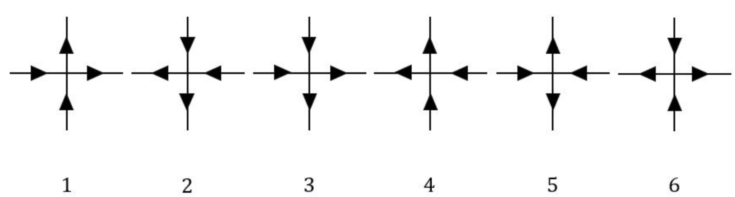

Figure 1: Vertex Arrangements

In the most general form each possible arrangement has its own distinct energy ε_i $(i = 1, \ldots, 6)$. We then form the partition function as

$$Z = \sum_s e^{-(n_1\varepsilon_1 + \cdots + n_6\varepsilon_6)/k_B T}, \tag{1.1}$$

where the sum is over all possible arrangements of vertices in the lattice, s, and n_i is the number of vertices of type $i = 1, \ldots, 6$ in an arrangemen, s t. Three particular cases of interest are the ice model where the energy of each lattice is independent of its arrangement so we choose $\varepsilon_1 = \cdots = \varepsilon_6 = 0$ for simplicity, the ferroelectric model where we choose $\varepsilon_1 = \varepsilon_2 = 0,\ \varepsilon_3 = \cdots = \varepsilon_6 > 0$ hence the ground state consists of vertices which are all in state 1 or all in state 2, and the anti-ferroelectric model where we choose $\varepsilon_1 = \cdots = \varepsilon_4 > 0,\ \varepsilon_5 = \varepsilon_6 = 0$ hence the ground state consists of vertices in state 5 and 6 in an alternating pattern. For the remainder of this section we will use $\varepsilon_1 = \varepsilon_2,\ \varepsilon_3 = \varepsilon_4,\ \varepsilon_5 = \varepsilon_6$ as this implies reversing all arrows does not change the system as expected in the absence of an external field.

We now introduce an alternate representation of a vertex. Each arrow is replaced with a binary value or *spin*, $i(j) = 0$ if a vertical (horizontal) arrow points down (right) and $i(j) = 1$ if it points up (left). This will make the connection to the XXZ spin chain clearer and so too its arbitrary spin generalisation. Then an arbitrary row-row transition of the lattice can be described by two sets of vertical spins $\mathbf{i} = \{i_1, \ldots, i_N\}$ and $\mathbf{i}' = \{i'_1, \ldots, i'_N\}$ and one set of horizontal spins $\mathbf{j} = \{j_1, \ldots, j_N\}$ (figure 2 shows the k_{th} site in some row). We now see that the *ice rule* proposed by Slater is equivalent to the conservation rule

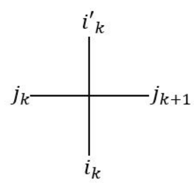

Figure 2: Vertex with Spin Representation

$$i_k + j_k = i'_k + j_{k+1}. \tag{1.2}$$

Summing (1.2) over k we arrive at the conservation of spin for each row of the lattice

$$\sum_{k=1}^{N} i_k = \sum_{k=1}^{N} i'_k := l. \tag{1.3}$$

It is also worth noting that (1.2) also reveals that for given $\mathbf{i}$ and $\mathbf{i}'$, $\mathbf{j}$ is uniquely determined by j_1.

The partition function (1.1) can now be reformulated as

$$Z = \operatorname{Tr}\left(\mathbf{T}^M\right), \tag{1.4}$$

where $\mathbf{T}$ is the transfer matrix which describes the transfer between two row states defined as

$$\mathbf{T}_{\mathbf{i},\mathbf{i}'} = \sum_{\mathbf{j}} e^{-(m_1\varepsilon_1 + \cdots + m_6\varepsilon_6)/k_B T}, \tag{1.5}$$

where m_i is the number of vertices in state i in the row-row transition described by the sets $\mathbf{i}$, $\mathbf{i}'$ and $\mathbf{j}$. This is a $2^N \times 2^N$ matrix since there 2^N choices of $\mathbf{i}$ and $\mathbf{i}'$. If $\mathbf{T}$ can be diagonalised then (1.4) reduces to the following simple expression

$$Z = \sum_{i=1}^{2^N} \lambda_i^M, \tag{1.6}$$

where λ_i are the eigenvalues of $\mathbf{T}$. Futhermore, in the thermodynamic limit $M \to \infty$ the largest eigenvalue dominates the sum (1.6) hence $Z \sim \lambda_{\text{max}}^M$.

In section 2 we will introduce a more generalised version of the six vertex model with arbitrary spin $I \in \mathbb{C}$, and show how its spectrum can be solved using a Q-operator which obeys a TQ relation. This approach was first introduced by Baxter in his original solution of the 8-vertex model [6]. In section 3 we examine two parallel constructions of Q-operators and the connection between them.

Before section 2 we will briefly recall some facts about the $U_q(sl(2))$ algebra. It is defined by the generators E, F and H which obey the relations

$$q^H E q^{-H} = q^2 E, \quad q^H F q^{-H} = q^{-2} F, \quad [E, F] = \frac{[q^H]}{[q]}, \tag{1.7}$$

where we use the notation

$$[x] = x - x^{-1}. \tag{1.8}$$

For any $I \in \mathbb{C}$ we can introduce an infinite-dimensional Verma module V_I^+ with basis $v_i, i \in \mathbb{Z}_+$. Then $U_q(sl(2))$ has an infinite dimensional representation $\pi_I^+ : U_q(sl(2)) \to \mathrm{End}(V_I^+)$ defined by the following action

$$Hv_i = (I - 2i)v_i, \quad Ev_i = \frac{[q^i]}{[q]}v_{i-1}, \quad Fv_i = \frac{[q^{I-i}]}{[q]}v_{i+1}. \tag{1.9}$$

Note above we write E, F and H instead of $\pi_I^+(E), \pi_I^+(F)$ and $\pi_I^+(H)$ for convenience.

2 Arbitrary spin generalisation of six-vertex model

In this section we introduce the arbitrary spin generalisation of the six vertex model or XXZ spin chain taken from [7]. For some $I \in \mathbb{C}$ we construct the $U_q(sl(2))$ L-operator acting in the tensor product $\mathbb{C}^2 \otimes V_I^+$ as

$$L(\lambda;\phi) = \begin{pmatrix} \phi^{-1}[\lambda q^{H/2}] & \phi^{-1}[q]F \\ \phi[q]E & \phi[\lambda q^{-H/2}] \end{pmatrix}, \tag{2.1}$$

where E, F and H are as per (1.7) and (1.9). λ is a spectral parameter and ϕ is the horizontal field. In the case $I \in \mathbb{Z}_+$ the representation π_I^+ is reducible. One can introduce a finite dimensional module $V_I \cong V_I^+/V_{-I-2}^+$ with the basis $\{v_0, \ldots, v_I\}$. The corresponding finite dimensional representation is denoted as π_I.

In the case $I \in \mathbb{C}$ we can now define the tranfer matrix $\mathbf{T}_I(\lambda;\phi)$ with periodic boundary conditions acting in the $(I+1)^N$ dimensional quantum space $W = \bigotimes_{i=1}^N V_I$ as

$$\mathbf{T}_I(\lambda;\phi) = \mathrm{Tr}\left(L_1(\lambda;\phi) \otimes \cdots \otimes L_N(\lambda,\phi)\right), \tag{2.2}$$

where the Trace is taken over the auxiliary space $\mathbb{C}^2$. The conservation law (1.3) still holds in the quantum space W except now $i_k, i'_k \in \mathbb{Z}_+$ are not restricted to binary values instead $i_k, i'_k = 0, \ldots, I$. This means only row states with the same total spin can interact with each other hence the transfer matrix (2.2) has block-diagonal form

$$\mathbf{T}_I(\lambda;\phi) = \bigoplus_{l=0}^{IN} \mathbf{T}_I^{(l)}(\lambda;\phi). \tag{2.3}$$

We will call the subspace of W with a fixed l the l_{th} sector and denote it W_l. Each $\mathbf{T}_I^{(l)}$ in (2.3) is a matrix acting invariantly in the subspace W_l.

We now notice that the form (2.3) is preserved for arbitrary complex spin $I \in \mathbb{C}$ except that now our quantum space is infinite dimensional, $W = \bigotimes_{i=1}^N V_I^+$. The direct sum in (2.3) now runs from 0 to infinity, however, for a fixed l the block $\mathbf{T}_I^{(l)}$ acting in the l_{th} sector is still finite dimensional.

We now briefly return to the standard six-vertex model to motivate the introduction of Baxter's Q-operators [6]. We introduce the following Boltzmann weights for convenience

$$a := e^{-\varepsilon_1/k_BT} = e^{-\varepsilon_2/k_BT}, \quad b := e^{-\varepsilon_3/k_BT} = e^{-\varepsilon_4/k_BT}, \quad c = e^{-\varepsilon_5/k_BT} = e^{-\varepsilon_6/k_BT}. \tag{2.4}$$

Now we parametrise the variables a, b and c by entire functions in new variables ρ, μ and v

$$a, b, c = \rho \sinh\frac{1}{2}(\mu - v), \rho\sinh\frac{1}{2}(\mu + v), \rho\sinh\mu. \tag{2.5}$$

Baxter showed that if we regard ρ and μ as constants then elements of the transfer matrix (1.5) are entire functions of v and that two transfer matrices commute for different $u, v \in \mathbb{C}$

$$[\mathbf{T}(v), \mathbf{T}(u)] = 0. \tag{2.6}$$

Baxter also showed that there was a Q-operator with matrix elements entire in $v \in \mathbb{C}$ which satisfied the following commutation relations for any $v, u \in \mathbb{C}$

$$[\mathbf{Q}(v), \mathbf{Q}(u)] = [\mathbf{T}(v), \mathbf{Q}(u)] = 0, \tag{2.7}$$

and the famous TQ-relation

$$\mathbf{T}(v)\mathbf{Q}(v) = \phi(\mu - v)\mathbf{Q}(v + 2\mu') + \phi(\mu + v)\mathbf{Q}(v - 2\mu'), \tag{2.8}$$

where

$$\phi(v) = \rho^N \sinh^N(v/2), \quad \mu' = \mu - i\pi. \tag{2.9}$$

From (2.7), the left and right hand sides of (2.8) can be simultaneously diagonalised giving the following defining relation for the eigenfunctions $\Lambda(v)$ of $\mathbf{T}(\mathbf{v})$

$$\Lambda(v) = \frac{\phi(\mu - v)q(v + 2\mu') + \phi(\mu + v)q(v - 2\mu')}{q(v)}, \tag{2.10}$$

where the eigenfunctions for, $q(v)$ of $\mathbf{Q}(v)$ are of the form

$$q(v) = \prod_{k=1}^{l} \sinh \frac{1}{2}(v - v_k), \tag{2.11}$$

for unknowns $v_1, \ldots, v_l$ and some fixed l as per (1.3). However, since $\Lambda(v)$ must be an entire function of v the numerator and denominator of (2.10) must both vanish at $v = v_i$ thus giving l the followings equations for $v_1, \ldots, v_l$

$$\frac{\phi(\mu - v_i)}{\phi(\mu + v_i)} = -\frac{q(v_i - 2\mu')}{q(v_i + 2\mu')}, \quad \text{for } i = 1, \ldots, l. \tag{2.12}$$

We now continue this Q-operator approach for the XXZ spin chain at arbitrary spin $I \in \mathbb{C}$. Mangazeev has constructed two operators $\mathbf{Q}^{(I)}(\lambda)$ [7,8] which satisfy the follow the following commutation relations for arbitrary $\lambda, \lambda' \in \mathbb{C}$

$$[\mathbf{Q}^{(I)}(\lambda), \mathbf{T}_I(\lambda'; \phi)] = [\mathbf{Q}^{(I)}(\lambda), \mathbf{Q}^{(I)}(\lambda')] = 0, \tag{2.13}$$

and satisfy the TQ-relation in a different form

$$\mathbf{T}_I(\lambda; \phi)\mathbf{Q}^{(I)}(\lambda) = \phi^N [\lambda/\zeta]^N \mathbf{Q}^{(I)}(q\lambda) + \phi^{-N} [\lambda\zeta]^N \mathbf{Q}^{(I)}(q^{-1}\lambda), \tag{2.14}$$

where we introduce the following variable for convenience

$$\zeta = q^{I/2}. \tag{2.15}$$

As before (2.13) and (2.14) are enough to define a scalar relation for the eigenfunctions of $\mathbf{T}_I(\lambda; \phi)$ which we will denote $\Lambda_I(\lambda; \phi)$

$$\Lambda_I(\lambda; \phi) = \frac{\phi^N [\lambda/\zeta]^N \mathcal{Q}^{(I)}(q\lambda) + \phi^{-N} [\lambda\zeta]^N \mathcal{Q}^{(I)}(q^{-1}\lambda)}{\mathcal{Q}^{(I)}(\lambda)}, \tag{2.16}$$

where for arbitrary $\phi \neq 1$ the eigenfunctions $\mathcal{Q}^{(I)}(\lambda)$ of $\mathbf{Q}^{(I)}$ in the l_{th} sector are now of the form

$$\mathcal{Q}^{(I)}(\lambda) = \frac{A}{\lambda^l} \prod_{k=1}^{l} (\lambda^2 - \lambda_k^2) \tag{2.17}$$

for any constant A and unknowns $\lambda_1, \ldots, \lambda_l$. We now use the same argument as before to deduce that the numerator of (2.15) vanishes for $\lambda = \pm\lambda_k$ giving the following $2l$ equations for $\lambda_1, \ldots, \lambda_l$

$$\phi^{2N} \frac{[\pm\lambda_i/\zeta]^N}{[\pm\lambda_i\zeta]^N} = -\frac{\mathcal{Q}^{(I)}(\pm\lambda_i/q)}{\mathcal{Q}^{(I)}(\pm\lambda_i q)}, \quad \text{for } i = 1, \ldots, l. \tag{2.18}$$

This is sufficient to define the eigenfunctions (2.16).

3 Construction of Q-Operators

In this section we introduce two parrallel construction of Q-operators which satisfy the relation (2.14). In 3.1 we introduce a Q-operator constructed as transfer matrices acting in the l_{th} sector using the q-oscillator algebra and in 3.2 we introduce a Q-operator constructed as an integral operator acting on a space of polynomials.

3.1 q-oscillator construction

The construction of the operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ comes from [7]. They are constructed as transfer matrices acting in the subspace W_l

$$\mathbf{A}_{\pm}^{(I)}(\lambda) = (1-\phi^{2N}q^{2l-IN}) \times \operatorname*{Tr}_{\mathcal{F}_q}\left(A_{\pm}^{(I)}(\lambda)_1 \otimes \cdots \otimes A_{\pm}^{(I)}(\lambda)_N\right). \tag{3.1}$$

Here the trace is taken over the auxiliary infinite dimensional Fock space $\mathcal{F}_q$, spanned by $\{|n\rangle \mid n = 0, 1, \dots\}$. The result (3.1) is in general a geometric series in ϕ and q which in general we cannot expect to converge. However, for appropriate $\phi \in \mathbb{C}$ it can be made to converge and then analytically continued for all values of ϕ.

The local operators $A_{\pm}^{(I)}(\lambda)$ act in the tensor product $\mathcal{F}_q \otimes V_I$ and have the form

$$\begin{aligned}\left[A_{+}^{(I)}(\lambda)\right]_{n,i}^{n',i'} &= \delta_{i+n',i'+n}\phi^{-2n}(-1)^{i+i'}\lambda^{-i}q^{\frac{1}{2}i(i+1)-\frac{1}{2}i'(i'+1)+i(I+i')+n(I-i-i')}\times \\ &\times \frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n(q^2;q^2)_i}\,{}_3\overline{\phi}_2\left(\begin{array}{c} q^{-2i};q^{-2i},\lambda^2 q^{-I} \\ q^{-2I},q^{2(1+n-i)} \end{array}\middle|\, q^2,q^2\right),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\left[A_{-}^{(I)}(\lambda)\right]_{n,i}^{n',i'} &= \delta_{i+n,i'+n'}\phi^{2n}\lambda^{i-I}q^{-\frac{1}{2}i(i-1)+\frac{1}{2}i'(i'-1)+i(I+i')+n(I-i-i')}\times \\ &\times \frac{(\lambda^2 q^{-I+2(i'-n)};q^2)_{I-i-i'}}{(q^2;q^2)_i}\,{}_3\overline{\phi}_2\left(\begin{array}{c} q^{-2i};q^{-2i},\lambda^2 q^{-I} \\ q^{-2I},q^{2(1+n-i')} \end{array}\middle|\, q^2,q^2\right),\end{aligned} \tag{3.3}$$

where we define the q-Pochammer symbol

$$(x;q)_n = \prod_{k=0}^{n-1}(1-xq^k), \tag{3.4}$$

and a *regularized* terminating basic hypergeometric series ${}_{r+1}\overline{\phi}_r$ as

$${}_{r+1}\overline{\phi}_r\left(\begin{array}{c} q^{-n};a_1,\dots,a_r \\ b_1,\dots,b_r \end{array}\middle|\, q,z\right) = \sum_{k=0}^{n} z^k\frac{(q^{-n};q)_k}{(q;q)_k}\prod_{s=1}^{r}(a_s,q)_k(b_s q^k;q)_{n-k}. \tag{3.5}$$

The δ-functions included in definitions (3.2) and (3.3) ensure that the that both Q-operators act invariantly within the subspaces W_l just as the transfer matrix $\mathbf{T}_I(\lambda;\phi)$ (2.2) does.

Q-operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ constructed this way obey the followng TQ relation

$$\mathbf{T}_I(\lambda;\phi)\mathbf{A}_{\pm}^{(I)}(\lambda) = \phi^{\pm N}[\lambda/\zeta]^N\mathbf{A}_{\pm}^{(I)}(q\lambda) + \phi^{\mp N}[\lambda\zeta]^N\mathbf{A}_{\pm}^{(I)}(q^{-1}\lambda), \tag{3.6}$$

as well as the commutation relations (2.13). In particular this means that the operator $\mathbf{A}_{+}^{(I)}(\lambda)$ satisfies the TQ-relation as per (2.14). For this reason we will mostly be concerned with the operator $\mathbf{A}_{+}^{(I)}(\lambda)$. The normlization factor in (3.1) means that it has the following asymptotic behaviour as $\lambda \to \infty$

$$\mathbf{A}_{+}^{(I)}(\lambda)|_{\lambda\to\infty} = -(-\lambda)^l\phi^{2N}q^{l-IM}(\mathbf{I}+O(\lambda^{-2})). \tag{3.7}$$

Another interesting case is when the spectral parameter takes the value $\lambda = \zeta$. In this case only the $k = 0$ term from the series ${}_3\overline{\phi}_2$ is non-zero and equation (3.2) simplifies greatly to

$$[A_+^{(I)}(\zeta)]_{n,i}^{n',i'} = \delta_{i+n',i'+n}(-1)^{i'} q^{ii'+\frac{1}{2}i(I+3-i)-\frac{1}{2}i'(i'+1)} \frac{(\zeta^{-4};q^2)_i}{(q^2;q^2)_i} \times \\ \times \frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n} \phi^{-2n} q^{n(I+i-i')} (q^{-2n};q^2)_i. \tag{3.8}$$

In [7] $\mathbf{A}_+^{(I)}(\zeta)$ was evaluated in closed form for the subspace W_l using the local operator (3.8)

$$\left[\mathbf{A}_+^{(I)}(\zeta)\right]_{\mathbf{i},\mathbf{i}'} = (-1)^{l+1}(1-\phi^{2N}q^{2l-IN})q^l\zeta^l \prod_{k=1}^{N} \frac{(\zeta^{-4};q^2)_{i_k}}{(q^2;q^2)_{i_k}} (\phi/\zeta)^{2+2k(i'_k-i_k)} \times \\ \times \sum_{s=0}^{l-i_1} \frac{(q^2;q^2)_{i_1}}{(\phi^{2N}\zeta^{-2N}q^{2s};q^2)_{i_1+1}} \frac{1}{s!}\frac{d^s}{dz^s} \prod_{m=2}^{N} (zq^{2+2\delta_m};q^2)_{i_m}\Bigg|_{z=0}, \tag{3.9}$$

where δ_m is defined as

$$\delta_m = \sum_{p=1}^{m-1} (i_p - i'_p). \tag{3.10}$$

In section 4 we will see the importance of this remarkably simple result (3.9) as it allows for a much simpler calculation of the operator $\mathbf{A}_+^{(I)}(\lambda)$ for arbitrary λ.

3.2 Integral operator construction

The construction of the operator $\mathbf{Q}_f(\lambda)$ comes from [8]. We begin by introducing a polynomial ring $K[x]$ in variable x over the field $\mathbb{C}$ and its multi-variable generalisation $K_N[X], X := \{x_1,\dots,x_N\}$. We now introduce a linear map $\varphi : W \to K_N[X]$ which is uniquely defined by its action on the basis vectors of W

$$\varphi\left(v_{i_1}\otimes\cdots\otimes v_{i_N}\right) = x_1^{i_1}\dots x_N^{i_N}. \tag{3.11}$$

This gives a clear way to identify row states in W as monomials in $K_N[X]$. Also note that $K_N[X]$ can be written in the direct sum form

$$K_N[X] = \bigoplus_{l\in\mathbb{Z}_+} K_N^{(l)}[X] \tag{3.12}$$

where $K_N^{(l)}[X]$ is generated by monomials in N variables of total degree l. It is clear that $\varphi|_{W_l}$ is an isomorphism between W_l and $K_N^{(l)}[X]$.

Now the fact that the algebra $U_q(sl(2))$ has a representation on the space of polynomials means that we can construct the transfer matrix, $\mathbf{T}_I(\lambda;\phi)$, as per (2.2) acting on the space $K_N[X]$ with the local L_k-operator (2.1) now acting in the tensor product $\mathbb{C}^2\otimes K[x_k]$. However, working in the space $K_N[X]$ allows us to construct Q-operators more generally as integral operators with factorized kernels.

In [8] Mangazeev proved that an integral operator defined[1] by the following action on polynomials of the form $x_k^{i_k}$

$$\mathbf{Q}_f(\lambda) : x_k^{i_k} \mapsto \left(\frac{\lambda}{\zeta}\right)^{i_k} \frac{(q^2;q^2)_{i_k}}{(\zeta^{-4};q^2)_{i_k}} \sum_{p=0}^{i_k} x_{k-1}^p x_k^{i_k-p} \left(\frac{\phi^2}{\lambda^2}\right)^p \frac{(\lambda^2\zeta^{-2};q^2)_p(\lambda^{-2}\zeta^{-2};q^2)_{i_k-p}}{(q^2;q^2)_p(q^2;q^2)_{i_k-p}}, \tag{3.13}$$

satisfies the TQ-relation (2.14) as well as the commutation relations (2.13). Notice that from (3.13), $\mathbf{Q}_f(\lambda)$ preserves the degree of a polynomial hence it acts invariantly in the subspace $K_N^{(l)}[X]$ as does $\mathbf{T}_I(\lambda;\phi)$. It is also clear setting $\lambda = \zeta$ in (3.13) that $\mathbf{Q}_f(\lambda)$ acts trivially hence

$$\mathbf{Q}_f(\zeta) = \mathbf{I}. \tag{3.14}$$

[1](3.13) is sufficient to define the action of $\mathbf{Q}_f(\lambda)$ on any monomial since it has a factorised kernel meaning that product of polynomials $x_k^{i_k}$ maps to a product of RHS's in (3.13).

This result (3.14) is another key ingredient in calculation of the operator $\mathbf{A}_+^{(I)}(\lambda)$ for arbitrary λ which we will see in section 4.

Using (3.13) the action of $\mathbf{Q}_f(\lambda)$ on an arbitrary monomial $x_1^{i_1}\dots x_N^{i_N}$ is

$$\mathbf{Q}_f(\lambda)\ \cdot\ x_1^{i_1}\dots x_N^{i_N} = \prod_{k=1}^{N}\left(\frac{\lambda}{\zeta}\right)^{i_k}\frac{(q^2;q^2)_{i_k}}{(\zeta^{-4};q^2)_{i_k}}\times$$
$$\times\sum_{p_1=0}^{i_1}\cdots\sum_{p_N=0}^{i_N}\prod_{k=1}^{N}\left(\frac{\phi^2}{\lambda^2}\right)^{p_k}\frac{(\lambda^2\zeta^{-2};q^2)_{p_k}(\lambda^{-2}\zeta^{-2};q^2)_{i_k-p_k}}{(q^2;q^2)_{p_k}(q^2;q^2)_{i_k-p_k}}x_k^{i_k-p_k+p_{k+1}}. \tag{3.15}$$

Now by taking the coefficient of the monomial $x_1^{i'_1}\dots x_N^{i'_N}$ we find the matrix elements of $\mathbf{Q}_f(\lambda)$

$$[\mathbf{Q}_f(\lambda)]_{\mathbf{i},\mathbf{i}'} = \left(\prod_{k=1}^{N}\left(\frac{\lambda}{\zeta}\right)^{i'_k}\frac{(q^2;q^2)_{i'_k}}{(\zeta^{-4};q^2)_{i'_k}}\right)\left(\sum_{p_1=0}^{i'_1}\prod_{k=1}^{N}\left(\frac{\phi^2}{\lambda^2}\right)^{p_1+\delta_k}\frac{(\lambda^2\zeta^{-2};q^2)_{p_1+\delta_k}(\lambda^{-2}\zeta^{-2};q^2)_{i'_k-p_1-\delta_k}}{(q^2;q^2)_{p_1+\delta_k}(q^2;q^2)_{i'_k-p_1-\delta_k}}\right), \tag{3.16}$$

where δ_k as is in (3.10). Note the factor $(q^{-2I},q^2)_{i'_k}$ in the denominator of (3.16) means that the Q-operator $\mathbf{Q}_f(\lambda)$ becomes singular in the case $I\in\mathbb{Z}_+$ since this factor has a simple zero for $i'_k>I$. The Q-operator as constructed in section 3.1 does not have this problem.

It will also be worthwhile to examine the asymptotic behaviour of $\mathbf{Q}_f(\lambda)$ in the limit $\lambda\to\infty$. In [8] it was shown that the finite matrix block of $\mathbf{Q}_f(\lambda)$ acting in the l_{th} sector behaves as

$$\mathbf{Q}_f(\lambda)|_{\lambda\to\infty} = \lambda^l\mathbf{Q}_\infty(1+O(\lambda^{-2})), \tag{3.17}$$

where

$$[\mathbf{Q}_\infty]_{\mathbf{i},\mathbf{i}'} = \zeta^{-l}\left(\prod_{k=1}^{N}\left(\frac{\lambda}{\zeta}\right)^{i'_k}\frac{(q^2;q^2)_{i'_k}}{(\zeta^{-4};q^2)_{i'_k}}\right)\left(\sum_{p_1=0}^{i'_1}\prod_{k=1}^{N}\frac{(q^{-2i'_k};q^2)_{p_1+\delta_k}}{(q^2;q^2)_{p_1+\delta_k}}(q^{i'_k}\phi\zeta^{-1})^{2(p_1+\delta_k)}\right). \tag{3.18}$$

Finally now that we have the expression (3.16) we can see that the operator can be constructed instead as a transfer matrix

$$\mathbf{Q}_f(\lambda) = \mathrm{Tr}\left(Q_f^{(1)}\otimes\cdots\otimes Q_f^{(N)}\right), \tag{3.19}$$

where local L-operators Q_f act in the tensor product $\mathcal{V}\otimes V_I$ where $\mathcal{V}$ is some space spanned by vectors $|p_k\rangle = 0,\dots,i'_k$. They have the form

$$[Q_f(\lambda)]_{p,i}^{p',i'} = \delta_{i'+p',i+p}\left(\frac{\lambda}{\zeta}\right)^{i'}\frac{(q^2;q^2)_{i'}}{(\zeta^{-4};q^2)_{i'}}\left(\frac{\phi^2}{\lambda^2}\right)^{p}\frac{(\lambda^2\zeta^{-2};q^2)_p(\lambda^{-2}\zeta^{-2};q^2)_{i'-p}}{(q^2;q^2)_p(q^2;q^2)_{i'-p}}. \tag{3.20}$$

Although this construction as a transfer matrix is not apparent until after the matrix elements of $\mathbf{Q}_f(\lambda)$ have been explicitly calculated it will be useful in the next section. Note that from (3.20) it appears that the local structure of $\mathbf{Q}_f(\lambda)$ is dependent only on the indices p and i' apart from in the conservation rule given by the delta function.

4 Connection between Q-operators

So far it has been alluded to that there is a connection between the Q-operators constructed in sections 3.1 and 3.2. In this section we will make that precise. Recall that the Q-operators $\mathbf{A}_+^{(I)}$ and $\mathbf{Q}_f(\lambda)$ obey the TQ-relation (2.14). Requiring that an operator satisfies (2.14), a second order difference equation, fixes it up to a constant matrix multiple. Therefore, we can write

$$\mathbf{A}_+^{(I)}(\lambda) = \mathbf{A}_0\mathbf{Q}_f(\lambda) \tag{4.1}$$

for some matrix $\mathbf{A}_0$ which is independent of λ. If we now set $\lambda = \zeta$ in (4.1) and use the identity (3.14) it is clear that

$$\mathbf{A}_0 = \mathbf{A}_+^{(I)}(\zeta), \tag{4.2}$$

hence (4.1) becomes

$$\mathbf{A}_+^{(I)}(\lambda) = \mathbf{A}_+^{(I)}(\zeta)\mathbf{Q}_f(\lambda) = \mathbf{Q}_f(\lambda)\mathbf{A}_+^{(I)}(\zeta).\ ^2 \tag{4.3}$$

Since we already have an explicit expression for $\mathbf{A}_+^{(I)}(\zeta)$ (3.9) and $\mathbf{Q}_f(\zeta)$ (3.16) in closed form, (4.3) gives us a remarkably simple way of calculating $\mathbf{A}_+^{(I)}(\lambda)$ for arbitrary λ. This is normally a very expensive calculation (3.1) as it involves an infinite sum over tensors of hypergeometric series ${}_3\overline{\phi}_2$ which fast becomes an unbearable calculation for even small systems. Furthermore, we can note that the poles in (3.16) for $I \in \mathbb{Z}_+$ which come from the factor $(\zeta^{-4}; q^2)_{i'_k}$ in the denominator cancel exactly with the factor $(\zeta^{-4}; q^2)_{i_k}$ in (3.9). The result is that the matrix product from (4.3) is perfectly well-defined in the finite dimensional case $I \in \mathbb{Z}_+$ as we should require for $\mathbf{A}_+^{(I)}(\lambda)$.

The relation (4.3) has been checked in this project for some small cases $N = 1, 2, 3,\ l = 0, 1, 2, 3$. However, (4.3) is a general relation which holds for any length of system N. For this reason it is believed that there is a local relation between the L-operators of the two Q-operators $\mathbf{A}_+^{(I)}(\lambda)$ and $\mathbf{Q}_f(\lambda)$ from which the result (4.3) can be derived. The main goal of this project was to investigate such a local relation, however, no significant progress was made on this question.

We now aim to construct a composite L-operator from $[Q_f(\lambda)]_{p,i}^{p',i'}$ and $[A_+^{(I)}(\zeta)]_{n,i}^{n',i'}$ in order to calculate $\mathbf{A}_+^{(I)}(\lambda)$, particularly in the limit $\lambda \to 0$ as this result is unknown. This composite L-operator is represented graphically in figure 3. We build the composite L operator by taking the product of the two L operators

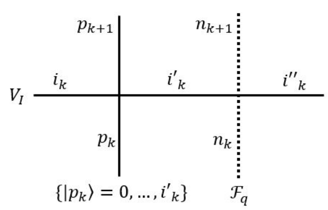

Figure 3: Composite L-operator

$$[Q_f A_+(\lambda)]_{i,i',i''}^{p,p';n,n'} := [Q_f(\lambda)]_{p,i}^{p',i'} [A_+^{(I)}(\zeta)]_{n,i'}^{n',i''} = \delta_{i'+p',i+p}\delta_{i'+n',i''+n}(-1)^{i''} q^{-\frac{i''}{2}(i''+1)+n(I-i'')} \phi^{2(p-n)} \lambda^{-2p} \frac{(\lambda^2\zeta^{-2}; q^2)_p}{(q^2; q^2)_p} \times$$

$$\times \frac{(q^{-2n}; q^2)_{i'}(\lambda^{-2}\zeta^{-2}; q^2)_{i'-p}}{(q^2; q^2)_{i'-p}} \left(\lambda q^{n+i''+\frac{3}{2}}\right)^{i'} \left(q^{-\frac{1}{2}}\right)^{i'^2}, \tag{4.4}$$

where the factor $\frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n}$ in (3.8) is superflous and can be removed by a similarity transformation.

Now using the L-operator (4.4), the matrix blocks $\mathbf{A}_+^{(I)}(\lambda)$ in W_l are constructed as follows

$$\mathbf{A}_\pm^{(I)}(\lambda) = (1 - \phi^{2N} q^{2l-IN}) \times \operatorname*{Tr}_{V_I}\left(\operatorname*{Tr}_{\mathcal{V}}\left(\operatorname*{Tr}_{\mathcal{F}_q}(Q_f A_+(\lambda)_1 \otimes \cdots \otimes Q_f A_+(\lambda)_N)\right)\right). \tag{4.5}$$

The first trace (denoted as over V_I) is a finite summation over the index $i'_k = 0, \ldots, l$. The ordering of the traces over the spaces $\mathcal{V}$ and $\mathcal{F}_q$ is not important. The calculation (4.5) has been performed for small cases $N = 1, 2, 3,\ l = 0, 1, 2, 3$ and it agrees with the matrix $\mathbf{A}_+^{(I)}(\lambda)$ calculated as per (3.1).

[2] The ordering is irrelevant since $[\mathbf{A}_+^{(I)}(\zeta), \mathbf{A}_+^{(I)}(\lambda)] = 0$.

The utility of the composite L-operator (4.4) can now be seen. As $\lambda \to 0$ (4.4) behaves as

$$[Q_f A_+(\lambda)]_{i,i',i''}^{p,p';n,n'}|_{\lambda\to 0} = \delta_{i'+p',i+p}\delta_{i'+n',i''+n}(-1)^{i''+p} q^{-\frac{i''}{2}(i''+1)+n(I-i'')+p(p+1)}\phi^{2(p-n)}\zeta^{2p}\frac{1}{(q^2;q^2)_p}\times$$
$$\times \frac{(q^{-2n};q^2)_{i'}}{(q^2;q^2)_{i'-p}}\left(-\lambda^{-1}q^{n+i''+\frac{1}{2}-I-2p}\right)^{i'}\left(q^{\frac{1}{2}}\right)^{i'^2}\left(1+O(\lambda^2)\right). \tag{4.6}$$

It is hoped this approach will allow for the calculation of the matrix $\mathbf{A}_+^{(I)}(\lambda)|_{\lambda\to 0}$ in closed form although it has not been fully developed yet.

Another interesting case worth examining is the relation (4.3) in the limit $\lambda \to \infty$. Using results (3.7) and (3.17) we see that a factor of λ^l cancels out of both sides giving a result which is well defined as $\lambda \to \infty$

$$\mathbf{A}_+^{(I)}(\zeta)\mathbf{Q}_\infty = (-1)^{l+1}\phi^{2M}q^{l-IM}\mathbf{I}. \tag{4.7}$$

That is, matrices $\mathbf{A}_+^{(I)}(\zeta)$ and $\mathbf{Q}_\infty$ are inverses of each other up tp a simple scalar factor. Inverting either of the matrices $\mathbf{A}_+^{(I)}(\zeta)$ and $\mathbf{Q}_\infty$ for any l is a highly non-trivial problem so this result is interesting as it gives an explicit answer for their inverses.

5 Conclusion

In this report we studied the six-vertex model and its higher spin generalisations. We give a brief introduction to the six-vertex model using an arrow representation and a spin representation and showed how the partition function could be reformulated in terms of the transfer matrix $\mathbf{T}$ (1.5) and its spectrum. We then introduced the higher spin six-vertex model or XXZ spin chain constructed in [7] with horizontal field ϕ. For spin $I \in \mathbb{Z}_+$ this gives a finite dimensional transfer matrix $\mathbf{T}_I(\lambda;\phi)$ acting on the finite dimensional quantum space W. Otherwise, the transfer matrix acts on W which is infinite dimensional for non-integer I. However, in this case the transfer matrix is still a direct sum finite dimensional blocks corresponding to the invariant action within the subspace with total spin l, $W_l \subset W$. We showed how Baxter's famous Q-operator approach allows for calculation of the spectrum of the transfer matrix first using the standard six-vertex model as an example (2.10) and then for the more higher spin model (2.16).

We then examined two parallel constructions of explicit Q-operators satisfying the TQ-relation (2.14). In the first construction from [7] $\mathbf{A_+}^{(I)}(\lambda)$ was a transfer matrix from q-oscillator L-operators acting in the tensor product $\mathcal{F}_q \otimes V_I$ where $\mathcal{F}_q$ was the infinite dimensional Fock space. We saw an explicit formula for this operator evaluated at the special value $\lambda = \zeta$ (3.9) as well as its limiting behaviour as $\lambda \to \infty$.

The second construction from [8] built the operator $\mathbf{Q}_f(\lambda)$ as an integral operator acting in the space of polynomials in N variables based on the idea that we could represent row states as monomials in N variables. We give an explicit formula for this operator (3.13) and saw that it reduces to the identity operator in the case $\lambda = \zeta$. We also see its limiting behaviour in the case $\lambda \to \infty$ and noted that this operator has the problem of being singular in the case $I \in \mathbb{Z}_+$ which $\mathbf{A}_+^{(I)}(\lambda)$ did not have.

Finally, we saw the relationship (4.3) between the operators. This is a remarkable result as it allows for simpler calculation of the matrix $\mathbf{A}_+^{(I)}(\lambda)$ (for fixed l) which in general involves a very non-trivial calculation. We used this relationship to build a *composite* L-operator to evaluate the matrix $\mathbf{A}_+^{(I)}$ in the hope that we could evaluate the particular limit $\lambda \to 0$ although this approach was not fully developed. This is potentially a direction for further study.

The nature of the relationship (4.3) i.e. being independent of the systems length, suggests that there is an underlying relationship between the local operators that build the two Q-operators. The main focus of this project was to investigate and find such a local relation, however, no significant progress was made on this endeavour. This is a particularly interesting feature of the Q-operators constructed and will undoubtedly be the subject of further study.

6 References

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