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An Unorientable Extension of the Temperley-Lieb algebra

Benjamin Morris[1]
Joint work with Dionne Ibarra[2], Gabriel Montoya-Vega[3], and Paul Martin[1] (supervisor)

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Junior London Algebra Colloqium, March 2024

Motivation/Aims/Set-up/Framing etc

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-Motivated by gram determinant of Skein modules...

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a concrete question was crystallized: Can you have a an extension of the Temperley-Lieb algebra (category), where you consider diagrams on non-orientable surfaces? YES! What about finite dimensional?...

Plan for the talk

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Hom-spaces are infinite dimensional! :(

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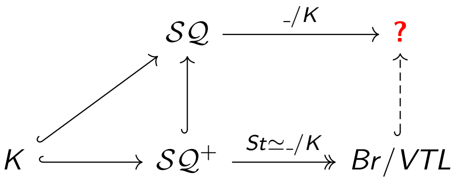

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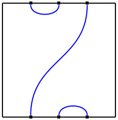
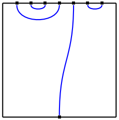


$\in \mathrm{Hom}(3, 3)$,



$\in \mathrm{Hom}(1, 7)$.

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Temperley-Lieb Diagrams

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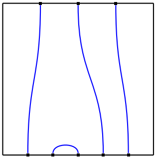
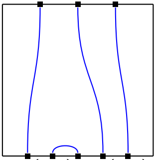
What we see:





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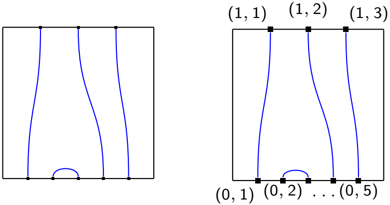

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$$\{\{(0,1),(1,1)\},\{(0,2),(0,3)\},\{(0,4),(1,2)\},\{(0,5),(1,3)\}\}$$

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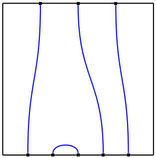

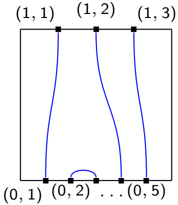

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What is a crossing?

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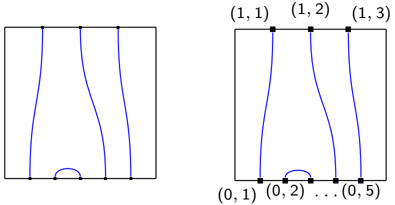

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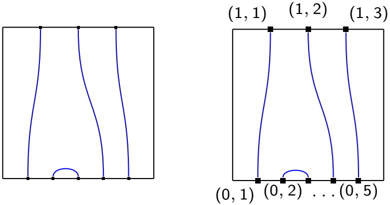

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What is a crossing? Order the vertices AC starting from $(0,1)$ as the minimum. Then $\{v, v'\}$ crosses $\{u, u'\}$ if $v \prec u \prec v' \prec u'$.

Temperley-Lieb Diagrams: Composition

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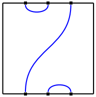
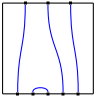
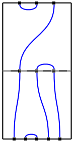
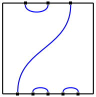
$D_2 \circ D_1 =$  $\circ$  $=$  $=$  $= D_2 \# D_1$.

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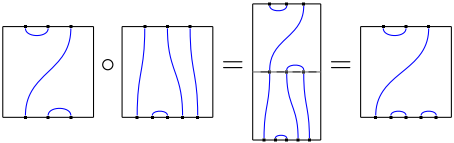
$D_2 \circ D_1 =$  $= D_2 \# D_1.$

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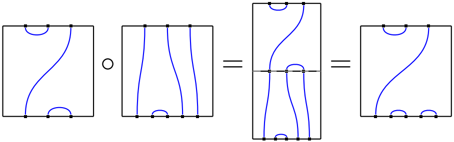
$D_2 \circ D_1 =$  $= D_2 \# D_1.$

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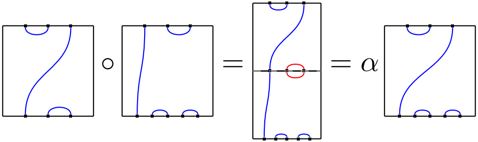
$D_2 \circ D_1' =$  $= \alpha D_2 \# D_1'$

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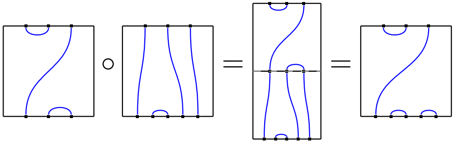
$D_2 \circ D_1 =$  $= D_2 \# D_1.$

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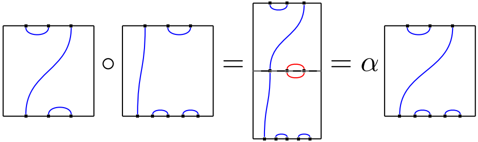
$D_2 \circ D_1' =$  $= \alpha D_2 \# D_1'$

Generically $D_2 \circ D_1 = \alpha^{L(D_1,D_2)}\ D_2 \# D_1.$

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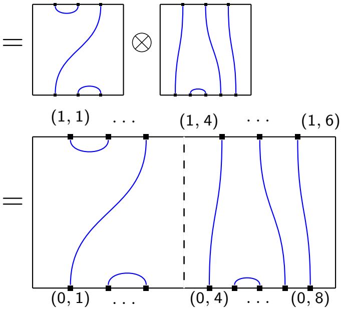

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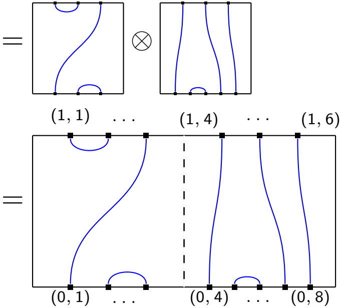
$D_2 \otimes D_1 =$ 

where $n_1 \otimes n_2 = n_1 + n_2$.

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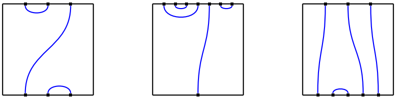
TL-diagrams can be drawn on a square frame



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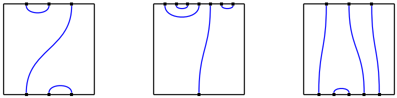

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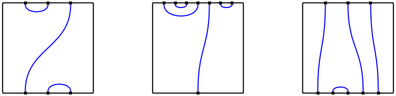
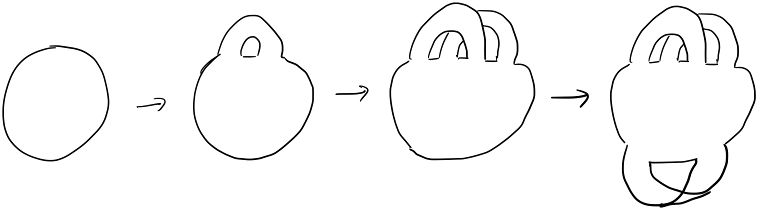


We would like to draw diagrams on different surfaces. Use a "disc with bands" model for surfaces:



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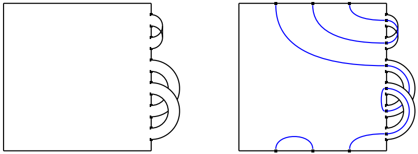

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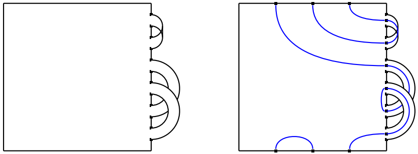

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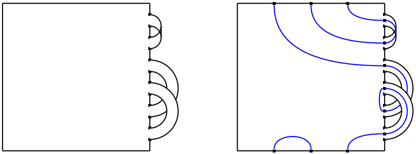

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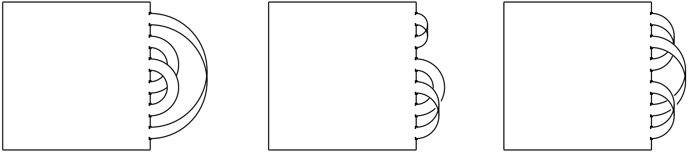

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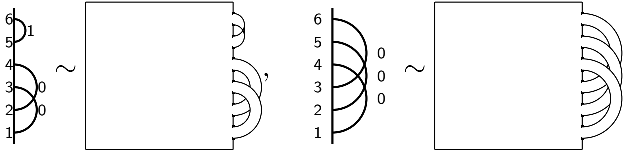

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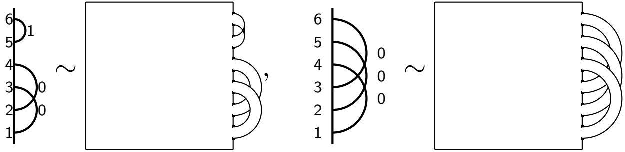

A diagram is **orientable** if $s(P) = \{0\}$.

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For two diagrams $(P_1, s_1) \in \mathcal{TC}_{N_1}$ and $(P_2, s_2) \in \mathcal{TC}_{N_2}$, their vertical juxtaposition is a twisted chord diagram of rank $N_1 + N_2$:

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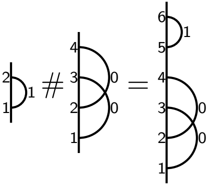

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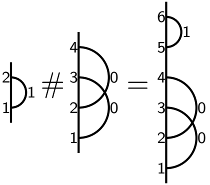

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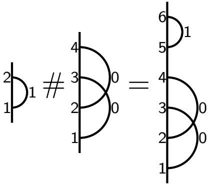

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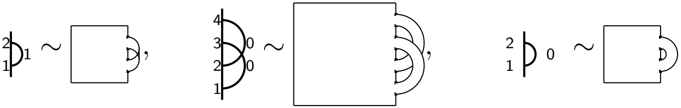

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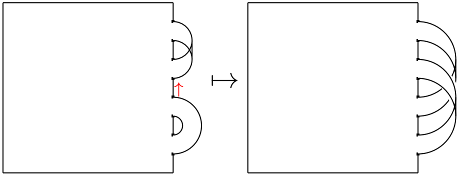

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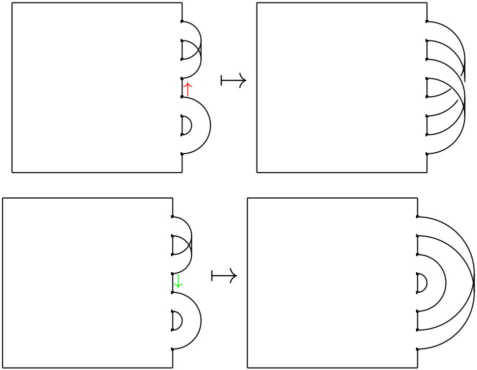

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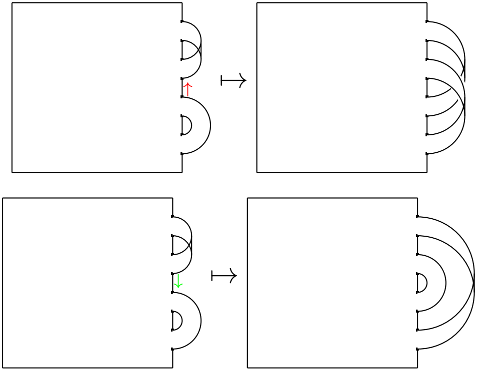

View this as a map $h_{(i,\pm1)} : \mathcal{TC}_N \to \mathcal{TC}_N$,
$h_{(i,\pm1)} : (P, s) \mapsto \left(\sigma(P), s' \circ \sigma^{-1}\right)$, $\sigma = \sigma_{(i\pm1,P,s)} \in \mathrm{Sym}_{2N}$.

Chordsliding

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Can define an equivalence on $\mathcal{TC}$ by $(P,s) \sim (P',s')$ if (P',s') is obtained from (P,s) by a finite sequence of chordslides.

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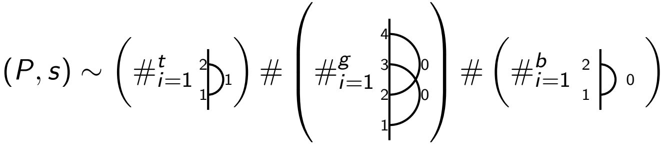

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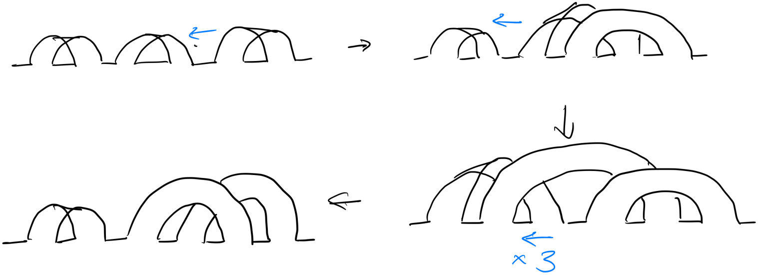

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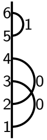

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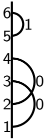

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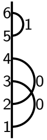

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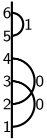

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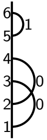

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SWB diagrams

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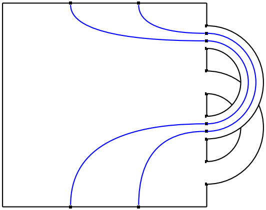

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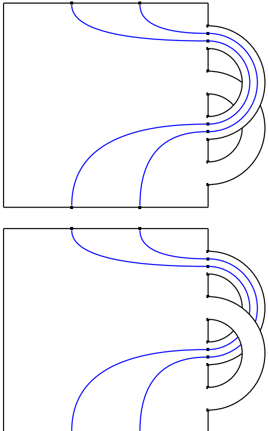

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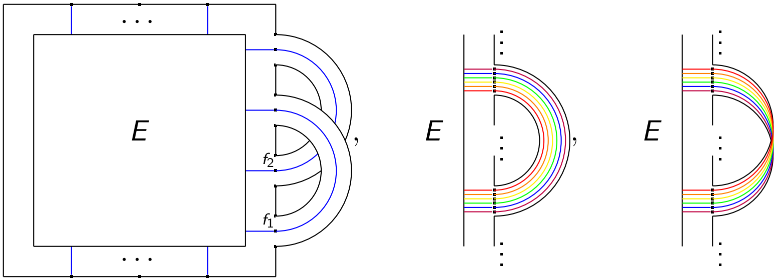

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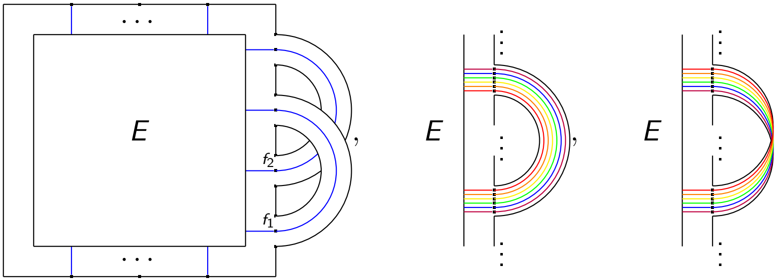

Form the graph $G(\Theta) = (V, E \cup D(\Theta))$, where $D(\Theta) = \{\{u, \iota_\Theta(u)\} \mid u \in V_I\}$ where $\iota_\Theta : V_I \to V_I$.

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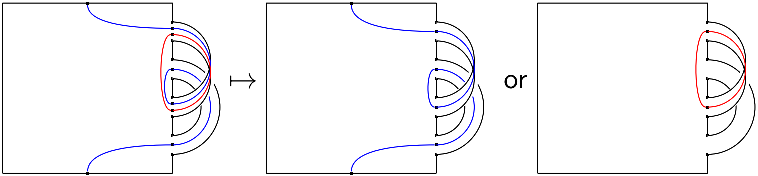
Example: We can "delete components" of $G(\Theta)$



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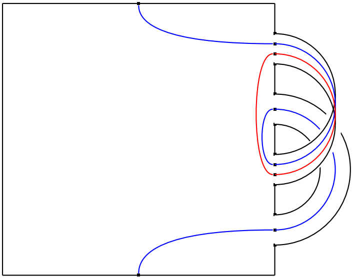

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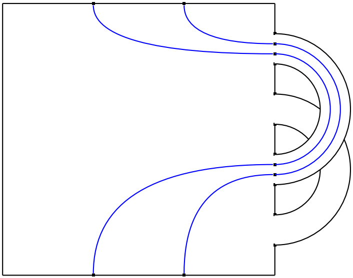

SWB diagrams - Vertical Juxtaposition

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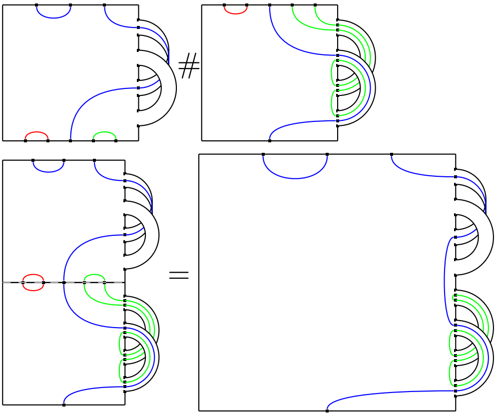
$\Theta_2 \# \Theta_1 =$ … $\#$ … $=$ … $=$ …



$L(\Theta_1, \Theta_2) = 1.$

SWB diagrams - "Isotopy"

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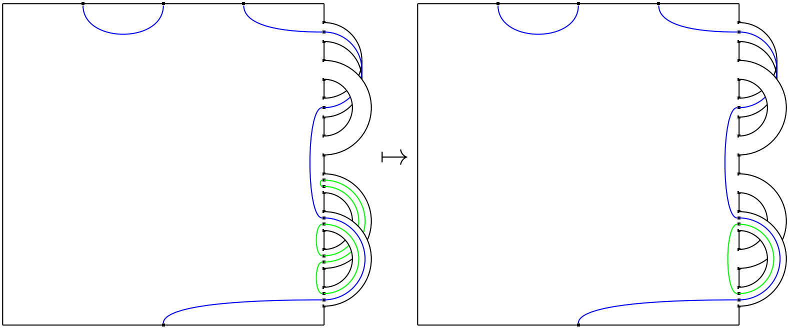
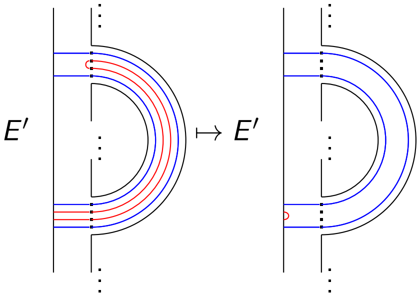


SWB diagrams - "Isotopy"

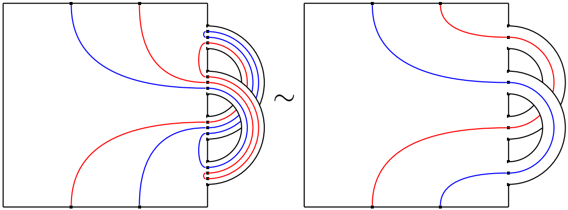
Generically, we can remove "turnbacks" by pull throughs



$$(P, s, f, E' \sqcup \{\text{"red cup"}\}) \mapsto (P, s, f', E'')$$

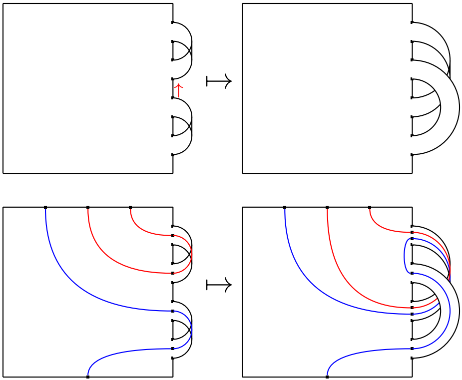
Can generate an equivalence relation with this move - **strong equivalence** of SWB diagrams.

Fact: If Θ has no internal components, then its strong equivalence class has a **unique** representative w/o turnbacks!



SWB diagrams - "Handlesliding"

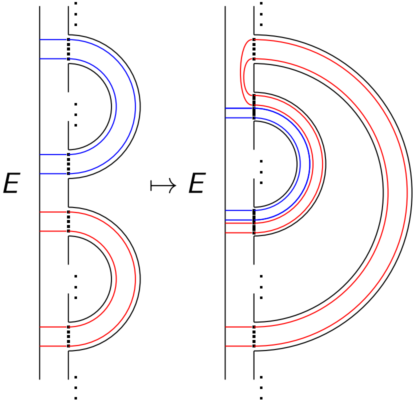
We have the "chordsliding" equivalence move on our surfaces



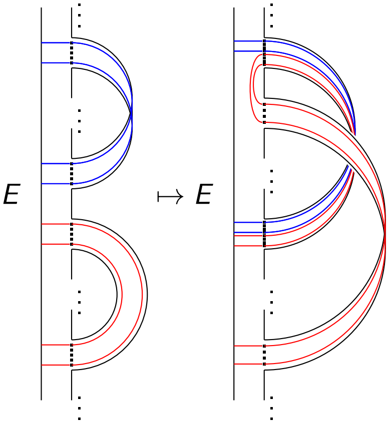
Now lets extend this to moves on our diagrams

SWB diagrams - “Handlesliding”

Generically: “Two bands involved”



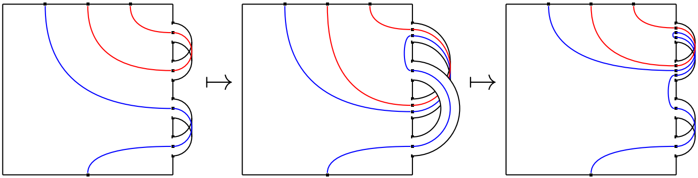
$$(P, s, f, E) \mapsto (h(P, s), f', E \cup \{\text{“new red cups”}\})$$


E
↦ E

SWB diagrams - "Handlesliding"

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P', s') by a finite sequence of chordslides.

What about on our diagrams? Suppose we define a relation by $\Theta \sim \Theta'$ if Θ' can be obtained from Θ by a finite sequence of handleslides: This won't be an equivalence, but...



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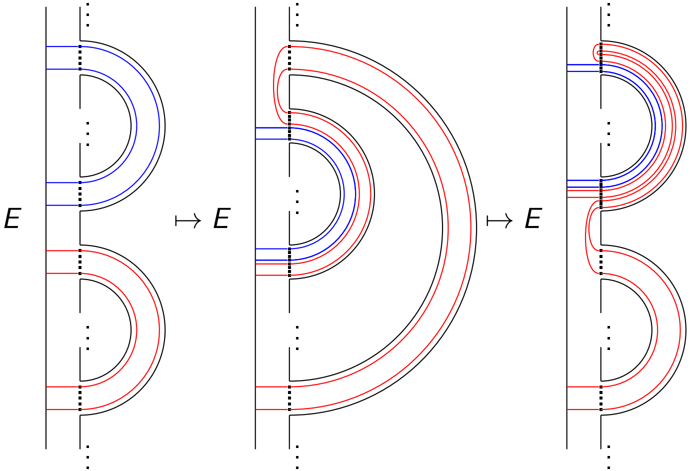


SWB diagrams - "Handlesliding"

Instead, we can define an equivalence relation on strong equivalence classes of SWB-diagrams by $[\Theta]_{\mathrm{st.}} \sim [\Theta']_{\mathrm{st.}}$ if Θ'' can be obtained from Θ by a sequence of handleslides where $[\Theta'']_{\mathrm{st.}} = [\Theta']_{\mathrm{st.}}$.

We will call this **weak equivalence** $[[\Theta]_{\mathrm{st.}}]_{\mathrm{w}} := \overline{\Theta}$

N.B.: There is a non-trivial check here to see that this is well defined!! Essentially boils down to finding appropriate "commutation" relations between the handleslide and pull-through moves.

The Category $\mathcal{SQ}$

Let R be a unital commutative ring with $\alpha, \beta \in R$. The category $\mathcal{SQ}(\alpha, \beta)$ is defined as the R-linear category with:

- Objects: Are non-negative integers
- Morphisms: $\mathrm{Hom}(n, m) = \{0\}$ if $n + m = 1 \mod 2$, and otherwise it is R-linear combinations of weak equivalence classes of SWB diagrams, $\overline{\Theta}$ **Modulo** the relations:

$$\Theta = \alpha\,(\Theta \setminus \Gamma), \qquad \Theta = \beta\,(\Theta \setminus \Lambda),$$

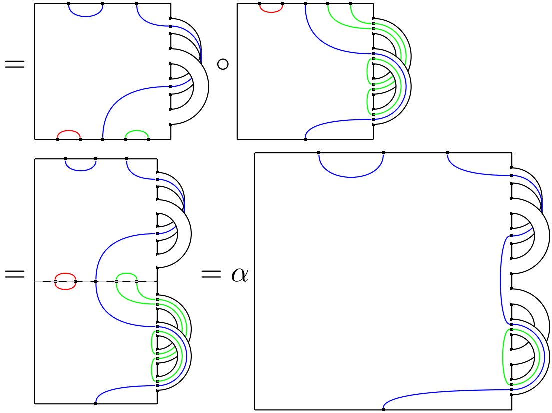
  where Γ, Λ are internal components with twist parameters $\tau_\Gamma = 0, \tau_\Lambda = 1$. e.g.
- Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ $((\phi, \psi) \mapsto \psi \circ \phi)$:

$$\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}$$

The Category $\mathcal{SQ}$

Examples:

$\Theta_2 \circ \Theta_1 =$



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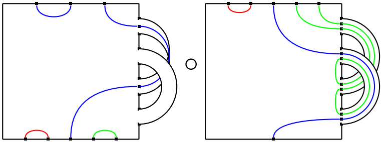
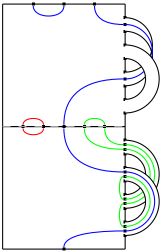
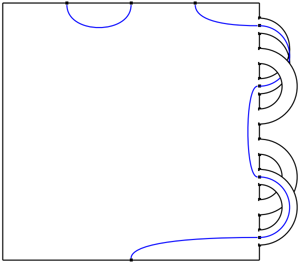


$=$



$= \alpha^2$



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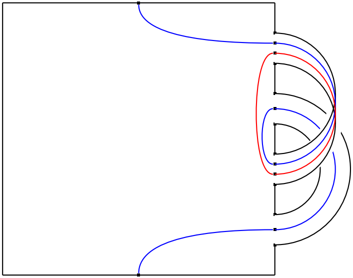
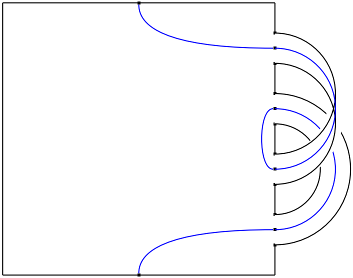


$= \beta$



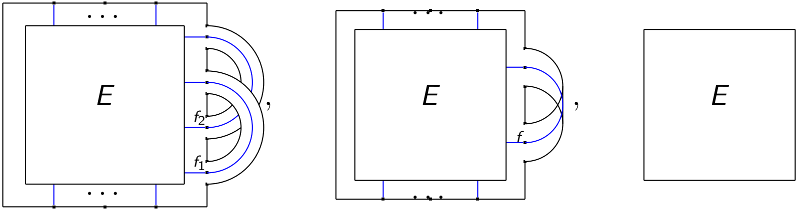
The Category $\mathcal{SQ}$

Fact 1: For any $\Theta \in Sq(n,m)$, there exist **unique** integers l_u and l_t such that:

$$\overline{\Theta} = \alpha^{l_u}\beta^{l_t}\,\overline{\Theta'} \in \mathrm{Hom}(n,m),$$

where $\Theta' \in Sq(n,m)$ has no internal components (i.e. the number of internal components of each type are well defined)!

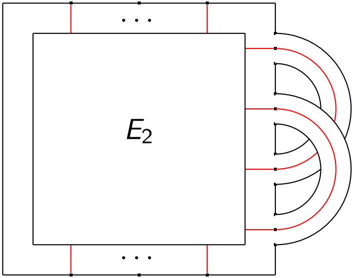
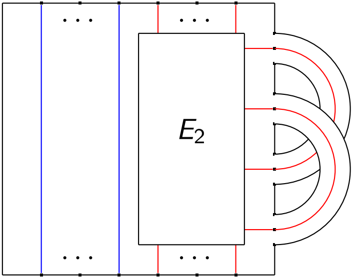
Fact 2: Any morphism $\overline{\Theta} \in \mathrm{Hom}(n,m)$ has a factorisation in terms of diagrams of the following form (using "class. of surf.")



The Category $\mathcal{SQ}$: Tensor Product

In the TL case we had $n_1 \otimes n_2 = n_1 + n_2$ on objects, and on morphisms "horizontal stacking" of diagrams:

How can we "horizontally stack" SWB?
However, we can add put a copy of the identity on the left...

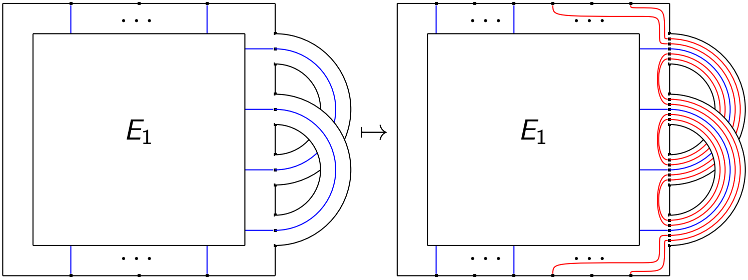
$\Theta_2 =$  $\mapsto$ 

$\stackrel{?}{=} \text{id} \otimes \Theta_2$

The Category $\mathcal{SQ}$: Tensor Product

If we can add a copy of the identity on the right, this would give us a candidate for a tensor product since it should follow

$$(\Theta_1 \otimes \text{id}) \circ (\text{id} \circ \Theta_2) \overset{?}{=} (\text{id} \circ \Theta_2) \circ (\Theta_1 \otimes \text{id}) := \Theta_1 \otimes \Theta_2$$

$\Theta_1 =$ 

$= \Theta_1 \otimes \text{id}$