Introduction to Quantum Groups

ASC Report - SCNC3101

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Abstract

This report presents an overview of a reading course in the basic theory of quantum groups. The introduction covers the background theory relevant to define and work with these objects. In section 2, the utility of this background theory is revealed by examining its role in the representation theory of algebras. In section 3 a specific example of a quantum group, $U_q(\mathfrak{sl}_2)$ is studied.

1 Introduction

The theory of quantum groups is a powerful tool for the mathematical physicist. Through their representation theory, they provide an algebraic path towards solutions to the Yang-Baxter equation; a rich, well studied equation with a clear physical interpretation. This report offers an overview of the basic theory of quantum groups. Since there is a large amount of background theory underpinning this topic, the remainder of this introduction is dedicated to this background theory. In section 2 we motivate some of this background material by considering its role in the representation theory of algebras and in section 3 we study $U_q(\mathfrak{sl}_2)$ a specific example of a quantum group.

1.1 Algebras

Algebras are a common structure in mathematics and physics. Formally, they are vector spaces with a bilinear and associative multiplication operation. Suppose that A is an algebra, that is, a vector space over field $\mathbb{K}$. Bilinearity of multiplcation means that it may be interpretted as a linear map $\mu : A \otimes A \to A$. We will adopt the convention $\mu(a_1 \otimes a_2) = a_1a_2$ for $a_1, a_2 \in A$ to denote multiplication. Associativity means that $\mu(\mu(a_1 \otimes a_2) \otimes a_3) = \mu(a_1 \otimes \mu(a_2 \otimes a_3))$ or simply $(a_1a_2)a_3 = a_1(a_2a_3)$ for any $a_1, a_2, a_3 \in A$. Even more succintly, we can simply say that the following diagram commutes.

$$\begin{array}{ccccc} & & A \otimes A \otimes A & & \\ & \swarrow^{\mu\otimes \mathrm{id}} & & \searrow^{\mathrm{id}\otimes\mu} & \\ A \otimes A & & & & A \otimes A \\ & \searrow^{\mu} & & \swarrow_{\mu} & \\ & & A & & \end{array} \tag{1.1}$$

A unital algebra is an algebra A which has a unit element 1_A such that $a1_A = 1_Aa = a$ for all $a \in A$. Uniqueness of the identity is clear from its defining property. Notice that we can equivalently regard the identity element as a map $\eta : \mathbb{K} \to A$ which satisfies a "unitarity" property described by the following commutative diagram,

$$\begin{array}{ccccc} & & A \otimes A & & \\ & \nearrow^{\eta\otimes\mathrm{id}} & \downarrow^{\mu} & \nwarrow^{\mathrm{id}\otimes\eta} & \\ \mathbb{K} \otimes A & \xleftarrow[\cong]{} & A & \xrightarrow{\cong} & A \otimes \mathbb{K} \end{array} \tag{1.2}$$

where we identify $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$ via $1 \otimes a = a = a \otimes 1$. With this description of a unitary algebra, it is clear that the unit element is simply $\eta(1) := 1_A \in A$. For our purposes, algebras will always be unitary meaning that an algebra A is simply a triple $A = (A, \eta, \mu)$ where μ and η are maps satisfying (1.1) and (1.2). It is worth noting that if $\mu : A \otimes A \to A$ satisfies (1.1) and (1.2) then so does the map $\mu_{\text{op}} := \mu \circ \sigma : A \otimes A \to A$, where σ is the permutation map $\sigma(a \otimes b) = b \otimes a$. This means that given an algebra $A = (A, \eta, \mu)$ we can create another algebra $A_{\text{op}} := (A, \eta, \mu_{\text{op}})$ called the opposite algebra of A. An algebra is commutative if $\mu_{\text{op}} = \mu$.

Finally, we examine the interaction between two algebras, say (A, η_A, μ_A) and (B, η_B, μ_B). Firstly, the natural structure preserving map between them is called an algebra homomorphism. It is a linear map $f : A \to B$ defined by the properties

$$\eta_B = f \circ \eta_A, \quad \mu_B \circ (f \otimes f) = f \circ \mu_A, \tag{1.3}$$

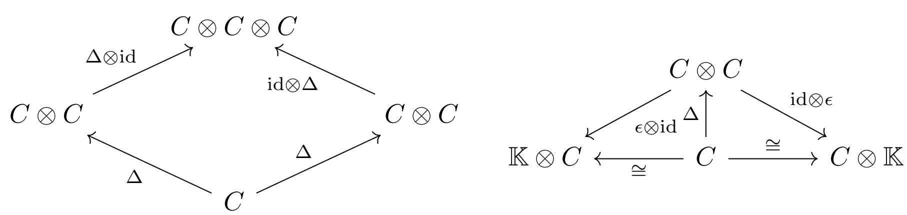
which again may be interpretted as commutative diagrams. Secondly, we can construct the tensor product algebra $A \otimes B = (A \otimes B, \eta_{A\otimes B}, \mu_{A\otimes B})$ where the product and unit maps are defined by

$$\mu_{A\otimes B}\left((a_1 \otimes b_1) \otimes (a_2 \otimes b_2)\right) = \mu_A(a_1 \otimes a_2) \otimes \mu_B(b_1 \otimes b_2), \quad \eta_{A\otimes B} = \eta_A \otimes \eta_B. \tag{1.4}$$

Notice we could equivalently write $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ \sigma_{(23)}$.

1.2 Coalgebras

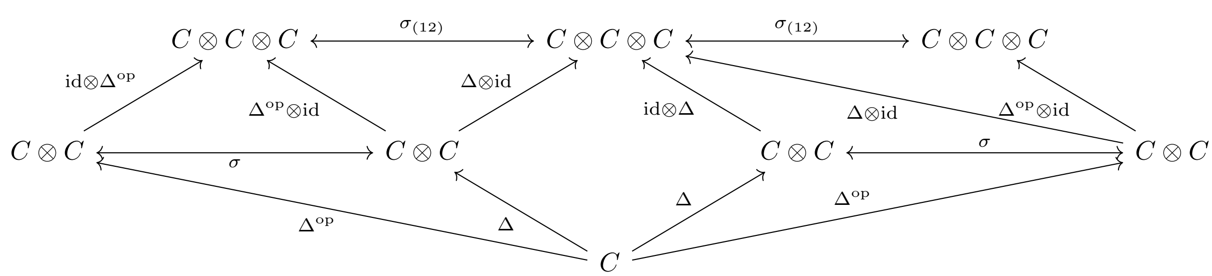
The language of commutative diagrams is unnecessary for dealing with algebras alone. However, its use is made clear when considering coalgebras. The definition of a coalgebra is best motivated by equipping a vector space with appropriate maps such that we can reverse all arrows in diagrams (1.1) and (1.2). With this in mind we define a coalgebra as a triple $C = (C, \epsilon, \Delta)$, where C is a vector space over a field $\mathbb{K}$, and ϵ and Δ are linear maps $\epsilon : C \to \mathbb{K}$ and $\Delta : C \to C \otimes C$ such that the following diagrams commute.



(1.5)

The map Δ is called the coproduct and ϵ the counit. The condition imposed by the first diagram above which mirrors (1.1) is known as coassociativity and that imposed by the second diagram is known as the counitarity.

As with algebras, given a coalgebra, (C, ϵ, Δ), the map $\Delta^{\text{op}} = \sigma \circ \Delta$ where σ is again the permutation map, also satisfies the conditions (1.5). For example, coassociativity follows from the following diagram,



(1.6)

where we note that the maps σ and $\sigma_{(12)}$ are their own inverses ($\sigma_{(12)}$ swaps the first and second tensor components of a triple tensor[1]). Equating the two "boundary" paths in (1.5) from C to the top right copy of $C \otimes C \otimes C$ then yields $\sigma_{(12)} \circ \sigma_{(12)} \circ (\text{id} \otimes \Delta^{\text{op}}) \circ \Delta^{\text{op}} = (\text{id} \otimes \Delta^{\text{op}}) \circ \Delta^{\text{op}} = (\Delta^{\text{op}} \otimes \text{id}) \circ \Delta^{\text{op}}$ as desired. Counitarity of Δ^{op} can be checked similarly. With this in mind we can construct the opposite coalgebra $C^{\text{op}} = (C, \epsilon, \Delta^{\text{op}})$ and similar to the case of algebras we call C cocommutative if $\Delta^{\text{op}} = \Delta$.

[1]More generally for any $\tau \in S_n$ we may consider the map σ_τ which acts on an n-fold tensor product, permuting tensor components according to the element τ.

We finish this subsection by noting that the interplay between two coalgebras, say $(C, \epsilon^C, \Delta^C)$ and $(B, \epsilon^B, \Delta^B)$, also parallels the case of algebras. We define a coalgebra homomorphism as a linear map $f : C \to B$ such that

$$\Delta^B \circ f = (f \otimes f) \circ \Delta^C, \quad \epsilon^B \circ f = \epsilon^C, \tag{1.7}$$

and we define the tensor product coalgebra $C \otimes B = (C \otimes B, \epsilon^{C\otimes B}, \Delta^{C\otimes B})$ by

$$\epsilon^{C\otimes B}(c \otimes b) = \epsilon^C(c)\epsilon^B(b), \quad \Delta^{C\otimes B}(c \otimes b) = \sum_{i,j} (c_i^{(1)} \otimes b_j^{(1)}) \otimes (c_i^{(2)} \otimes b_j^{(2)}), \tag{1.8}$$

where $\Delta^C(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)}$ and likewise for $\Delta^B(b)$. Notice that we could also write this coproduct as $\Delta^{C\otimes B} = \sigma_{(23)} \circ (\Delta^C \otimes \Delta^B)$.

1.3 Bialgebras

Now that we have defined coalgebras, which were saw were a way to dualise algebras, it is natural to ask if we can have both structures at once. To achieve this, we now define bialgebras. A bialgebra over a field $\mathbb{K}$ is a quintuple $(B, \eta, \mu, \epsilon, \Delta)$ where (B, η, μ) is an algebra and (B, ϵ, Δ) is a coalgebra. Furthermore, we also ask that these operations respect each other in the following sense:

1. Comultiplication $\Delta : A \to A \otimes A$ and counit $\epsilon : A \to \mathbb{K}$ are algebra homomorphisms.
2. Multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{K} \to A$ are coalgebra homomorphisms.

It turns out that conditions 1 and 2 above are both equivalent to the following four conditions (and hence equivalent to each other)

$$\Delta \circ \mu = (\mu \otimes \mu) \circ \sigma_{(23)} \circ (\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon \circ \mu = \epsilon \otimes \epsilon, \quad \epsilon \circ \mu = \mathrm{id}_{\mathbb{K}}, \tag{1.9}$$

which all have clear interpretations as commutative diagrams. Given a bialgebra $(B, \eta, \mu, \epsilon, \Delta)$, the maps μ_{op} and Δ^{op} previously defined also satisfy the above conditions so we can define three more associated bialgebras

$$B_{\mathrm{op}} = (B, \eta, \mu_{\mathrm{op}}, \epsilon, \Delta), \quad B^{\mathrm{op}} = (B, \eta, \mu, \epsilon, \Delta^{\mathrm{op}}), \quad B^{\mathrm{op}}_{\mathrm{op}} = (B, \eta, \mu_{\mathrm{op}}, \epsilon, \Delta^{\mathrm{op}}). \tag{1.10}$$

A bialgebra homomorphism is a linear map between bialgebras which is both an algebra and coalgebra homomorphism with respect to the relevant structure. We can also create the tensor product of two bialgebras $(B, \eta_B, \mu_B, \epsilon_B, \Delta_B)$ and $(A, \eta_A, \mu_A, \epsilon_A, \Delta_A)$ simply as $(B \otimes A, \eta_{B\otimes A}, \mu_{B\otimes A}, \epsilon_{B\otimes A}, \Delta_{B\otimes A})$ with the previously defined maps.

1.4 Hopf Algebras

In this section we introduce Hopf algebras which provide a way to relate the independent algebra and coalgebra structure of a bialgebra. A Hopf algebra over a field $\mathbb{K}$ is a bialgebra $H = (H, \eta, \mu, \epsilon, \Delta)$ equipped with a bijective $\mathbb{K}$-linear map $S : H \to H$ such that

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{1.11}$$

Here S is known as the antipode. The defining property is equivalent to asking that the following diagrams commute.

$$\begin{array}{ccccc} H \otimes H & & \xrightarrow{\mathrm{id}\otimes S} & & H \otimes H \\ \Delta\uparrow & & & & \downarrow\mu \\ H & \xrightarrow{\epsilon} & \mathbb{K} & \xrightarrow{\eta} & H \end{array} \qquad \begin{array}{ccccc} H \otimes H & & \xrightarrow{S\otimes \mathrm{id}} & & H \otimes H \\ \Delta\uparrow & & & & \downarrow\mu \\ H & \xrightarrow{\epsilon} & \mathbb{K} & \xrightarrow{\eta} & H \end{array} \tag{1.12}$$

Homomorphisms of Hopf algebras are bialgebra homomorphisms $T : H \to H'$ such that $T \circ S = S' \circ T$ (we use an apostrophe to denote the structural maps of the Hopf algebra H'). Interestingly, this property is no weaker than simply requiring T to be a bialgebra homomorphism as the following calculation shows

$$\begin{aligned}\mu' \circ ((T \circ S - S' \circ T) \otimes T) \circ \Delta &= \mu' \circ (T \otimes T) \circ (S \otimes \mathrm{id}) \circ \Delta - \mu' \circ (S' \otimes \mathrm{id}) \circ (T \otimes T)\Delta \\ &= T \circ (\mu \circ (S \otimes \mathrm{id}) \circ \Delta) - (\mu' \circ (S' \otimes \mathrm{id}) \circ \Delta') \circ T \\ &= T \circ \eta \circ \epsilon - \eta' \circ \epsilon' \circ T \\ &= \eta' \circ \epsilon - \eta' \circ \epsilon = 0, \end{aligned} \tag{1.13}$$

which implies $T \circ S - S' \circ T = 0$ as desired. The tensor product of two Hopf algebras, H and H', is simply their tensor product $H \otimes H'$ as bialgebras with the antipode $S \otimes S'$.

In order to further probe the structure of Hopf algebras we will now introduce a helpful notation. Recall that for any $a \in H$, $\Delta(a) \in H \otimes H$ and thus can be written as a linear combination of tensors as follows

$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}. \tag{1.14}$$

In order to better keep track of components we will adopt the Swindler convention of writting the above as a summation over the element a instead:

$$\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}. \tag{1.15}$$

The coassociativity property then looks like

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_{(a)} \left(\sum_{(a^{(1)})} a^{(1)(1)} \otimes a^{(1)(2)} \right) \otimes a^{(2)} = \sum_{(a)} a^{(1)} \otimes \left(\sum_{(a^{(2)})} a^{(2)(1)} \otimes a^{(2)(2)} \right) = (\mathrm{id} \otimes \Delta) \circ \Delta(a). \tag{1.16}$$

The power of this notation becomes clear when we adopt the following shorthand

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) := \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes a^{(3)}. \tag{1.17}$$

Here the summation over (a) is hiding the double summation; the coassociativity property is exactly what makes this well defined. In fact the above is generalised the above for an n fold coproduct as follows

$$(\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \cdots \circ (\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes \cdots \otimes a^{(n+1)}. \tag{1.18}$$

With this notation, the counit and antipode properties are respectively written as

$$\sum_{(a)} \epsilon(a^{(1)})a^{(2)} = a = \sum_{(a)} a^{(1)}\epsilon(a^{(2)}), \quad \sum_{(a)} S(a^{(1)})a^{(2)} = \epsilon(a)1 = \sum_{(a)} a^{(1)}S(a^{(2)}). \tag{1.19}$$

Using this notation we can begin to prove some important facts about Hopf algebras. The first is that the antipode is unique: suppose that S and S' are both linear maps satisfying the defining property (1.11). The following calculation then shows that they are indeed the same

$$\begin{aligned} S(a) &= \sum_{(a)} \epsilon(a^{(1)})S(a^{(2)}) \\ &= \sum_{(a)} (\sum_{(a^{(1)})} S'(a^{(1)(1)})a^{(1)(2)})S(a^{(2)}) \\ &= \sum_{(a)} S'(a^{(1)}) \left(\sum_{(a^{(2)})} a^{(2)(1)}S(a^{(2)(2)}) \right) \\ &= \sum_{(a)} S'(a^{(1)})\epsilon(a^{(2)}) \\ &= S'(a). \end{aligned} \tag{1.20}$$

Next we consider how the antipode interacts with the given bialgebra structure. As shown in [4] $S : H \to H$ is an anti-algebra and anti-coalgebra homomorphism (we omit the proof here), which means precisely the following

$$S(ab) = S(b)S(a), \quad S \circ \eta = \eta, \quad (S \otimes S) \circ \Delta = \Delta^{\mathrm{op}} \circ S, \quad \epsilon \circ S = \epsilon. \tag{1.21}$$

It is relatively simple to show that an antipode S for a Hopf algebra H is also an antipode for the bialgebra $H^{\mathrm{op}}_{\mathrm{op}}$ and thus the above is exactly the statement that $S : H \to H^{\mathrm{op}}_{\mathrm{op}}$ is a Hopf algebra homomorphism. As S was defined to be bijective, it follows that this is also an isomorphism. Furthermore, the bialgebras H^{op} and H_{op} admit a Hopf algebra structure with the antipode S^{-1}, and are isomorphic via S. For example we check that S^{-1} makes H^{op} into a Hopf algebra:

$$\begin{aligned}\mu \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta^{\mathrm{op}}(a) &= \mu \circ (S^{-1} \otimes \mathrm{id}) \circ \Delta^{\mathrm{op}} \circ S(b) \\ &= \mu \circ (S^{-1} \otimes \mathrm{id}) \circ (S \otimes S) \circ \Delta(b) \\ &= \mu \circ (\mathrm{id} \otimes S) \circ \Delta(b) = \epsilon(b) = \epsilon(S(b)) = \epsilon(a).\end{aligned} \tag{1.22}$$

We used bijectivity to write $a = S(b)$ for some $b \in H$.

One immediate consequence of this theory is that if H is commutative or co-commutative and hence $H = H_{\mathrm{op}}$ or $H = H^{\mathrm{op}}$, then uniqueness of the antipode requires that $S = S^{-1}$ or $S^2 = \mathrm{id}$. Because of this property, it is clear that cocommutativity (and commutativity) place extremely strong constraints on Hopf algebras, yet fully general Hopf algebras do not give us enough tools to do physics with. The next series of definitions are intended to address this by weakening the condition of cocommutativity.

A bialgebra B (or Hopf algebra H) over a field $\mathbb{K}$ is said to be almost cocommutative if there is an invertible element $\mathcal{R} \in B \otimes B$ such that

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{1.23}$$

for all $a \in B$. Let us denote such an element $\mathcal{R}$ (and its inverse) as

$$\mathcal{R} = \sum_i x_i \otimes y_i, \quad \mathcal{R}^{-1} = \sum_i x'_i \otimes y'_i. \tag{1.24}$$

For convenience, we denote by $\mathcal{R}_{ij} \in B^{\otimes N}$, the element of the N-fold tensor product which contains the first (second) component of $\mathcal{R}$ in the i-th (j-th) tensor component. With this notation, we make the following the series of definitions: An almost cocommutative bialgebra (Hopf algebra) is said to be

1. Coboundary, if it satisfies $\mathcal{R}_{21} = \mathcal{R}^{-1}$ and $(\epsilon \otimes \epsilon)(\mathcal{R}) = 1$;

2. Quasitriangular if

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{1.25}$$

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}; \tag{1.26}$$

3. Triangular if it is Quasitriangular and satisfies $\mathcal{R}_{21} = \mathcal{R}^{-1}$.

An element $\mathcal{R}$ satisfying (1.23), (1.25) and (1.26) is called a universal R-matrix. If H is a quasitriangular Hopf algebra, it is immediately clear that given a universal R-matrix $\mathcal{R}$, the element $\mathcal{R}_{21}^{-1}$ is also a universal R-matrix. Property (1.23) for $\mathcal{R}_{21}^{-1}$ follows by rewriting equation (1.23) for $\mathcal{R}$ and applying a permutation

$$\Delta(a) = \mathcal{R}^{-1}\Delta^{\mathrm{op}}(a)\mathcal{R} \overset{\sigma_{(12)}}{\Longrightarrow} \Delta^{\mathrm{op}}(a) = \mathcal{R}_{21}^{-1}\Delta(a)\mathcal{R}_{21}. \tag{1.27}$$

Property (1.26) follows by using multiplicativitity of the co-product to commute taking an inverse with $(\Delta \otimes \mathrm{id})$ and then applying a permutation

$$(\Delta \otimes \mathrm{id})(\mathcal{R}^{-1}) = (\mathcal{R}_{13}\mathcal{R}_{23})^{-1} = \mathcal{R}_{23}^{-1}\mathcal{R}_{13}^{-1} \overset{\sigma_{(123)}}{\Longrightarrow} (\mathrm{id} \otimes \Delta)(\mathcal{R}_{21}^{-1}) = \mathcal{R}_{31}^{-1}\mathcal{R}_{21}^{-1} = (\mathcal{R}_{21}^{-1})_{13}(\mathcal{R}_{21}^{-1})_{12}, \tag{1.28}$$

and (1.25) follows similarly. We can also prove that $\mathcal{R}_{21}$ and $\mathcal{R}^{-1}$ are universal R-matrices for H_{op} and H^{op} using similar tactics. The introduction of quasitriangular Hopf algebras, specifically as universal envoloping algebras of classical and affine Lie algebras known as quantum groups, is the work of Russian mathematician V.G. Drinfeld [2]. The following major theorem reveals the power of these structures:

If H is a quasitriangular Hopf algebra with universal R-matrix, $\mathcal{R}$, then the following hold

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{1.29}$$

$$(\epsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \epsilon)(\mathcal{R}) = 1, \tag{1.30}$$

$$(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\mathrm{id} \otimes S^{-1})(\mathcal{R}), \tag{1.31}$$

$$(S \otimes S)(\mathcal{R}) = \mathcal{R}. \tag{1.32}$$

These results are relatively simple to demonstrate

$$\begin{aligned}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) &= \sum_i (\mathcal{R}\Delta(x_i)) \otimes y_i = \sum_i (\Delta^{\mathrm{op}}(x_i)\mathcal{R}) \otimes y_i \\ &= (\sigma_{(12)} \circ (\Delta \otimes \mathrm{id})(\mathcal{R}))\mathcal{R}_{12} = (\sigma_{(12)}(\mathcal{R}_{13}\mathcal{R}_{23}))\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.\end{aligned} \tag{1.33}$$

For (1.30) we use the counit property and (1.26)

$$\mathcal{R} = (\mathrm{id} \otimes \epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta)(\mathcal{R}) = (\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{12}) = \mathcal{R}(\sum_i \epsilon(y_i)x_i \otimes 1). \tag{1.34}$$

Uniqueness of the unit then gives $1 = \sum_i \epsilon(y_i)x_i = (\mathrm{id} \otimes \epsilon)(\mathcal{R})$ as desired (a similar argument works for $(\epsilon \otimes \mathrm{id})(\mathcal{R})$). Now we verify (1.31) as follows

$$\begin{aligned}(S \otimes \mathrm{id})(\mathcal{R}) \cdot \mathcal{R} &= (\mu \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23}) = (\mu \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R}) \\ &= ((\mu \circ (S \otimes \mathrm{id}) \circ \Delta) \otimes \mathrm{id})(\mathcal{R}) = (\eta \otimes \mathrm{id})(\epsilon \otimes \mathrm{id})(\mathcal{R}) = \eta(1) \otimes 1 = 1_{H \otimes H}. \quad \text{(Use (1.30).)}\end{aligned}$$

Then since H^{op} is also a Hopf algebra with universal R-matrix R_{21} and antipode S^{-1}, we permute the above result for this algebra to get

$$(S^{-1} \otimes \mathrm{id})(\mathcal{R}_{21}) = \mathcal{R}_{21}^{-1} \overset{\sigma_{(12)}}{\Longrightarrow} (\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}. \tag{1.35}$$

It is clear that (1.32) now follows as a consequence of (1.31)

$$(S \otimes S)(\mathcal{R}) = (\mathrm{id} \otimes S) \circ (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S) \circ (\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \mathcal{R}. \tag{1.36}$$

The equation (1.29) here is the famous Yang-Baxter equation. It was invented by C.N. Yang as a self consistency condition for relativistic, purely elastic scattering [5] and by R.J. Baxter as a sufficient condition for commutativity of transfer matrices in 2-dimensional lattice models [1]. It is a well studied physical equation with beautiful solutions under pinned by rich theory.

The final result we will prove about quasitriangular Hopf algebras is another important one. It contains an explicit description of the map S^2 and produces a Casimir element (a central element). We start with the element u defined by

$$u = \mu(S \otimes \mathrm{id})(\mathcal{R}_{21}) = \sum_i S(y_i)x_i. \tag{1.37}$$

We will show that u is invertible soon however, it will be more convenient to show this after we have seen that $ua = S^2(a)u$. To do this we consider the following

$$\begin{aligned}
\mu_{\rm op}\circ(\mathrm{id}\otimes\mu_{\rm op})\circ(\mathrm{id}\otimes S\otimes S^2)(\mathcal{R}_{12}(\Delta\otimes\mathrm{id})(\Delta(a))) &= \mu_{\rm op}(\sum_{i,(a)} x_i a^{(1)}\otimes\mu_{\rm op}((S(y_i a^{(2)})\otimes S^2(a^{(3)})))\\
&= \mu_{\rm op}(\sum_{i,(a)} x_i a^{(1)}\otimes S(S(a^{(3)}))S(y_i a^{(2)}))\\
&= \mu_{\rm op}(\sum_{i} x_i a^{(1)}\otimes S(y_i a^{(2)}S(a^{(3)})))\\
&= \mu_{\rm op}(\mathrm{id}\otimes S)\left(\sum_i (x_i\otimes y_i)\left(\sum_{(a)} a^{(1)}\otimes a^{(2)}S(a^{(3)})\right)\right)\\
&= \mu_{\rm op}(\mathrm{id}\otimes S)\left(\sum_i (x_i\otimes y_i)\left(\sum_{(a)} \epsilon(a^{(2)})a^{(1)}\otimes 1\right)\right)\\
&= \mu_{\rm op}(\mathrm{id}\otimes S)(\sum_i (x_i\otimes y_i)(a\otimes 1)) = S(y_i)x_i a = ua,
\end{aligned} \tag{1.38}$$

yet on the other hand we can use the defining property of the R-matrix to find

$$\begin{aligned}
ua &= \mu_{\rm op}\circ(\mathrm{id}\otimes\mu_{\rm op})\circ(\mathrm{id}\otimes S\otimes S^2)(\mathcal{R}_{12}(\Delta\otimes\mathrm{id})(\Delta(a)))\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)\circ(\mu_{\rm op}\otimes\mathrm{id})\circ(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathcal{R}_{12}(\Delta\otimes\mathrm{id})(\Delta(a)))\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)\circ(\mu_{\rm op}\otimes\mathrm{id})\circ(\mathrm{id}\otimes S\otimes\mathrm{id})((\Delta^{\rm op}\otimes\mathrm{id})(\Delta(a))\mathcal{R}_{12})\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)(\sum_{i,(a)}\mu_{\rm op}(a^{(2)}x_i\otimes S(a^{(1)}y_i))\otimes a^{(3)})\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)(\sum_{i,(a)} S(y_i)S(a^{(1)})a^{(2)}x_i\otimes a^{(3)})\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)\left(\sum_i (S(y_i)\otimes 1)\left(\sum_{(a)} S(a^{(1)})a^{(2)}\otimes a^{(3)}\right)(x_i\otimes 1)\right)\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)\left(\sum_i (S(y_i)\otimes 1)\left(\sum_{(a)} 1\otimes\epsilon(a^{(1)})a^{(2)}\right)(x_i\otimes 1)\right)\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)\sum_i S(y_i)x_i\otimes a = \mu_{\rm op}\circ(\mathrm{id}\otimes S^2)u\otimes a = S^2(a)u.
\end{aligned} \tag{1.39}$$

This gives $ua = S^2(a)u$ as desired. Using this we now show that u has the inverse $v = \mu\otimes(\mathrm{id}\otimes S^2)(\mathcal{R}_{21}) = \sum_i y_i S^2(x_i)$:

$$\begin{aligned}
uv &= \sum_i u y_i S^2(x_i) = \sum_i S^2(y_i)uS^2(x_i) \overset{(1.32)}{=} \sum_i S(y_i)uS(x_i) = \sum_{i,j} S(y_i)S(y_j)x_jS(x_i)\\
&= \sum_{i,j} S(y_jy_i)x_jS(x_i) = \mu_{\rm op}\circ(\mathrm{id}\otimes S)(\sum_{i,j} x_jS(x_i)\otimes y_jy_i)\\
&= \mu_{\rm op}\circ(\mathrm{id}\otimes S)((\sum_j x_j\otimes y_j)(\sum_i S(x_i)\otimes y_i)) = \mu_{\rm op}\circ(\mathrm{id}\otimes S)(\mathcal{R}(S\otimes\mathrm{id})(\mathcal{R}))\\
&\overset{(1.31)}{=} \mu_{\rm op}\circ(\mathrm{id}\otimes S)(\mathcal{R}\mathcal{R}^{-1}) = \mu_{\rm op}(\mathrm{id}\otimes S)(1) = 1.
\end{aligned} \tag{1.40}$$

Then the result $1 = uv = S^2(v)u$ gives a left inverse automatically, i.e. $v = u^{-1}$. Thus we have determined

$S^2(a) = uau^{-1}$ for all $a \in H$. We use this to now show that $S(u)u$ is a central element

$$aS(u)u = S(b)S(u)u = S(ub)u = S(S^2(b)u)u = S(u)S(S^2(b))u = S(u)S^2(S(b))u = S(u)uau^{-1}u = S(u)ua, \tag{1.41}$$

where we have used the fact that S is a bijection to write $a = S(b)$ for some $b \in H$. Taking $a = u^{-1}$ in (1.41) shows that $S(u)u = uS(u)$. This central element $uS(u)$ is often referred to as the quantum Casimir element of the quasitriangular Hopf algebra H.

2 Representation Theory

In this section we discuss the representation theory of algebras as motivation for the previous definitions. In the introduction, we started by defining algebras with the language of commutative diagrams. We then dualised this process in defining coalgebras and introduced bialgebras so that we may have both algebras and coalgebras simultaneously. In order to relate these two structures of a bialgebra we introduced Hopf algebras and finally to generalise cocommutativity we introduced (quasi)triangular Hopf algebras. A small study of these objects revealed they were non-trivial with rich theory. However, it may not seem apparent why one would embark on this journey in the first place. Representation theory provides an answer for this.

Given a vector space V, a representation of an algebra A (both over the same field $\mathbb{K}$) on V is an algebra homomorphism $\rho : A \to \mathrm{End}(V)$, i.e. we can identify an element of our algebra A with a linear transformation of V in a way which respects the multiplicative structure of A. Linear transformations of a vector space form an algebra with composition as the product. Given a representation of A on V it is actually completely equivalent to regard V as an A-module with A action given by

$$a \cdot v = \rho(a)(v). \tag{2.1}$$

Often module language is simpler to use, so we use them interchangeably. A homomorphism of representations, otherwise an A-module or A-linear map, is a linear map $f : V \to W$ which commutes with A action in the following sense

$$\rho_W(a) \circ T = T \circ \rho_V(a). \tag{2.2}$$

A-module maps are the natural structure preserving maps between representations of A.

Given an A-module V, a vector subspace $W \subset V$ is a A-submodule of V if it is invariant under A action, that is for any $w \in W$ and $a \in A$ we have $a \cdot w \in W$. An A-module V is irreducible if its only submodules are $\{0\}$ and V. Given two A-modules V and W, we can make the direct sum $V \oplus W$ into an A-module with the action

$$a \cdot (v \oplus w) = a \cdot v \oplus a \cdot w, \quad \text{or} \quad \rho_{V\oplus W}(a) = \rho_V(a) \oplus \rho_W(a). \tag{2.3}$$

Given an algebra A, the classification of irreducible representations of A is useful as these are in some sense building blocks representations of A; we can form many more representations from their direct sums. However, given two vector spaces V and W on which A has a representation (say ρ_V and ρ_W respectively), there are many associated vector spaces we can build other than just their direct sum and it is natural to ask if there are representations of A on these spaces. Two such vector spaces include the tensor product $V \otimes W$, and the space of linear maps $\mathrm{Hom}(V, W)$ (including the dual space V^* as a special case when $W = \mathbb{K}$). However, in general there is no way to form representations of A on these spaces. In fact there is no trivial representation in general, that is, a representation on the 1 dimensional vector space $\mathbb{K}$.

Instead of working with a completely general algebra A let us insist on some more structure, namely suppose that $H = (H, \eta, \mu, \epsilon, \Delta)$ is a Hopf algebra with antipode S. We observe now that this extra structure is precisely what is needed to define these elusive representations! We start by noting that the counit $\epsilon : H \to \mathbb{K}$ is precisely a trivial representation of H. We construct a representation of H on $V \otimes W$ as follows

$$\rho_{V\otimes W}(a) = (\rho_V \otimes \rho_W)\Delta(a). \tag{2.4}$$

That Δ is an algebra homomorphism is precisely what we need to ensure that $\rho_{V\otimes W}$ is a representation. The coassociativity and counit properties of Δ and ϵ ensure that the canonical vector space isomorphisms

$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ and $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$ are A-module maps. Next we construct a representation of H on the space $\mathrm{Hom}(V, W)$ according to the action

$$\rho(a)(T) = \sum_{(a)} \rho_W(a^{(1)}) \circ T \circ \rho_V(S(a^{(2)}), \tag{2.5}$$

for $T : V \to W$ linear. This is a homomorphism due to the anti-multiplicative nature of the antipode

$$\begin{aligned}\rho(a)\rho(b)(T) &= \sum_{(a),(b)} \rho_W(a^{(1)})\rho_W(b^{(1)}) \circ T \circ \rho_V(S(b^{(2)}))\rho_V(S(a^{(2)})) = \sum_{(a),(b)} \rho_W(a^{(1)}b^{(1)}) \circ T \circ \rho_V(S(a^{(2)}b^{(2)})) \\ &= \sum_{(ab)} \rho_W((ab)^{(1)}) \circ T \circ \rho_V(S((ab)^{(2)})) = \rho(ab)(T).\end{aligned} \tag{2.6}$$

In particular, the representation on the dual space is

$$\rho_{V^*}(a)(\phi)(v) = \sum_{(a)} \epsilon(a^{(1)})\phi(\rho_V(S(a^{(2)}))(v)) = \phi(\rho_V(S(\sum_{(a)} \epsilon(a^{(1)})a^2))(v)) = \phi(\rho_V(S(a))(v)), \tag{2.7}$$

for $\phi \in V^*, v \in V$. Thus we see that Hopf algebras are exactly the structures which let us define representations on the desired vector spaces.

Let us note that the vector spaces $V \otimes W$ and $W \otimes V$ are isomorphic, as vector spaces, via the permutation map $\sigma_{V,W}(v \otimes w) = w \otimes v$. Furthermore, notice that $\sigma_{W,V} \circ \sigma_{V,W}. = \mathrm{id}$ which loosely says that the permutation map squares to the identity. Although this map is $\mathbb{K}$-linear, it is not H-linear in general:

$$\sigma_{V,W} \circ \rho_{V \otimes W}(a)(v \otimes w) = \sum_{(a)} \rho_W(a^{(2)})(w)) \otimes (\sum_{(a)} \rho_V(a^{(1)})(v)) = (\rho_W \otimes \rho_V)(\Delta^{\mathrm{op}}(a))(w \otimes v) \neq \rho_{W \otimes V}(a) \circ \sigma_{V,W}(v \otimes w). \tag{2.8}$$

Clearly it will be H-linear if H is cocommutative, but as we have already seen such an assumption is too strong. Instead, now suppose that H is a quasitriangular (or triangular) Hopf algebra with universal R-matrix $\mathcal{R} \in H \otimes H$ and define a linear map $c_{V,W} : V \otimes W \to W \otimes V$ as follows

$$c_{V,W} = \sigma_{V,W} \circ (\rho_V \otimes \rho_W)(\mathcal{R}). \tag{2.9}$$

Since both $\sigma_{V,W}$ and $(\rho_V \otimes \rho_W)(\mathcal{R})$ are vector space isomorphisms (clear from invertibility of $\mathcal{R}$), it follows that $c_{V,W}$ is a vector space isomorphism. However, it is also H-linear as we now demonstrate

$$\begin{aligned}c_{V,W} \circ \rho_{V \otimes W}(a) &= \sigma_{V,W}\left[(\rho_V \otimes \rho_W(\mathcal{R})) \cdot (\rho_V \otimes \rho_W(\Delta(a)))\right] \\ &= \sigma_{V,W}\left[(\rho_V \otimes \rho_W)(\mathcal{R}\Delta(a))\right] = \sigma_{V,W}\left[(\rho_V \otimes \rho_W)(\Delta^{\mathrm{op}}(a)\mathcal{R})\right] \\ &= (\rho_W \otimes \rho_V)(\Delta(a)) \circ \sigma_{V,W}((\rho_V \otimes \rho_W(\mathcal{R})) = \rho_{W \otimes V} \circ c_{V,W}.\end{aligned} \tag{2.10}$$

It can be checked directly that $c_{W,V} \circ c_{V,W} = (\rho_V \otimes \rho_W)(\mathcal{R}_{21}\mathcal{R})$. This means that for triangular Hopf algebras we preserve the squaring to identity property of the permutation map, yet this is not the case for quasitriagular Hopf algebras. For quasitriangular we can think of the map $c_{V,W}$ as braiding[2] the representations V and W. As shown in [3] this braiding is natural in that for A-module maps $f : V \to V'$ and $g : W \to W'$, $c_{V,W}$ and $c_{V',W'}$ obey

$$c_{V',W'} \circ (f \otimes W) = (g \otimes f) \circ c_{V,W}. \tag{2.11}$$

A consequence of this naturality and the defining properties (1.25) and (1.26) is that for H-modules U, V and W, we have

$$(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}). \tag{2.12}$$

[2]This can be made more precise by introducing representations of the braid group on the n-fold tensor product $V^{\otimes n}$ but this will not be necessary for our purposes.

Finally taking $U = V = W$ and setting $\hat{R} = \sigma \circ (\rho_V \otimes \rho_V)(\mathcal{R})$ in the above we arrive at an equivalent form of the Yang-Baxter equation

$$\hat{R}_{12} \circ \hat{R}_{23} \circ \hat{R}_{12} = \hat{R}_{23} \circ \hat{R}_{12} \circ \hat{R}_{23}, \tag{2.13}$$

and thus we have a method of generating solutions to the Yang-Baxter equation.

Before we consider a concrete example of a Hopf algebra in the next section, we will briefly consider the role of Casimir elements in representation theory. If $c \in H$ is a Casimir element and V is an irreducible H module, then the map $\rho_V(c) : V \to V$ is an A-module map since $\rho_V(c) \circ \rho_V(a) = \rho_V(ca) = \rho_V(ac) = \rho_V(a) \circ \rho_V(c)$ for all $a \in H$. Working over an algebraically closed field $\mathbb{K}$ guarantees the existence of an eigenvector $\rho_V(c)(v) = \lambda v$ for some $v \in V, \lambda \in \mathbb{K}$. Since linear combinations of A module maps are also A-module maps it follows that $(\rho_V(c) - \lambda \cdot \mathrm{id}_V) : V \to V$ is such a map, and furthermore it has a non-trivial kernel. It is clear that kernels of A-module maps on V are submodules of V since A action commutes with these maps. As such $\mathrm{Ker}((\rho_V(c) - \lambda \mathrm{id}_V))$ is a submodule of V and hence by irreducibility must be all of V as it is non-trivial. This says exactly that $\rho_V(c) = \lambda \cdot \mathrm{id}_V$ for any Casimir element $c \in H$.

3 The Hopf Algebra $U_q(\mathfrak{sl}_2)$

In the previous two sections we examined some introductory theory of Hopf algebras, in particular quasitriangular Hopf algebras. Whilst the theory is extremely rich, this report would be too dry without examining a concrete example. That is the purpose of this section. For a complex parameter $q \in \mathbb{C} \setminus \{0, \pm 1\}$ we define $U_q(\mathfrak{sl}_2)$ to be the algebra (over $\mathbb{C}$) generated by E, F, K and K^{-1} subject to the following commutation relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \tag{3.1}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{3.2}$$

If we introduce the following q-deformed integer and factorial notation

$$[n] := [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [1][2] \dots [n], \tag{3.3}$$

then we can show by induction, the following commutation relations for powers of E and F

$$[E^n, F] = [n]E^{n-1}\frac{(q^{n-1}K - q^{1-n}K^{-1})}{q - q^{-1}}, \quad [E, F^m] = [m]\frac{(q^{m-1}K - q^{1-m}K^{-1})}{q - q^{-1}}F^{m-1}. \tag{3.4}$$

For example we will the first of these relations. The base case $n = 1$ is exactly (3.2). Now suppose that the formula holds for all $k < n$ and observe

$$\begin{aligned} [E^n, F] &= E^{n-1}[E, F] + [E^{n-1}, F]E = E^{n-1}\frac{(K - K^{-1})}{q - q^{-1}} + E^{n-2}[n-1]\frac{(q^{n-2}K - q^{2-n}K^{-1})}{q - q^{-1}}E \\ &\overset{KE = q^2EK}{=} \frac{E^{n-1}}{q - q^{-1}}\left((1 + [n-1]q^n)K - (1 + [n-1]q^{-n})K^{-1}\right) \end{aligned} \tag{3.5}$$

It is then a simple matter of checking that

$$1 + q^{\pm n}[n-1] = \frac{q - q^{-1} + q^{\pm n}(q^{n-1} - q^{1-n})}{q - q^{-1}} = \pm\frac{q^{\pm(2n-1)} - q^{\mp}}{q - q^{-1}} = q^{\pm(n-1)}[n], \tag{3.6}$$

giving the desired result in (3.5). With these commutation relations, any element $x \in U_q(\mathfrak{sl}_2)$ which is in general a linear combination of words in E, F, K and K^{-1} can be rearranged into a unique sum of terms of the form $E^nK^mF^l$ for $n, l \in \mathbb{N}$ and $m \in \mathbb{Z}$, that is $\{E^nK^mF^l \mid n, l \in \mathbb{N}; m \in \mathbb{Z}\}$ is a basis (a Poincare-Birkhoff-Witt basis to be precise) for $U_q(\mathfrak{sl}_2)$.

As alluded to, $U_q(\mathfrak{sl}_2)$ is not just an algebra but in fact a Hopf algebra (In fact a quasitriangular Hopf algebra!). To see this we need to introduce counit, coproduct and antipode maps. We start with the counit and coproduct. These are defined by linearly and multiplicatively extending the following action on generators

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F \tag{3.7}$$

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0. \tag{3.8}$$

It is a simple matter of checking that these satisfy all the bialgebra relations on all generators. For example we check coassociativity on E and we check that the coproduct respects the formula $KF = q^{-2}FK$

$$\begin{aligned}(\Delta \otimes \mathrm{id})(\Delta(E)) &= \Delta(E) \otimes K + \Delta(1) \otimes K = E \otimes K \otimes K + 1 \otimes E \otimes K + 1 \otimes 1 \otimes E \\ (\mathrm{id} \otimes \Delta)(\Delta(E)) &= E \otimes \Delta(K) + 1 \otimes \Delta(E) = E \otimes K \otimes K + 1 \otimes E \otimes K + 1 \otimes 1 \otimes E = (\Delta \otimes \mathrm{id})(\Delta(E)),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\Delta(KF) &= \Delta(K)\Delta(F) = (K \otimes K)(F \otimes 1 + K^{-1} \otimes F) = KF \otimes K + 1 \otimes KF \\ &= q^{-2}(FK \otimes K + 1 \otimes FK) = q^{-2}(F \otimes 1 + K^{-1} \otimes F)(K \otimes K) = q^{-2}\Delta(F)\Delta(K) = \Delta(q^{-2}FK).\end{aligned} \tag{3.10}$$

The antipode is now uniquely determined by its defining property:

$$\eta \circ \epsilon(K) = 1 = \mu \circ (S \otimes \mathrm{id}) \circ \Delta(K) = S(K)K \Rightarrow S(K) = K^{-1} \Rightarrow S(K^{-1}) = (S(K))^{-1} = (K^{-1})^{-1} = K, \tag{3.11}$$

$$\eta \circ \epsilon(E) = 0 = \mu \circ (\mathrm{id} \otimes S) \circ \Delta(E) = E \cdot S(K) + 1 \cdot S(E) = EK^{-1} + S(E) \Rightarrow S(E) = -EK^{-1}, \tag{3.12}$$

$$\mu \circ \epsilon(F) = 0 = \mu \circ (S \otimes \mathrm{id}) \circ \Delta(F) = S(F) \cdot 1 + S(K^{-1}) \cdot F = S(F) + KF \Rightarrow S(F) = -KF. \tag{3.13}$$

Thus we have equipped $U_q(\mathfrak{sl}_2)$ with a Hopf algebra structure.

An equivalent definition of the algebra $U_q(\mathfrak{sl}_2)$ is the so called $U_h(\mathfrak{sl}_2)$ algebra for a parameter $h \neq 0 \in \mathbb{C}$. It is generated by the elements E, F and H subject to the relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}} = \frac{\sinh(hH)}{\sinh(h)}, \tag{3.14}$$

where we understand exponentials as formal power series. In this form it is easier to see the correspondence to the classical Lie algebra $\mathfrak{sl}_2$; in the limit $h \to 0$ the above commutation relations are exactly those for the Lie algebra $\mathfrak{sl}_2$. It is also clearer that $U_h(\mathfrak{sl}_2)$ is a universal enveloping algebra for $\mathfrak{sl}_2$ and hence a quantum group. We go between $U_h(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ via the identification

$$q = e^h, \quad K = e^{hH} = q^H. \tag{3.15}$$

Another reason for introducing this equivalent description is that we have a nice formula for the universal R-matrix which makes $U_q(\mathfrak{sl}_2)$ into a quasitriangular Hopf algebra

$$\mathcal{R} := e^{h(H \otimes H)/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(1 - q^{-2})}{[n]_q!} E^n \otimes F^n. \tag{3.16}$$

We present this without proof of the properties (1.23), (1.25) and (1.26). We should remark that the above is a formal power series which belongs to the proper completion of the tensor product $U_h(\mathfrak{sl}_2) \otimes U_h(\mathfrak{sl}_2)$. If we were to try to write this element in the $U_q(\mathfrak{sl}_2)$ formulation in terms of K we would fail; there is no way to write it in rational functions of q.

3.1 Representation Theory of $U_q(\mathfrak{sl}_2)$

In this section we classify all finite dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$ when q is not a root of unity. This process is essentially identical to that of the classical Lie algebra $\mathfrak{sl}_2$. Suppose that V is a d-dimensional irreducible $U_q(\mathfrak{sl}_2)$ module. Since V is a complex vector space we are guaranteed a non-zero eigenvector $v \in V$ such that $H \cdot v_\lambda = \lambda v_\lambda$ for some $\lambda \in \mathbb{C}$. This gives $K.v_\lambda = q^H.v_\lambda = q^\lambda \cdot v_\lambda$ and we will say that v_λ has weight λ. Now if we act by F on v observe that we lower the weight

$$K \cdot (F \cdot v_\lambda)) = (KF) \cdot v_\lambda = q^{-2} FK \cdot v_\lambda = q^{\lambda-2} F v_\lambda. \tag{3.17}$$

Repeating this process we generate a set of eigenvectors with non-degenerate eigenvalues (this is guaranteed since q is not a root of unity). By finite dimensionality this process must terminate meaning that there is some lowest weight vector $w_1 := F^{l_1} v_\lambda \neq 0 \in V$ with weight $\lambda - 2l_1$ for some $l_1 \in \mathbb{N}$ such that $F \cdot w_1 = 0$. The same process shows that E raises the weight $K \cdot (Ev_\lambda) = q^{\lambda+2}(Ev_\lambda)$ and we come to the conclusion that there is some highest weight vector $w_2 := E^{l_2} v_\lambda$ with weight $\lambda + 2l_2$ for some $l_2 \in \mathbb{N}$ such that $E \cdot w_2 = 0$. Then $\{F^{l_2} v_\lambda, \dots, F v_\lambda, v_\lambda, E v_\lambda, \dots, E^{l_2} v_\lambda\}$ is a set of eigenvectors for K. Uniqueness of eigenvalues ensures this set is linearly independent and its span is invariant under $U_q(\mathfrak{sl}_2)$ action, i.e. it is a non-trivial submodule so by irreducibility it spans V. Thus $\{F^{l_1} v_\lambda, \dots, F v_\lambda, v_\lambda, E v_\lambda, \dots, E^{l_2} v_\lambda\}$ is a basis for V and hence $l_1 + l_2 + 1 = d$.

Let us now call the highest weight $\nu := \lambda + 2l_2$. Labelling eigenvectors by their weight we know we can form a K eigenbasis of V, $\{v_{\nu-2(d-1)}, \dots, v_\lambda, \dots, v_\nu\}$, but we require a definition for weight vectors and need to require that the commutation relation (3.2) is met. Define weight vectors by

$$E \cdot v_{\alpha-2} = a(\alpha, \nu) v_\alpha, \quad F \cdot v_\alpha = a(\alpha, \nu) v_{\alpha-2}, \tag{3.18}$$

for some set of coefficients $a(\alpha, \nu)$. Then the commutation relation (3.2) impose the recurrence relation $a(\alpha, \nu)^2 + a(\alpha + 2, \nu)^2 = [\alpha]$ with boundary $a(\nu, \nu)^2 = [\nu]$ as shown in [4]. This has the solution

$$a(\alpha, \nu) = \sqrt{\left[\frac{\nu + \alpha}{2}\right] \left[\frac{\nu - \alpha}{2} + 1\right]}. \tag{3.19}$$

Finite dimensionality requires that $a(\alpha, \nu)$ is 0 for small enough α and from (3.19) this is only the case when $\alpha = -\nu$. This means the lowest weight is $\nu - 2(d-1) = -\nu \Rightarrow \nu = d - 1$. Thus we have determined that the d-dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$ are (up to change a basis) exactly the $U_q(\mathfrak{sl}_2)$-modules, V with basis $\{v_{1-d}, v_{3-d}, \dots, v_{d-3}, v_{d-1}\}$ and action is defined by

$$K \cdot v_\alpha = q^\alpha v_\alpha, \quad E \cdot v_{\alpha-2} = \sqrt{\left[\frac{d+1+\alpha}{2}\right] \left[\frac{d+3-\alpha}{2}\right]} v_\alpha, \quad F \cdot v_\alpha = \sqrt{\left[\frac{d+1+\alpha}{2}\right] \left[\frac{d+3-\alpha}{2}\right]} v_{\alpha-2}. \tag{3.20}$$

3.2 Root of Unity Case

In this section we study the algebra $U_q(\mathfrak{sl}_2)$ at its most elusive, that is when q is a root of unity. Suppose that e is the smallest integer such that $q^e = \pm 1$. Then we will show that the elements E^e, F^e and K^e are central. For K^e this is clear from the defining commutation relations with generators E and F

$$K^e E = q^{2e} E K^e = E K^e, \quad K^e F = q^{-2e} F K^e = F K^e. \tag{3.21}$$

Showing that E^e and F^e commute with the generator K is similar. The fact that that E^e (F^e) commutes with the generator F (E) is immediate from (3.4) noting that

$$[e] = \frac{q^e - q^{-e}}{q - q^{-1}} = \frac{q^{-e}(q^{2e} - 1)}{q - q^{-1}} = 0. \tag{3.22}$$

This finishes the claim that E^e, F^e and K^e are central.

Let us now remark that the representation (3.20) is still a valid representation for $d < e$. This is because in section 3.1, the assumption that q was not a root of unity was only used to conclude that eigenvalues were non-degenerate. The spectrum of K in (3.20) reveals that eigenvalues of different basis elements differ by a factor of q^{2l} where $1 \leq l \leq d - 1$. Thus when e is odd and $d < e$ it is clear that $2l$ is not a multiple of e hence $q^{2l} \neq \pm 1$. If e is even and $d < e$ then we require $q^e = -1$ so $q^{2l} \neq 1$ for any $1 < l < d - 1$. Thus section 3.1 shows that the irreducible representations of $U_q(\mathfrak{sl}_2)$ of dimension $d < e$ are exactly the representations (3.20).

Now let us examine the irreducible representations of $U_q(\mathfrak{sl}_2)$ of dimension $d > e$. In fact there are none! To show this, suppose for a contradiction that V is such a $U_q(\mathfrak{sl}_2)$-module. If there exists a non-zero eigenvector of K (say $K \cdot v = \kappa v$ for $\kappa \in \mathbb{C}, v \in V$) such that $F \cdot v = 0$, then we claim that the set $\mathcal{B} = \{v, E \cdot v, \ldots, E^{e-1} \cdot v\}$ spans a submodule of at most dimension e. To show this it suffices to show that the span is preserved under the action of the generators. As we saw in section 3.1 the defining commutation relations mean that applying E to an eigenvector of K produces another eigenvector and hence $\mathcal{B}$ is a set of eigenvectors for K, so K preserves its span. It is also clear that E action also preserves the span as $E(\mathcal{B}) = \{E \cdot v, E^2 \cdot v, \ldots, E^{e-1} \cdot v, E^e \cdot v\}$ and centrality of E^e implies that it acts as a multiple of the identity. Finally, $\mathcal{B}$ is preserved under F-action since,

$$FE^k \cdot v = E^k F \cdot v - [E^k, F] = 0 - [k]E^{k-1}\frac{(q^{k-1}K - q^{1-k}K^{-1})}{q - q^{-1}} \cdot v = \frac{[k](q^{1-k}\kappa^{-1} - q^{k-1}\kappa)}{q - q^{-1}} E^{k-1} \cdot v, \quad (3.23)$$

which is clearly in the span of $\mathcal{B}$ for $0 \leq k \leq e-1$. If there is no eigenvector v of K which vanishes under F, then starting with any eigenvector $w \in V$ of K (existence is guaranteed), similar arguments show that the set $\{w, F \cdot w, \ldots, F^{e-1} \cdot w\}$ spans a submodule of at most dimension e. Thus in either case we have derived a contradiction..

Finally we consider the most interesting case: e-dimensional representations of $U_q(\mathfrak{sl}_2)$. Instead of deriving it here, we will take the approach of simply stating what the e-dimensional representations are and verifying as in [3]. The claim is that for three complex parameters $\mu, a, b \in \mathbb{C}$ ($\mu \neq 0$) we can define a family of $U_q((sl)_2)$-modules $W(\mu; a, b)$ with basis $w_0, w_1, \ldots w_{e-1}$ and action

$$\begin{aligned}
K \cdot w_j &= \mu q^{-2j} w_j && \text{for all } j, \\
F \cdot w_j &= w_{j+1} && \text{for } 0 \leq j \leq e-2, \\
F \cdot w_{e-1} &= b w_0, \\
E \cdot w_j &= \left(ab + \frac{[j]_q}{q - q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\right) w_{j-1} && \text{for } 1 \leq j \leq e-1, \\
E \cdot w_0 &= a w_{e-1}.
\end{aligned} \quad (3.24)$$

The defining relations (3.1) are trivial to check, so we need only check (3.2). For $1 \leq j \leq e-2$ this is simple

$$\begin{aligned}
(EF - FE) \cdot w_j &= E w_{j+1} - F\left(ab + \frac{[j]_q}{q - q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\right) w_{j-1} \\
&= \left(ab + \frac{[j+1]_q}{q - q^{-1}}(\mu q^{-j} - \mu^{-1} q^{j}) - ab - \frac{[j]_q}{q - q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\right) w_j \\
&= \frac{1}{(q - q^{-1})^2}\left((q^{j+1} - q^{-j-1})(\mu q^{-j} - \mu^{-1} q^j) - (q^j - q^{-j})(\mu q^{1-j} - \mu^{-1} q^{j-1})\right) w_j \\
&= \frac{1}{(q - q^{-1})^2}\left(\mu q^{-2j}(q - q^{-1}) - \mu^{-1} q^{2j}(q - q^{-1})\right) w_j \\
&= \frac{\mu q^{-2j} - \mu^{-1} q^{2j}}{q - q^{-1}} = \frac{K - K^{-1}}{q - q^{-1}} w_j.
\end{aligned} \quad (3.25)$$

The $j = 0$ case is as follows

$$(EF - FE) w_0 = E w_1 - a F w_{e-1} = \left(ab + \frac{\mu - \mu^{-1}}{q - q^{-1}} - ab\right) w_0 = \frac{\mu - \mu^{-1}}{q - q^{-1}} w_0 = \frac{K - K^{-1}}{q - q^{-1}} w_0. \quad (3.26)$$

The case $j = e-1$ is similar so we omit it here. Since we know that E^e, F^e and K^e are all central, we know they act as multiples of the identity map. It is clear from the action described above that $F^e = b \cdot \mathrm{id}$ and $K^e = \mu^e \cdot \mathrm{id}$. The value of E^e is not as simple to write yet it can still be found by acting on w_0 (or any basis vector)

$$E^e = a \prod_{j=1}^{e-1}\left(ab + \frac{[j]_q}{q - q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\right) \cdot \mathrm{id}. \quad (3.27)$$

Finally we will note that while $b \neq 0$ the representation is irreducible since repeated F-action on a K-eigenvector can always generate the basis. However, suppose that $b = 0$. Then the vector w_{e-1} is invariant under F and K action. We may attempt to generate the basis by repeated E action, however if for some $1 \leq j \leq e-1$ we have

$$\frac{[j]_q}{q-q^{-1}}(\mu q^{1-j} - \mu^{-1}q^{j-1}) = 0 \Rightarrow (\mu q^{1-j} - \mu^{-1}q^{j-1}) = 0 \Rightarrow \mu^2 = q^{2(j-1)}, \tag{3.28}$$

then this process is prematurely stopped at the basis vector w_j; we have found a $U_q(\mathfrak{sl}_2)$ invariant subspace. That is for $b = 0$ and $\mu = \pm q^{j-1}$ $(1 \leq j \leq e-1)$, $W(\mu; a, b)$ is reducible with a non-trivial submodule spanned by $\{w_j, \ldots, w_{e-1}\}$.

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