Introduction to Quantum Groups - Appendix

ASC Report - SCNC3101

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1 Hopf Ideals and Quotients

When working with familiar algebraic objects (groups, rings, algebras, and vector spaces) a natural thing one may wish to consider are quotients of said objects, which comes with a universal mapping property. In order to define quotients of objects we need some notion of an ideal (normal subgroup, subspace etc). This process is again possible for Hopf algebras, however, we note that as algebraic structures become more complex so to do notions of their ideals.

Suppose that $H = (H, \eta, \mu, \epsilon, \Delta)$ is a Hopf algebra with antipode S. We say that a vector subspace $I \subset H$, is a Hopf ideal if it is:

1. a two-sided algebra ideal for the algebra (H, η, μ), that is,

$$\mu(H \otimes I) \subset I, \quad \mu(I \otimes H) \subset I, \tag{1.1}$$

2. a coideal for the coalgebra (H, ϵ, Δ), that is,

$$\Delta(I) \subset I \otimes H + H \otimes I, \quad \epsilon|_I = 0, \tag{1.2}$$

3. antipode invariant, that is,

$$S(I) \subset I. \tag{1.3}$$

If $I \subset H$ is a Hopf ideal we will write $I \lhd H$. As is the case with notions of ideals on more familiar objects, the kernels of Hopf algebra homomorphisms are Hopf ideals as we now show:

Let $f : H \to H'$ be a Hopf algebra homorphism and denote the vector subpace $\ker(f) \subset H$ by I (we will denote structural maps on H' with an apostrophe). Conditions (1.1), (1.2) and (1.3) are all basically immediate from the definitions of a Hopf algebra homomorphism. For (1.1) we note that for $a \in H, x \in I$ we have $f(\mu(a \otimes x)) = \mu'(f(a) \otimes f(x)) = \mu'(f(a) \otimes 0) = 0$ and likewise $f(\mu(x \otimes a)) = 0$. For (1.2) since $\epsilon = \epsilon' \circ f$ the second condition is immediate. Similarly for $x \in I$, $f \otimes f(\Delta(x)) = \Delta(f(x)) = 0$ which is only possible if $\Delta(x) \in I \otimes H + H \otimes I$. Finally, antipode invariance is clear from the condition $f \circ S = S' \circ f$.

We have now seen that the kernels of Hopf algebra homomorphisms are Hopf ideals. Now let us consider the converse question; given a Hopf ideal $I \lhd H$, can we find a homomorphism $f : H \to H'$ such that the kernel of f is exactly I? Of course in order to achieve this we need to find not only the map f but also the Hopf algebra H'. This is exactly what leads us to define the quotient of a Hopf algebra.

Let $I \lhd H$ be a Hopf ideal and let H/I be the vector space quotient, that is, the set of equivalence classes $x + I$. We then equip the quotient space H/I with a Hopf algebra structure $(H/I, \tilde{\eta}, \tilde{\mu}, \tilde{\epsilon}, \tilde{\Delta})$ and antipode $\tilde{S}$ defined by

$$\tilde{\eta}(c) = \eta(c) + I, \qquad \tilde{\mu}((x + I) \otimes (y + I)) = \mu(x \otimes y) + I, \tag{1.4}$$

$$\tilde{\epsilon}(x + I) = \epsilon(x), \qquad \tilde{\Delta}(x + I) = \sum_{(x)} (x^{(1)} + I) \otimes (x^{(2)} + I), \tag{1.5}$$

$$\tilde{S}(x + I) = S(x) + I \tag{1.6}$$

The fact that $\tilde{\epsilon}$ and $\tilde{S}$ are well defined is clear using $x+I = y+I \Leftrightarrow x-y \in I$. Then by (1.2) ϵ acts trivially on $x-y$ giving $\tilde{e}(x+I) = \tilde{e}(y+I)$, and similarly $S(x-y) \in I$ meaning exactly $\tilde{S}(x+I) = \tilde{S}(y+I)$. To see that $\tilde{\mu}$ is well defined in its first slot suppose that $x+I = x'+I$. Then since $x - x' \in I$ it follows that

$$\begin{aligned}\tilde{\mu}((x+I)\otimes(y+I)) - \tilde{\mu}((x'+I)\otimes(y+I)) = \mu((x-x')\otimes y) + I \overset{(1.1)}{=} 0 + I \\ \Rightarrow \tilde{\mu}((x+I)\otimes(y+I)) = \tilde{\mu}((x'+I)\otimes(y+I)).\end{aligned} \tag{1.7}$$

It is then immediate that $\tilde{\mu}$ is well defined in its second slot by taking $\mu = \mu_{\mathrm{op}}$. Finally, to see that $\Delta(x+I)$ is well defined suppose that $x+I = y+I \Leftrightarrow x-y \in I$. Then expanding $\Delta(x-y)$ using swindler notation, it follows from property (1.2) that in each summand either $(x-y)^{(1)} \in I$ or $(x-y)^{(2)} \in I$, meaning at least one component of each summand in the expansion of $\tilde{\Delta}(x-y+I)$ belongs to the equivalence class $0+I$. Thus by linearity $\tilde{\Delta}(x-y+I) = 0+I \Leftrightarrow \tilde{\Delta}(x+I) = \tilde{\Delta}(y+I)$ as desired.

We should also check that $(H/I, \tilde{\eta}, \tilde{\mu}, \tilde{\epsilon}, \tilde{\Delta})$ with antipode $\tilde{S}$ obey the Hopf algebra axioms, although this is almost immediate since it was true for $H = (H, \eta, \mu, \epsilon, \Delta)$. For example we check the defining property of the antipode

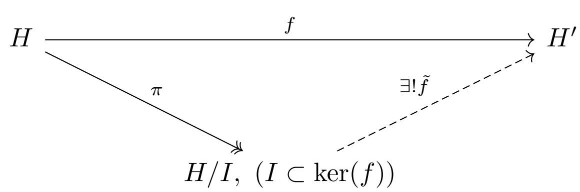
$$\tilde{\mu}\circ(\tilde{S}\otimes\mathrm{id})\circ\tilde{\Delta}(x+I) = \tilde{\mu}\left(\sum_{(x)}(S(x^{(1)})+I)\otimes(x^{(2)}+I)\right) = \mu\left(\sum_{(x)} S(x^{(1)})\otimes x^{(2)}\right) + I = \eta(\epsilon(x)) + I = \tilde{\eta}\circ\tilde{\epsilon}(x+I). \tag{1.8}$$

We now return to the question of finding a homomorphism $f : H \to H'$ with a kernel of $I \lhd H$ to find that all the work has been done. For given $I \lhd H$, denote by $\pi : H \to H/I$ the canonical project map $x \mapsto x+I$. The definitions (1.4)-(1.6) are exactly what it means for π to be a homomorphism. Now observe that

$$x \in \ker(\pi) \Leftrightarrow \pi(x) = x+I = 0+I \Leftrightarrow x \in I, \tag{1.9}$$

thus the kernel of π is exactly I as desired.

To conclude this section we mention the universal property of quotients. Suppose that $f : H \to H'$ is a Hopf algebra homomorphism. Then for any Hopf ideal $H \rhd I \subset \ker(f)$, there exists a unique homomorphism $\tilde{f} : H/I \to H'$ such that $f = \pi \circ \tilde{f}$. This is perhaps best understood by the following commutative diagram.



(1.10)

The map $\tilde{f}$ is defined by $x+I \mapsto f(x)$ and is manifestly unique from the diagram. The fact that it is well defined is checked easily using the property that $I \subset \ker(f)$, and the fact that it is a homomorphism is checked easily using the fact that f is a homomorphism. For example we show it respects multplication as follows

$$\tilde{f}\circ\tilde{\mu}((x+I)\otimes(y+I)) = \tilde{f}(\mu(x\otimes y)+I) = f(\mu(x\otimes y)) = \mu'(\tilde{f}(x)\otimes\tilde{f}(y)) = \mu'\circ(f\otimes f)((x+I)\otimes(y+I)). \tag{1.11}$$

2 The Algebra $U_q(\mathfrak{sl}_2)$

In this section we return to our study of the quantum group $U_q(\mathfrak{sl}_2)$ when q is a root of unity. We will show that in this case it has a finite dimensional quotient which is quasitriangular and from its representation theory, we will construct a solution to the Yang-Baxter equation. However, before this we will take a detour to prove a helpful result.

2.1 q-Binomial Theorem

Before we show that $U_q(\mathfrak{sl}_2)$ has a finite dimensional quotient, it will be useful to prove a generalisation of the binomial theorem. To do this we define the q-integer, $[\![n]\!]_q$, the q-factorial, $[\![n]\!]_q!$, and the q-binomial coefficient, $[\![{n \atop k}]\!]_q$, as follows

$$[\![n]\!]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}, \quad [\![n]\!]_q! = [\![1]\!]_q[\![2]\!]_q \dots [\![n]\!]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[\![n]\!]!_q}{[\![k]\!]!_q[\![n-k]\!]!_q}, \tag{2.1}$$

for some scalar parameter q (generally a complex number), and n a non-negative integer. We will note the following special cases:

$$[\![0]\!]_q = 0, \quad [\![1]\!]_q = 1, \quad [\![1]\!]_q! = [\![0]\!]_q! = 1, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1, \text{ for all } n. \tag{2.2}$$

The q-binomial theorem now says that for an algebra A over a field $\mathbb{K}$ with elements $a, b \in A$ satisfying $ab = qba$ for some $q \in \mathbb{K}$, the following holds

$$(a+b)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q b^j a^{n-j}. \tag{2.3}$$

This is easily verified by induction. The base case $n = 1$ is clear using the fourth identity in (2.2). Now we suppose that (2.3) holds for all $n < k$ for some $k > 1$ and observe

$$\begin{aligned}
(a+b)^k = (a+b)(a+b)^{k-1} &= \sum_{j=0}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q (a+b) b^j a^{k-1-j} \\
&= \sum_{j=0}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q q^j b^j a^{k-j} + \sum_{j=0}^{k-1} \begin{bmatrix} k-1 \\ j \end{bmatrix}_q b^{j+1} a^{k-(j+1)} \qquad \text{(Use } ab = qba\text{)} \\
&= a^k + \sum_{j=1}^{k-1} \left(q^j \begin{bmatrix} k-1 \\ j \end{bmatrix}_q + \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \right) b^j a^{k-j} + b^k \\
&= a^k + \sum_{j=1}^{k-1} \begin{bmatrix} k \\ j \end{bmatrix}_q b^j a^{k-j} + b^k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q b^j a^{k-j},
\end{aligned} \tag{2.4}$$

where in the last line we used the following identity

$$\begin{aligned}
q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_q + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q &= \frac{[\![n-1]\!]!_q}{[\![k]\!]!_q[\![n-k]\!]!_q} \left(q^j [\![n-j]\!]_q + [\![j]\!]_q \right) = \frac{[\![n-1]\!]!_q}{[\![k]\!]!_q[\![n-k]\!]!_q} \left(\frac{q^j(1-q^{n-j}) + 1 - q^j}{1-q} \right) \\
&= \frac{[\![n-1]\!]!_q}{[\![k]\!]!_q[\![n-k]\!]!_q} \left(\frac{1-q^n}{1-q} \right) = \frac{[\![n-1]\!]!_q [\![n]\!]}{[\![k]\!]!_q[\![n-k]\!]!_q} = \begin{bmatrix} n \\ j \end{bmatrix}_q.
\end{aligned} \tag{2.5}$$

Let us now explore an important consequence of (2.3). Suppose that q is a root of unity and let e be the smallest positive integer such that $q^e = 1$. It is then clear that $[\![ne]\!]_q = 0$ for any $n \in \mathbb{N}$ and that $[\![l]\!]!_q = 0$ if and only if $l \geq e$. It thus follows that

$$\begin{bmatrix} e \\ n \end{bmatrix}_q = 0, \text{ for any } 0 < n < e. \tag{2.6}$$

We do require that $n \neq 0$ or e since in this case the fourth identity in (2.2) still holds (as an indeterminate limit). A consequence of (2.6) and (2.3) is that if $ab = qba$ where q is a primitive e-th root of unity, then we have

$$(a+b)^e \overset{(2.3)}{=} \sum_{j=0}^{e} \begin{bmatrix} e \\ j \end{bmatrix}_q b^j a^{e-j} \overset{(2.6)}{=} a^e + b^e. \tag{2.7}$$

2.2 A Quotient of $U_q(\mathfrak{sl}_2)$

We recall briefly that for a complex parameter $q \in \mathbb{C}$ $U_q(\mathfrak{sl}_2)$ is the Hopf algebra generated by E, F, K, and K^{-1} with the relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.8}$$

The coproduct, Δ, and counit, ϵ, are as follows

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(K) = K \otimes K, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \tag{2.9}$$
$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0, \tag{2.10}$$

which uniquely determine the antipode, S,

$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF. \tag{2.11}$$

Now suppose that q is a root of unity and that e is the smallest number such that $q^e \in \{1, -1\}$. Now let I be the two sided algebra ideal generated by E^e, F^e, and $K^e - 1$. Let us show that I is a Hopf ideal. The property $\epsilon|_I = 0$ is almost immediate and not particular to the root of unity case

$$\epsilon(E^e) = \epsilon(E)^e = 0 = \epsilon(F)^e = \epsilon(F^e), \quad \epsilon(K^e - 1) = \epsilon(K)^e - \epsilon(1) = 1 - 1 = 0. \tag{2.12}$$

Similarly, antipode invariance is almost as immediate and again does not rely on q being a root of unity

$$S(K^e - 1) = S(K)^e - S(1) = K^{-e} - 1 = -K^{-e}(K^e - 1) \in I, \tag{2.13}$$
$$S(E^e) = S(E)^e = (-EK^{-1})^e = (-1)^e(EK^{-1}EK^{-1}\ldots EK^{-1}) \overset{(2.8)}{=} (-1)^e q^{e(e+1)} K^{-e}E^e \in I, \tag{2.14}$$
$$S(F^e) = S(F)^e = (-KF)^e = (-1)^e(KFKF\ldots KF) \overset{(2.8)}{=} (-1)^e q^{e(e-1)} K^e F^e \in I, \tag{2.15}$$

as is the case with the relation $\Delta(I) \subset I \otimes H + H \otimes I$ for the generator $K^e - 1$

$$\Delta(K^e - 1) = \Delta(K)^e - \Delta(1) = K^e \otimes K^e - 1 \otimes 1 = (K^e - 1) \otimes K^e + 1 \otimes (K^e - 1) \in I \otimes H + H \otimes I. \tag{2.16}$$

We now finally require the assumption that q is a root of unity to show the first condition in (1.2) is also true for E^e and F^e. Let us remark that with the given assumptions that q is a root of unity and e is the smallest positive integer such that $q^e \in \{1, -1\}$, it must be the case that q^2 is a primitve e-th root of unity. This follows from a simple contradiction argument. Suppose that q^2 is not a primitive e-th root of unity. Since it is necessarily an e-th root of unity, we must have some $n < e$ such that $q^{2n} = 1$. Minimality of e means that $2n > e$ so write $2n = e + k$ for some $0 < k < e$, which says that

$$1 = q^{2n} = q^{e+k} = \pm q^k, \tag{2.17}$$

contradicting the minimality of e. Since q^2 is a primitive e-th root of unity it follows that so too is q^{-2}. We now observe that

$$\Delta(E^e) = \Delta(E)^e = (E \otimes K + 1 \otimes E)^e \overset{(2.7)}{=} E^e \otimes K^e + 1 \otimes E^e \in I \otimes H + H \otimes I,$$
$$\Delta(F^e) = \Delta(F)^e = (F \otimes 1 + K^{-1} \otimes F)^e \overset{(2.7)}{=} F^e \otimes 1 + K^{-e} \otimes F^e \in I \otimes H + H \otimes I, \tag{2.18}$$

where the formula (2.7) applies since

$$(E \otimes K)(1 \otimes E) = E \otimes KE = q^2(E \otimes EK) = q^2(1 \otimes E)(E \otimes K), \tag{2.19}$$
$$(F \otimes 1)(K^{-1} \otimes F) = FK^{-1} \otimes F = q^{-2}(K^{-1}F \otimes F) = q^{-2}(K^{-1} \otimes F)(F \otimes 1), \tag{2.20}$$

and q^2 and q^{-2} are primitive e-th roots of unity.

Now that we have shown that I is a Hopf ideal, let us consider the quotient Hopf algebra $\widetilde{U_q(\mathfrak{sl}_2)} \equiv U_q(\mathfrak{sl}_2)/I$. We will construct a basis for $\widetilde{U_q(\mathfrak{sl}_2)}$ by projecting a basis for $U_q(\mathfrak{sl}_2)$ and removing redundancies.

Let $E^nK^mF^l \in U_q(\mathfrak{sl}_2)$ be a Poincaré-Birkhoff-Witt basis element. Under projection it is sent to $E^nK^mF^l + I \in \widetilde{U_q(\mathfrak{sl}_2)}$. If $n > e$ or $l > e$ we have $E^nK^mF^l \in I$ or $E^nK^mF^l + I = 0 + I$, so we can restrict to the case $0 \le n, l \le e-1$. Now observe the following

$$E^nK^mF^l + I = E^nK^mF^l + E^nK^m(K^e - 1)F^l + I = E^nK^{m+e}F^l + I. \tag{2.21}$$

By repeatedly applying the above relation, we see that we can reduce any power of K to its remainder modulo e. Thus the set $\mathcal{B} = \{E^nK^mF^l \mid 0 \le n, m, l \le e-1\}$ is a spanning set for $\widetilde{U_q(\mathfrak{sl}_2)}$ under projection π. To be sure that $\pi(\mathcal{B})$ is a basis, i.e. that we have removed all redundancies, it will be enough to show that $\mathrm{Span}(\mathcal{B}) \cap I = \{0\}$. This is immediately clear since $\mathrm{Span}(\mathcal{B})$ does not contain appropriately large enough powers of E and F, and powers of K can differ by at most $e-1$ so no element in $\mathrm{Span}(\mathcal{B})$ can factor as $a(K^e-1)b$. Thus $\widetilde{U_q(\mathfrak{sl}_2)}$ is an e^3-dimensional (an thus finite dimensional) quotient of $U_q(\mathfrak{sl}_2)$. Now the formalitites are covered we note that we can arrive at this quotient by simply imposing the relations $E^e = F^e = 0$ and $K^e = 1$.

2.3 Solution to Yang-Baxter Equation

In this section we will show that the quotient Hopf algebra $\widetilde{U_q(\mathfrak{sl}_2)}$ we saw in the previous section has a quasitriangular structure by giving an explicit formula for the R-matrix and then from its representation theory we construct a solution to the Yang-Baxter equation. Before we get into calculations we mention that instead of writing $x + I$ for an element of $\widetilde{U_q(\mathfrak{sl}_2)}$, which is cumbersome, we will abuse notation and write simply x instead leaving the projection implicit. With this notation, quotienting by I will simply impose the relations $E^e = F^e = 0$ and $K^e = 1$.

As in [1] we claim that the following is a universal R-matrix for $\widetilde{U_q(\mathfrak{sl}_2)}$

$$\mathcal{R} = \frac{1}{e}\sum_{i,j,k=0}^{e-1} \frac{(q-q^{-1})^k}{[k]_q!} q^{k(k-1)/2+2k(i-j)-2ij} E^kK^i \otimes F^kK^j, \tag{2.22}$$

where as in the main report we use the notation $[n]_q = (q^n - q^{-n})/(q-q^{-1}) = q^{1-n}[\![n]\!]_{q^2}$ and $[n]_q! = [1]_q[2]_q \dots [n]_q = q^{-k(k-1)/2}[\![n]\!]_{q^2}!$. For convenience we make the following definition

$$c_{i,j,k} := \frac{1}{e}\frac{(q-q^{-1})^k}{[k]_q!} q^{k(k-1)/2+2k(i-j)-2ij}. \tag{2.23}$$

Now let us provide some justification for the claim that (2.22) is a universal R-matrix. First we will show that $\mathcal{R}$ satisfies the braiding condition $\Delta^{\mathrm{op}}(a)\mathcal{R} = \mathcal{R}\Delta(a)$, for the generators $a = K$ and $a = E$. This is clear for K since $\Delta(K) = \Delta^{\mathrm{op}}(K) = K \otimes K$ and this commutes with (2.22) since commuting a K with the factor E^kK^i in the first slot we pick up a factor $q^{\pm 2k}$ and commuting with the factor F^kK^j in the second slot we pick up its inverse $q^{\mp 2k}$. Now for E observe

$$\begin{aligned}\Delta^{\mathrm{op}}(E)\mathcal{R} = (E \otimes 1 + K \otimes E)\mathcal{R} &= \sum_{i,j,k=0}^{e-1} c_{i,j,k}\left(E^{k+1}K^i \otimes F^kK^j + KE^kK^i \otimes EF^kK^j\right) \\ &= \sum_{i,j,k=0}^{e-1} c_{i,j,k}\left(E^{k+1}K^i \otimes F^kK^j + q^{2k}E^kK^{i+1} \otimes EF^kK^j\right)\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathcal{R}\Delta(E) = \mathcal{R}(E \otimes K + 1 \otimes E) &= \sum_{i,j,k=0}^{e-1} c_{i,j,k}\left(E^kK^iE \otimes F^kK^{j+1} + E^kK^i \otimes F^kK^jE\right) \\ &= \sum_{i,j,k=0}^{e-1} c_{i,j,k}\left(q^{2i}E^{k+1}K^i \otimes F^kK^{j+1} + q^{2j}E^kK^i \otimes F^kEK^j\right) \\ &= \sum_{i,j,k=0}^{e-1} c_{i,j,k}q^{2i}E^{k+1}K^i \otimes F^kK^{j+1} + \sum_{i=-1}^{e-2}\sum_{j,k=0}^{e-1} c_{i+1,j,k}q^{2j}E^kK^{i+1} \otimes F^kEK^j\end{aligned} \tag{2.25}$$

We now use the fact that $c_{i,j,k} = c_{i+e,j,k}$ and that $K^0 = 1 = K^e$ to change the $i = -1$ term in the second term in (2.25) to be a $i = e - 1$ term. Noting this, and the fact that $q^{2j} c_{i+1,j,k} = q^{2k} c_{i,j,k}$ the difference becomes

$$
\begin{aligned}
\Delta^{\mathrm{op}}(E)\mathcal{R} - \mathcal{R}\Delta(E) &= \sum_{i,j,k=0}^{e-1} \left(c_{i,j,k} E^{k+1} K^i \otimes F^k K^j + c_{i+1,j,k} q^{2j} E^k K^{i+1} \otimes E F^k K^j \right) \\
&- \sum_{i,j,k=0}^{e-1} \left(c_{i,j,k} q^{2i} E^{k+1} K^i \otimes F^k K^{j+1} + c_{i+1,j,k} q^{2j} E^k K^{i+1} \otimes F^k E K^j \right) \\
&= \sum_{i,j,k=0}^{e-1} \left(c_{i,j,k} \left(E^{k+1} K^i \otimes F^k K^j - q^{2i} E^{k+1} K^i \otimes F^k K^{j+1} \right) + c_{i+1,j,k} q^{2j} E^k K^{i+1} \otimes [E, F^k] K^j \right) \\
&= \sum_{i,j,k=0}^{e-1} \left((c_{i,j,k} - q^{2i} c_{i,j-1,k}) E^{k+1} K^i \otimes F^k K^j + c_{i,j,k} q^{2j} E^k K^i \otimes [E, F^k] K^j \right) \\
&= \sum_{i,j,k=0}^{e-1} \left((c_{i,j,k} - q^{-2k} c_{i,j-2,k}) E^{k+1} K^i \otimes F^k K^j + c_{i,j,k} q^{2j} E^k K^i \otimes [E, F^k] K^j \right) \quad (2.26)
\end{aligned}
$$

where in the second last line, we have used the modulo e invariance of coefficients $c_{i,j,k}$ in i and j and the cyclic property of powers of K to shift sums without consequence, and in the last line we used the identity $q^{2i} c_{i,j,k} = q^{-2k} c_{i,j-1.k}$. Now recall from the main report (equation (3.4)), the following commutator

$$
[E, F^m] = [m]_q \frac{(q^{m-1} K - q^{1-m} K^{-1})}{q - q^{-1}} F^{m-1} = [m]_q F^{m-1} \frac{(q^{1-m} K - q^{m-1} K^{-1})}{q - q^{-1}}. \quad (2.27)
$$

Now noting the following identity

$$
\begin{aligned}
\frac{[k]_q}{q - q^{-1}} c_{i,j,k} &= \frac{[k]_q}{q - q^{-1}} \left(\frac{1}{e} \frac{(q - q^{-1})^k}{[k]_q!} q^{k(k-1)/2 + 2k(i-j) - 2ij} \right) = \frac{1}{e} \frac{(q - q^{-1})^{k-1}}{[k-1]_q!} q^{k(k-1)/2 + 2k(i-j) - 2ij} \\
&= q^{2(i-j)+(k-1)} \left(\frac{1}{e} \frac{(q - q^{-1})^{k-1}}{[k-1]_q!} q^{(k-2)(k-1)/2 + 2(k-1)(i-j) - 2ij} \right) = q^{2(i-j)+(k-1)} c_{i,j,k-1}, \quad (2.28)
\end{aligned}
$$

putting equation (2.27) into (2.26) gives

$$
\begin{aligned}
\Delta^{\mathrm{op}}(E)\mathcal{R} - \mathcal{R}\Delta(E) &= \sum_{i,j,k=0}^{e-1} \Big((c_{i,j,k} - q^{-2k} c_{i,j-2,k}) E^{k+1} K^i \otimes F^k K^j \\
&+ c_{i,j,k-1} q^{2i+(k-1)} E^k K^i \otimes F^{k-1} (q^{-(k-1)} K - q^{k-1} K^{-1}) K^j \Big) \\
&= \sum_{i,j,k=0}^{e-1} \Big((c_{i,j,k} - q^{-2k} c_{i,j-2,k}) E^{k+1} K^i \otimes F^k K^j + c_{i,j,k} q^{2i+k} E^{k+1} K^i \otimes F^k (q^{-k} K^{j+1} - q^k K^{j-1}) \Big) \\
&\qquad (\star) \\
&= \sum_{i,j,k=0}^{e-1} \Big((c_{i,j,k} - q^{-2k} c_{i,j-2,k}) E^{k+1} K^i \otimes F^k K^j + (c_{i,j-1,k} q^{2i} - c_{i,j+1,k} q^{2i+2k}) E^{k+1} K^i \otimes F^k K^j \Big) \\
&= \sum_{i,j,k=0}^{e-1} \Big((c_{i,j,k} - q^{-2k} c_{i,j-2,k}) E^{k+1} K^i \otimes F^k K^j + (c_{i,j-2,k} q^{-2k} - c_{i,j,k}) E^{k+1} K^i \otimes F^k K^j \Big) \\
&= 0, \quad (2.29)
\end{aligned}
$$

where in the line marked $(\star)$ we shifted the index $k \mapsto k + 1$ in the second term since the $k = 0$ term in the previous line vanished (clear from (2.26)), and the $k = e - 1$ in line $(\star)$ vanishes since $E^e = 0$. In the lines following this we again utilise the cyclic nature of summation over j to shift indices appropriately, and the identity $q^{2i}c_{i,j,k} = q^{-2k}c_{i,j-1.k}$. Thus we conclude that $\Delta^{\text{op}}(E)\mathcal{R} = \mathcal{R}\Delta(E)$. The calculation for the generator F is similar so we will not include it. Invertibility of $\mathcal{R}$ follows from the result (1.31) in the main report. We should also show that $\mathcal{R}$ satisfies the relations $(\text{id}\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$ and $(\Delta\otimes\text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, however these calculations elude the author.

Now let us briefly discuss the representation theory of the quotient $\widetilde{U_q(\mathfrak{sl}_2)}$. Fortunately this is a simple task. First we note that for any representation of $\widetilde{U_q(\mathfrak{sl}_2)}$ on V, $\rho : \widetilde{U_q(\mathfrak{sl}_2)} \to End(V)$, we get a representation of $U_q(\mathfrak{sl}_2)$ on V by precomposing with projection $\rho\circ\pi$ (composition of homomorphisms is a homomorphism). Conversely for a representation of $U_q(\mathfrak{sl}_2)$ which acts trivially on the elements E^e, F^e and $K^e - 1$, we have a representation of the quotient $\widetilde{U_q(\mathfrak{sl}_2)}$ by the universal quotient property (1.10) (can be applied for just an algebra homomorphism). Thus by studying the representation theory of $U_q(\mathfrak{sl}_2)$, we obtain the representation theory for its quotient for free.

Consider the 2-dimensional representation of $U_q(\mathfrak{sl}_2)$ with basis $\{v_{-1}, v_1\}$ and defining action

$$Kv_{\pm1} = q^{\pm1}v_{\pm1}, \quad Ev_{-1} = v_1, \quad Ev_1 = 0, \quad Fv_{-1} = 0, \quad Fv_1 = v_{-1}. \tag{2.30}$$

Now suppose in our set up we had $q^e = 1$ for e odd (only consider $e > 1$), then in particular the relations $K^e = \text{id}$, $E^e = 0$ and $F^e = 0$ all hold (in fact E^2 and F^2 act as 0). Thus this representation factors to a representation of the quotient. Then by the result (2.13) from the main report we can construct a solution to the Yang-Baxter equation as $R = (\rho\otimes\rho)(\mathcal{R})$. Let us calculate this as a matrix with respect to the tensor basis elements $\{v_{-1}\otimes v_{-1}, v_1\otimes v_{-1}, v_{-1}\otimes v_1, v_1\otimes v_1\}$

$$\begin{aligned}
\mathcal{R}v_{\pm1}(\otimes v_{\pm1}) &= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij}K^i\otimes K^j(v_{\pm1}\otimes v_{\pm1})\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij\pm j\pm i}(v_{\pm1}\otimes v_{\pm1})\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{i(-2j\pm1)\pm j}(v_{\pm1}\otimes v_{\pm1})\right)\\
&= \frac{1}{e}\left(\sum_{j=0}^{e-1} q^{\pm j}\frac{(1-q^{-e(2j\mp1)})}{1-q^{-(2j\mp1)}}(v_{-1}\otimes v_{-1})\right)\\
&= q^{(1\pm e)/2}(v_{-1}\otimes v_{-1}) = q^{(1+e)/2}(v_{-1}\otimes v_{-1}),
\end{aligned} \tag{2.31}$$

Where we have used the fact that $\frac{1-q^{-e(2j+1)}}{1-q^{-(2j+1)}}$ is 0 unless $(2j+1)$ is a multiple of e in which case it is e. Since e was odd there is one occasion this occurs (namely $j = (e\pm1)/2$). Using this same idea we calculate

$$\begin{aligned}
\mathcal{R}(v_1\otimes v_{-1}) &= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij}K^i\otimes K^j(v_1\otimes v_{-1})\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij+i-j}(v_1\otimes v_{-1})\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{i(1-2j)-j}(v_1\otimes v_{-1})\right)\\
&= \frac{1}{e}\left(\sum_{j=0}^{e-1} q^{-j}\frac{1-q^{e(1-2j)}}{1-q^{(1-2j)}}(v_1\otimes v_{-1})\right)\\
&= q^{-(e+1)/2}(v_1\otimes v_{-1}) = q^{(e-1)/2}(v_1\otimes v_{-1}),
\end{aligned} \tag{2.32}$$

and

$$
\begin{aligned}
\mathcal{R}(v_{-1}\otimes v_1) &= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij}K^i\otimes K^j(v_{-1}\otimes v_1) + \sum_{i,j=0}^{e-1}(q-q^{-1})q^{2(i-j)-2ij}EK^i\otimes FK^j(v_{-1}\otimes v_1)\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{-2ij-i+j}(v_{-1}\otimes v_1) + (q-q^{-1})\sum_{i,j=0}^{e-1} q^{i-j-2ij}(v_1\otimes v_{-1})\right)\\
&= \frac{1}{e}\left(\sum_{i,j=0}^{e-1} q^{i(-1-2j)+j}(v_{-1}\otimes v_1) + (q-q^{-1})\sum_{i,j=0}^{e-1} q^{i(1-2j)-j}(v_1\otimes v_{-1})\right)\\
&= \frac{1}{e}\left(\sum_{j=0}^{e-1} q^j\frac{1-q^{-e(2j+1)}}{1-q^{-(2j+1)}}(v_{-1}\otimes v_1) + (q-q^{-1})\sum_{j=0}^{e-1} q^{-j}\frac{1-q^{e(1-2j)}}{1-q^{(1-2j)}}(v_1\otimes v_{-1})\right)\\
&= q^{(e-1)/2}(v_{-1}\otimes v_1) + q^{(e-1)/2}(q-q^{-1})(v_1\otimes v_{-1}) \qquad (2.33)
\end{aligned}
$$

Thus we have

$$
R = \begin{pmatrix} q^{(e+1)/2} & 0 & 0 & 0\\ 0 & q^{(e-1)/2} & q^{(e-1)/2}(q-q^{-1}) & 0\\ 0 & 0 & q^{(e-1)/2} & 0\\ 0 & 0 & 0 & q^{(e+1)/2}\end{pmatrix} = q^{(e-1)/2}\begin{pmatrix} q & 0 & 0 & 0\\ 0 & 1 & (q-q^{-1}) & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & q\end{pmatrix}, \qquad (2.34)
$$

which is easily verified as a solution of the Yang-Baxter equation.

References

[1] K. Kytölä. *Introduction to Hopf algebras and representations.* Department of Mathematics and Statistics University of Helsinki, 2011.