Introduction to Quantum Groups - Calculation

ASC Report - SCNC3101

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1 Missing calculation from Appendix

Let us now show that $\mathcal{R}$ given by

$$\mathcal{R} = \sum_{i,j,k=0}^{e-1} c_{i,j,k} E^k K^i \otimes F^k K^j, \qquad c_{i,j,k} = \frac{1}{e}\frac{(q-q^{-1})^k}{[k]_q!} q^{k(k-1)/2+2k(i-j)-2ij} \tag{1.1}$$

satisfies the relations $(\mathrm{id}\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$. First let us note the following identities

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}} = q^{\pm(1-n)}\left(\frac{q^{\pm 2n}-1}{q^{\pm 2}-1}\right) = q^{\pm(1-n)}[\![n]\!]_{q^{\pm 2}} \tag{1.2}$$

$$[n]_q! = [1]_q[2]_q \dots [n]_q = q^{\mp(\sum_{k=1}^{n-1} k)}[\![n]\!]_{q^{\pm 2}}! = q^{\mp k(k-1)/2}[\![n]\!]_{q^{\pm 2}}!$$

$$\begin{aligned}
c_{i,j,k}\begin{bmatrix}\!\begin{bmatrix} k \\ l \end{bmatrix}\!\end{bmatrix}_{q^{-2}} &= c_{i,j,k}\frac{[k]_q!}{[l]_q![k-l]_q!} q^{-k(k-1)/2} q^{l(l-1)/2} q^{(k-l)((k-l)-1)/2} \\
&= \frac{1}{e}\frac{(q-q^{-1})^k}{[l]_q![k-l]_q!} q^{2k(i-j)-2ij} q^{l(l-1)/2} q^{(k-l)((k-l)-1)/2} \\
&= e\left(\frac{1}{e}\frac{(q-q^{-1})^l}{[l]_q!} q^{l(l-1)/2+2l(i-j)-2ij}\right)\left(\frac{1}{e}\frac{(q-q^{-1})^{k-l}}{[k-l]_q!} q^{(k-l)(k-l-1)/2+2(k-l)(i-j)-2ij}\right) q^{2ij} \\
&= e c_{i,j,l} c_{i,j,k-l} q^{2ij}
\end{aligned} \tag{1.3}$$

$$c_{i,j,k} = c_{i+l,j,k} q^{-2kl+2lj} \qquad c_{i,j,k} = c_{i,j+l,k} q^{2l(k+i)} \tag{1.4}$$

Now we find,

$$\begin{aligned}
(\mathrm{id}\otimes\Delta)(\mathcal{R}) &= \sum_{i,j,k=0}^{e-1} c_{i,j,k} E^k K^i \otimes \Delta(F)^k \Delta(K)^j \\
&= \sum_{i,j,k=0}^{e-1} c_{i,j,k} E^k K^i \otimes (F\otimes 1 + K^{-1}\otimes F)^k (K^j \otimes K^j) \\
&= \sum_{i,j,k=0}^{e-1} c_{i,j,k} E^k K^i \otimes \left(\sum_{l=0}^{k} \begin{bmatrix}\!\begin{bmatrix} k \\ l \end{bmatrix}\!\end{bmatrix}_{q^{-2}} K^{-l} F^{k-l} K^j \otimes F^l K^j\right) \\
&= \sum_{i,j,k=0}^{e-1} c_{i,j,k} E^k K^i \otimes \left(\sum_{l=0}^{k} \begin{bmatrix}\!\begin{bmatrix} k \\ l \end{bmatrix}\!\end{bmatrix}_{q^{-2}} q^{2l(k-l)} F^{k-l} K^{j-l} \otimes F^l K^j\right) \\
&= e\sum_{i,j,k=0}^{e-1}\sum_{l=0}^{k} c_{i,j,l} c_{i,j,k-l} q^{2ij+2l(k-l)} E^k K^i \otimes F^{k-l} K^{j-l} \otimes F^l K^j \\
&= e\sum_{i,j,a,b=0}^{e-1} c_{i,j,b} c_{i,j,a} q^{2ij+2ab} E^{a+b} K^i \otimes F^a K^{j-b} \otimes F^b K^j,
\end{aligned} \tag{1.5}$$

where in line 3 we have used the q-binomial theorem to expand $(a+b)^k = \sum_{l=0}^{k} \left[\!\left[\begin{smallmatrix} k \\ l \end{smallmatrix} \right]\!\right]_{q^{-2}} b^l a^{k-l}$ for

$$a = F \otimes 1, \quad b = K^{-1} \otimes F, \quad \text{and} \quad ab = q^{-2} ba. \tag{1.6}$$

In the second last line we expanded the term in brackets and used (1.3) and in the last line we wrote $k = a+b$ and let $a+b$ freely vary over $0, 1, \ldots, e-1$ since the condition $a+b = k \leq e-1$ is enforced by the term E^{a+b} which is 0 whenever this is not the case. Now consider the left hand side of the desired equality. This can be manipulated as follows

$$\begin{aligned}
\mathcal{R}_{13}\mathcal{R}_{12} &= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} c_{i_1,j_1,k_1} c_{i_2,j_2,k_2} (E^{k_1}K^{i_1})(E^{k_2}K^{i_2}) \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} \\
&= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} c_{i_1,j_1,k_1} c_{i_2,j_2,k_2} q^{2i_1k_2} E^{k_1+k_2}K^{i_1+i_2} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} \\
&= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} (c_{i_1+i_2,j_1,k_1} q^{-2k_1i_2+2i_2j_1})(c_{i_2+i_1,j_2,k_2} q^{-2k_2i_1+2i_1j_2}) q^{2i_1k_2} \times \\
&\times E^{k_1+k_2}K^{i_1+i_2} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} && (1.4) \\
&= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} c_{i,j_1,k_1} c_{i,j_2,k_2} q^{2i_1j_2+2i_2j_1-2k_1i_2} E^{k_1+k_2}K^{i} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} && (i = i_1+i_2) \\
&= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2i_1j_2+2i_2j_1-2k_1i_2+2(j_1-j_2)(i+k_2)} E^{k_1+k_2}K^{i} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} && (1.4) \\
&= \sum_{\substack{i_1,j_1,k_1,i_2,\\ j_2,k_2=0}}^{e-1} c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2k_2(j_1-j_2)+i(3j_1-j_2-k_1)+\iota(j_2-(j_1-k_1))} E^{k_1+k_2}K^{i} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} \\
& && (\iota = i_1 - i_2) \\
&= \sum_{\substack{i,j_1,k_1,\\ j_2,k_2=0}}^{e-1} \left(\sum_{\iota=0}^{e-1} q^{\iota(j_2-(j_1-k_1))} \right) c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2k_2(j_1-j_2)+i(3j_1-j_2-k_1)} E^{k_1+k_2}K^{i} \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1},
\end{aligned} \tag{1.7}$$

where we have noted that summing over i_1, i_2 is the same as summing over i and ι (modulo e) as these uniquely determine i_1, i_2 modulo e, and powers of q and K are invariant modulo e and hence the coefficients $c_{i,j,k}$ are invariant for i modulo e. Now observe that

$$x = \sum_{i=0}^{e-1} q^{ip} \Rightarrow q^p x = \sum_{i=0}^{e-1} q^{(i+1)p} = \sum_{i=1}^{e-1} q^{ip} + q^{ep} = 1 + \sum_{i=0}^{e-1} q^{(i+1)p} = x. \tag{1.8}$$

This implies that $x = 0$ unless $q^p = 1$ that is $p = 0 \mod e$. If $q^p = 1$ then it is clear that $x = e$. Hence we may write $x = e\delta_{0,p \mod e}$. In other words x multiplies by e and enforces the equality $p = 0$ to hold modulo

p when summed over. Thus (1.7) becomes

$$\begin{aligned}
\mathcal{R}_{13}\mathcal{R}_{12} &= \sum_{\substack{i,j_1,k_1,\\ j_2,k_2=0}}^{e-1} \delta_{0,j_2-(j_1-k_1)} c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2k_2(j_1-j_2)+i(3j_1-j_2-k_1)} E^{k_1+k_2}K^i \otimes F^{k_2}K^{j_2} \otimes F^{k_1}K^{j_1} \\
&= e \sum_{i,j_1,k_1,k_2=0}^{e-1} c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2k_2(j_1-(j_1-k_1))+i(3j_1-(j_1-k_1)-k_1)} E^{k_1+k_2}K^i \otimes F^{k_2}K^{j_1-k_1} \otimes F^{k_1}K^{j_1} \\
&= e \sum_{i,j_1,k_1,k_2=0}^{e-1} c_{i,j_1,k_1} c_{i,j_1,k_2} q^{2k_2k_1+2ij_1} E^{k_1+k_2}K^i \otimes F^{k_2}K^{j_1-k_1} \otimes F^{k_1}K^{j_1}, \qquad (1.9)
\end{aligned}$$

where in the second line we have enforced the equality $j_2 = j_1 - k_1 \mod e$ using the delta function. Clearly this is the same as (1.5) up to a change of label $k_2 \leftrightarrow a$, $k_1 \leftrightarrow b$, $j_1 \leftrightarrow j$.