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Towards a Factorised Solution of the Yang-Baxter Equation with $U_q(\mathfrak{sl}_n)$ Symmetry

Benjamin Morris[1]
Based on Honours thesis sup. by Prof. V. Mangazeev[2]

[1]School of Mathematics,
University of Leeds
[2]Department of Theoretical Physics,
Australian National University

SIDE14.2, Warsaw, June 2023

Introduction

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  - Symmetric Group Relations

Yang-Baxter Equation

The (parameter dependent) YBE on End$(V_1 \otimes V_2 \otimes V_3)$ is

$$R_{V_1,V_2}(u_1,u_2)R_{V_1,V_3}(u_1,u_3)R_{V_2,V_3}(u_2,u_3)$$
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Additive dependence $\Rightarrow R_{V_i,V_j}(u_i,u_j) = R_{V_i,V_j}(u_i - u_j)$

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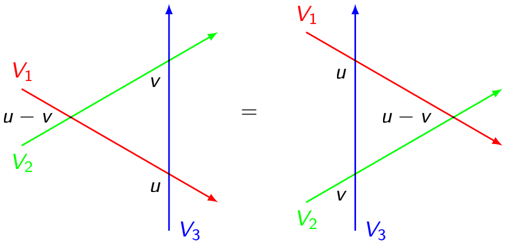

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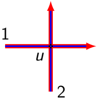
Our Goal: construct an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$,

$\mathcal{R}_{12}(u) =$ 

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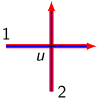
$\mathcal{R}_{12}(u) =$ 

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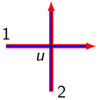
$\mathcal{R}_{12}(u) =$ 

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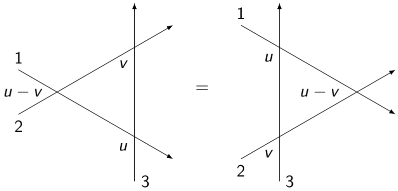

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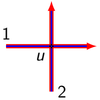
$\mathcal{R}_{12}(u) =$ 

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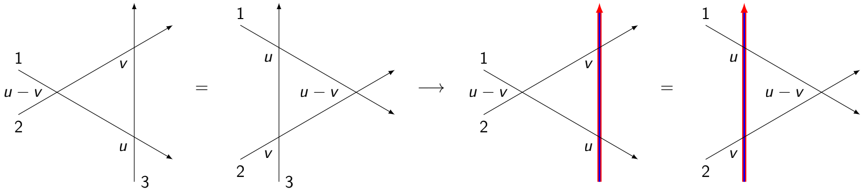

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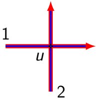
$\mathcal{R}_{12}(u) =$ 

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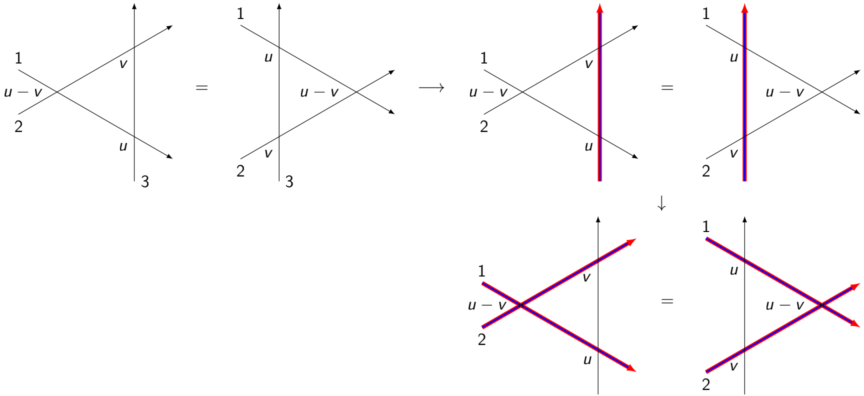

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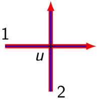
$\mathcal{R}_{12}(u) =$ 

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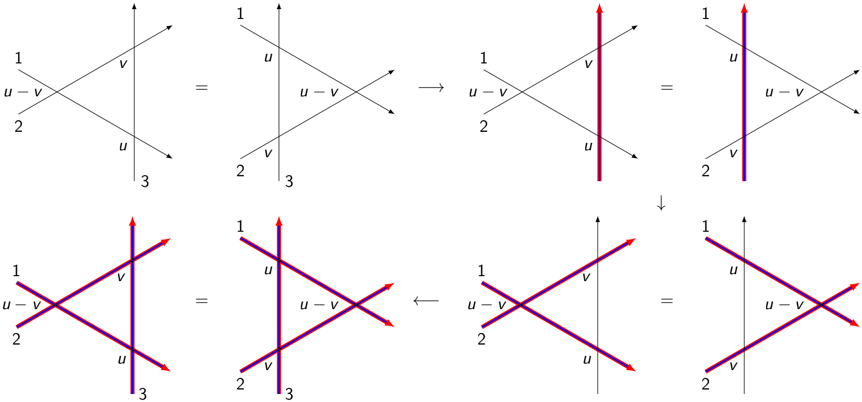

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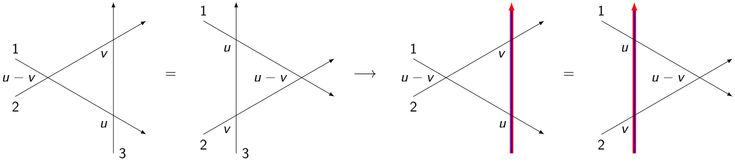
Defining R-Matrix and L-operators



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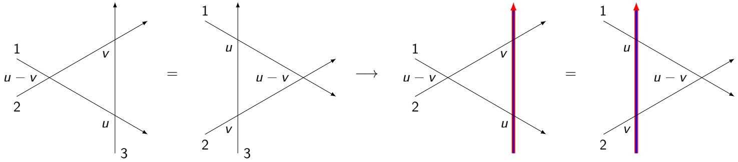

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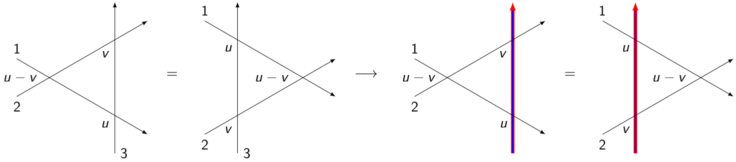

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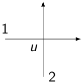
- $R_{12}(u) =$  $\in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (an $n^2 \times n^2$ matrix).

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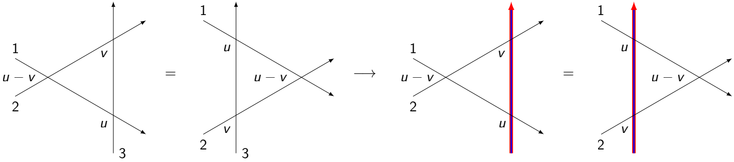

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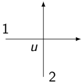
- $R_{12}(u) =$ 

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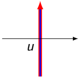
- $L(u) =$ 

$\in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$, where $\mathcal{A} \subset \mathrm{End}(\mathcal{V})$. An $n \times n$ matrix with values in $\mathcal{A}$.

Defining R-Matrix and Universal L-operators

RLL relation in $\mathrm{End}\left(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{V}\right)$:

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v).$$

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Why YBE for R? This is a consistency condition for associativity of $\mathcal{A}$.

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The universal enveloping algebra (UEA) $\mathcal{A} = U(\mathfrak{sl}_n)$ has a defining R-matrix

$$R_{12}(u) = u \cdot \mathrm{id}_{n^2} + P_{12} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n,$$

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$$L(u) = u \cdot \mathrm{id}_n \otimes 1_{\mathcal{A}} + \sum_{i,j=1}^{n} e_{ij} \otimes E_{ji},$$

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where e_{ij} is the matrix unit. Here $\{E_{ij}\}$ is the Cartan-Weyl basis for $\mathfrak{sl}_n$:

$$h_i = E_{ii} - E_{i+1,i+1}, \quad \textstyle\sum_i E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i,$$
$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{ik} E_{lj}.$$

Differential Representation of $\mathfrak{sl}_n$

For n-parameters $\boldsymbol{\rho} \in \mathbb{C}^n$ with $\sum_i \rho_i = n(n-1)/2$, we can define a representation on $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]$

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[Derkachov and Manashov, 2006]

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Differential Representation of $\mathfrak{sl}_n$

E.g. $n = 2$ case: Taking $N_x = x\partial x$ and $m = \rho_2 - \rho_1 + 1$,

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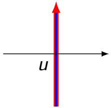
- It is reducible if some $m_i \in \mathbb{Z}_{\leq 0}$. It contains a finite dimensional irreducible subrep iff true for all m_i.
- It has a factorised L-operator!

$$L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1} =$$



,

$\boldsymbol{u} = (u_i)$, where $u_i = u - \rho_i$.

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$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_i,$$
$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$
$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$
$$g_i^2 g_{i\pm1} - (q + q^{-1}) g_i g_{i\pm1} g_i + g_{i\pm1} g_i^2 = 0,$$

$g_i = e_i, f_i$.

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$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm 1} - (q + q^{-1}) g_i g_{i\pm 1} g_i + g_{i\pm 1} g_i^2 = 0,$$

$g_i = e_i, f_i$. The a_{ij} are components of the A_n Cartan matrix.

q-Deformed Case: $U_q(\mathfrak{sl}_n)$

The q-deformed UEA $U_q(\mathfrak{sl}_n)$: For some $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$

- Generators: e_i, f_i, and invertible $k_i = q^{h_i}$ for $i = 1, 2, \ldots, n-1$
- Relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_i,$$
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- Notation: $[x]_q = (q^x - q^{-x})/(q - q^{-1})$

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$$L(u) = q^u L^+ + q^{-u} L^- \in \mathrm{End}(\mathbb{C}^n) \otimes U_q(\mathfrak{sl}_n),$$

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Now specialise:

Is there an analogous class of representations for $U_q(\mathfrak{sl}_n)$? How about a factorised L-operator?

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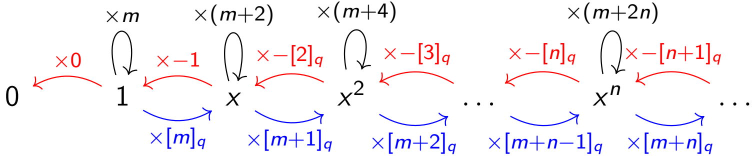
<u>$n = 2$ case:</u> Just one variable $x_{21} = x$

$$f = -D_x, \quad e = x[m + N_x]_q, \quad h = 2N_x + m,$$



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$$E_{ii}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji}+1)\,,$$

$$f_i^{(n)} = -D_{i+1,i} q^{\sum_{j=1}^{i-1}(N_{ij}-N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j} q^{\sum_{k=1}^{j-1}(N_{ik}-N_{i+1,k})}\,,$$

$$e_i^{(n)} = \begin{array}{c} x_{i+1,i}\Big[\, m_i + N_{i+1,i} + \sum_{j=i+2}^{n}(N_{ji}-N_{j,i+1})\,\Big]_q + q^{-m_i}\sum_{j=i+2}^{n} x_{ji} D_{j,i+1} q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ -q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij} q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{array},$$

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[Awata, Noumi, and Odake, 1994]

Factorised L-operator?

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Factorised L-operator?

$\underline{\mathfrak{sl}_n:}$ $L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(\boldsymbol{u}) = \begin{pmatrix} u_n & P_{21} & P_{31} & \dots & P_{n1} \\ & u_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & P_{n,n-1} \\ & & & & u_1 \end{pmatrix},$$

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$\underline{U_q(\mathfrak{sl}_n)}$: Postulate $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1}$

$$D(\boldsymbol{u}) = \begin{pmatrix} [u_n]_q q^{b_1} & P_{21} & \dots & P_{n1} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1}} & P_{n,n-1} \\ & & & [u_1]_q q^{b_n} \end{pmatrix},$$

$$P_{ij} = -D_{ij} q^{b_{ij}} - \sum_{k=i+1}^{n} x_{ki} D_{kj} q^{b_{ijk}}, \quad Z_i(\boldsymbol{u}) = \begin{pmatrix} 1 & & & \\ x_{21} q^{a^{(i)}_{21}} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{n1} q^{a^{(i)}_{n1}} & \dots & x_{n,n-1} q^{a^{(i)}_{n,n-1}} & 1 \end{pmatrix},$$

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n=2: Yes [Derkachov, Karakhanyan, and Kirschner, 2007]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1 - N_x} x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x q^{N_x} \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{u_2 - N_x} x & 1 \end{pmatrix}.$$

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n=3: Yes [Valinevich et al., 2008], $L(u_1, u_2, u_3) = Z_1 D Z_2^{-1}$ with

$$D = \begin{pmatrix} [u_3]_q q^{-N_{21}+N_{31}} & (D_{21} + x_{32} D_{31} q^{N_{31}-N_{32}-1}) q^{N_{21}+N_{31}} & D_{31} q^{N_{31}} \\ 0 & [u_2]_q q^{N_{21}-N_{32}} & D_{32} q^{u_2 - N_{31} + N_{32}} \\ 0 & 0 & [u_1]_q q^{N_{32}+N_{31}} \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ q^{u_2 - N_{31} + N_{32} - N_{21}} x_{21} & 1 & 0 \\ q^{-u_1 - N_{31} + N_{32}} x_{31} & q^{u_1 - u_2 - N_{32}} x_{32} & 1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ q^{c_{21}} x_{21} & 1 & 0 \\ q^{c_{31}} x_{31} & q^{c_{32}} x_{32} & 1 \end{pmatrix},$$

$$c_{21} = u_3 - N_{21}, \; c_{31} = -u_3 - N_{31} - N_{21} - 1, \; c_{32} = N_{21} + N_{31} - N_{32}.$$

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"Controlled deformation" breaks - We have "pure quantum phenomena" in the Cartan-Weyl elements:

$$E_{42} = [f_3, f_2]_q = -D_{42} q^{N_{21}-N_{32}-N_{41}-1} - x_{21} D_{41} q^{-(1+N_{31})} + (q-q^{-1}) x_{31} D_{41} D_{32} q^{N_{21}-N_{31}-1}.$$

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Such terms cannot arise from our ansatz.

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Factorised L-operator?

General n: Order of highest term in $(q - q^{-1})$

$$\mathcal{O}(L^+(\boldsymbol{u})) \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & 0 & 1 & 2 & 2 & 2 \\ & & & 0 & 0 & 1 & 2 & 3 \\ & & & & 0 & 0 & 1 & 2 \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}$$

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$\Rightarrow$ factorisation involves higher terms in $(q - q^{-1})$.

Q: Factor L-operator with near diagonal matrices which are only first order in $(q - q^{-1})$.

Parameter Permutations and YBE

For $\check{\mathcal{R}}(u) := P \circ \mathcal{R}(u) \in \mathrm{End}(\mathcal{V}_\rho \otimes \mathcal{V}_\sigma)$

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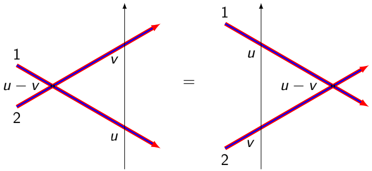

$$\sim \quad \check{\mathcal{R}}(u-v)L_1(\boldsymbol{u})L_2(\boldsymbol{v}) = L_1(\boldsymbol{v})L_2(\boldsymbol{u})\check{\mathcal{R}}(u-v).$$

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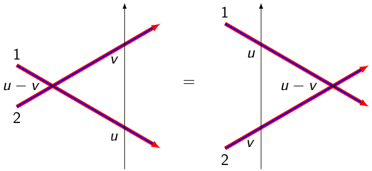

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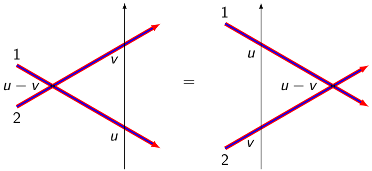

$$\sim \quad \begin{array}{c} \check{\mathcal{R}}(u-v)L_1(\boldsymbol{u})L_2(\boldsymbol{v}) \\ = L_1(\boldsymbol{v})L_2(\boldsymbol{u})\check{\mathcal{R}}(u-v) \end{array}.$$

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IDEA: Factorise $\check{\mathcal{R}}(u-v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$

$$\mathcal{S}_i L_{12}(\boldsymbol{u},\boldsymbol{v}) = L_{12}(s_i(\boldsymbol{u},\boldsymbol{v}))\mathcal{S}_i, \quad (L_{12}(\boldsymbol{u},\boldsymbol{v}) = L_1(\boldsymbol{u})L_2(\boldsymbol{v}))$$

$(s_i(\alpha_1,\ldots\alpha_{2n}) = (\alpha_1,\ldots,\alpha_{i+1},\alpha_i,\ldots\alpha_{2n}))$ for $i = 1,\ldots,2n-1$.

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$$\mathcal{S}_i L_{12}(\boldsymbol{u},\boldsymbol{v}) = L_{12}(s_i(\boldsymbol{u},\boldsymbol{v}))\mathcal{S}_i, \quad (L_{12}(\boldsymbol{u},\boldsymbol{v}) = L_1(\boldsymbol{u})L_2(\boldsymbol{v}))$$

$(s_i(\alpha_1,\dots\alpha_{2n}) = (\alpha_1,\dots,\alpha_{i+1},\alpha_i,\dots\alpha_{2n}))$ for $i = 1,\dots,2n-1$.

Simplification: Can just find $n-1$-"intertwining" operators $\mathcal{T}_i \in \mathrm{End}(\mathcal{V}_\rho)$:

$$\mathcal{T}_i(\boldsymbol{u})L_1(\boldsymbol{u}) = L_1(s_i\boldsymbol{u})\mathcal{T}_i(\boldsymbol{u}),$$

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and a single "exchange" operator:

$$\mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(u_1,\ldots,u_{n-1},v_1,u_n,v_2,\ldots,v_n).$$

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These operators should define an action of S_{2n}, *i.e.*,

$$s_{i_j} \dots s_{i_2} s_{i_1} \mapsto \mathcal{S}_{i_j}(s_{i_{j-1}} \dots s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \dots \mathcal{S}_{i_2}(s_{i_1}(\boldsymbol{u}, \boldsymbol{v}))\, \mathcal{S}_{i_1}(\boldsymbol{u}, \boldsymbol{v}),$$

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YBE then follows from equivalence of the decompositions in $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \xmapsto{\check{\mathcal{R}}_{12}} (\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \xmapsto{\check{\mathcal{R}}_{23}} (\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) \xmapsto{\check{\mathcal{R}}_{12}} (\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}),$$

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$U_q(\mathfrak{sl}_2)$ Case: [Derkachov, Karakhanyan, and Kirschner, 2007]

$U_q(\mathfrak{sl}_3)$ Case: [Valinevich et al., 2008]

q-Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ ($|q| < 1$) L-operator are given by

$$\mathcal{T}^{(n)}_{n-i}(\alpha) = \left(\Lambda^{(n)}_{n-i}\right)^{\alpha} \frac{e_{q^2}(q^{2(N_{i+1,i}+1)}\boldsymbol{X}^{(n)}_{n-i})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}^{(n)}_{n-i})},$$

$$e_{q^2}(\boldsymbol{Z}) = ((\boldsymbol{Z}; q^2)_\infty)^{-1} = \left[(1-\boldsymbol{Z})(1-q^2\boldsymbol{Z})(1-q^{2\cdot 2}\boldsymbol{Z})\dots\right]^{-1},$$

$$\frac{e_{q^2}(\boldsymbol{Z})}{e_{q^2}(q^{-\alpha}\boldsymbol{Z})} = \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j}\boldsymbol{Z}^j, \qquad \Lambda^{(n)}_{n-i} = (x_{i+1,i})^{-1} q^{\beta_i}$$

where $\alpha = u_{n-i} - u_{n+1-i}$, and

$$\boldsymbol{X}^{(n)}_{n-i} = 1 + x_{i+1,i}\sum_{j=i+2}^{n} \frac{x_{j,i+1}}{x_{ji}}(q^{N_{ij}} - q^{-N_{ij}})q^{\gamma_i}.$$

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Obtained using an approach from [Valinevich et al., 2008].

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Proof.

The only non-trivial relation is the braid relation

$$\mathcal{T}_i(\alpha)\mathcal{T}_{i+1}(\alpha+\beta)\mathcal{T}_i(\beta) = \mathcal{T}_{i+1}(\beta)\mathcal{T}_i(\alpha+\beta)\mathcal{T}_{i+1}(\alpha).$$

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After a series expansion it is reduced to a family of (terminating) q-series identity relating rank $i+1$ and rank $2i-1$ q-Lauricella series. □

q-Series Identity

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(Type D) q-Lauricella Function: q-Lauricella functions are a family of multivariable hypergeometric series:

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$$\Phi_D^{(n)}[b; a_1, \ldots, a_n; c; q; x_1, \ldots, x_n] = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(b;q)_M (a_1;q)_{m_1} \ldots (a_n;q)_{m_n}}{(c;q)_M (q;q)_{m_1} \ldots (q;q)_{m_n}} x_1^{m_1} \ldots x_n^{m_n}, \qquad (\star)$$

where $M = \sum_{i=1}^{n} m_i$ and

$$(x;q)_m = (1-x)(1-qx)\ldots(1-q^{m-1}x).$$

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[Andrews, 1972] gives a general transformation formula allowing us to rewrite $(\star)$ in terms of a ${}_{n+1}\phi_n$ hypergeometric series.

q-Series Identity

q-Series Identity

For $n \geq 1$ and non-negative integer tuples

$$\boldsymbol{k} = (k_0, \ldots, k_n) = (k_0, \tilde{\boldsymbol{k}}), \quad \boldsymbol{l} = (l_1, \ldots, l_n), \quad \boldsymbol{m} = (m_1, \ldots, m_{n-1}),$$

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with $K = \sum_{j=0}^{n} k_j$ and L, M. Define n-tuples $\boldsymbol{r} = (r_i)$ and $\boldsymbol{p} = (p_i)$

$$r_i = 1 + \sum_{a=1}^{i}(k_a - (l_a + m_a)), \quad p_i = 1 - \sum_{a=i}^{n}(k_a - (l_a + m_a)).$$

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The identity we need is the equality $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \frac{(\xi; q)_{L+M}}{(\xi\,\zeta; q)_{L+M}} \Phi_D^{(2n-1)}\big[\, \zeta; q^{-\boldsymbol{l}}, q^{-\boldsymbol{m}}\, ; q^{1-L-M}/\xi;\, q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{(r_i,\hat{r}_n)+\boldsymbol{m}} \,\big],$$

$$\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \zeta^{k_0} \frac{(\xi; q)_K}{(\xi\,\zeta; q)_K} \Phi_D^{(n+1)}\big[\, \zeta; q^{-\boldsymbol{k}}\, ; q^{1-K}/\xi;\, q^{1+k_0-K}/(\xi\,\zeta), q^{\boldsymbol{p}+\tilde{\boldsymbol{k}}} \,\big],$$

for arbitrary complex parameters ξ, ζ.

Exchange Operator

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The defining relation for the exchange operator $\mathcal{S}_n$ is

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Recall the (postulated) factorisation for $L(\boldsymbol{u})$. This can be put into the form:

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Now we can reduce the defining relation to

$$\begin{aligned} Z_2^{(x,\tilde{\boldsymbol{u}})}(v_1) \left[(D^{(x,\tilde{\boldsymbol{u}})})^{-1} \mathcal{S}_n \, D^{(x,\tilde{\boldsymbol{u}})} \right] \left(Z_2^{(x,\tilde{\boldsymbol{u}})}(u_n) \right)^{-1} \\ = Z_1^{(y,\tilde{\boldsymbol{v}})}(u_n) \left[D^{(y,\tilde{\boldsymbol{v}})} \mathcal{S}_n \, (D^{(y,\tilde{\boldsymbol{v}})})^{-1} \right] \left(Z_1^{(y,\tilde{\boldsymbol{v}})}(v_1) \right)^{-1}, \end{aligned}$$

if $\mathcal{S}_n^{(x,y)}$ commutes (element wise) with $Z_1^{(x)}$ and $Z_2^{(y)}$.

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Recall in the $n \geq 4$ case the postulated ansatz for the factorisation was inconsistent - the outer most factors will now have q-difference terms.

This seems to represent a serious obstruction to constructing the exchange operator - unclear whether to expect a multiplication operator (by shifted variables) to work or not

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- We introduced the *RLL*-method as a means for obtaining solutions to the YBE in the class of differential (q-difference) representations of $\mathfrak{sl}_n$ ($U_q(\mathfrak{sl}_n)$). A key feature here is a factorisation property of the L-operators.

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- We explain how the failure of the factorisation property for the $U_q(\mathfrak{sl}_4)$ L-operator represents an obstruction to constructing the missing "exchange" operator.

Thank You!

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Questions?

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