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Temperley-Lieb categories on Non-Orientable Surfaces

Benjamin Morris[1]
Joint work with Dionne Ibarra[2] and Gabriel Montoya-Vega[3]

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The Yang-Baxter Equation and all that, Bedlewo, June 2025

LONDON MATHEMATICAL SOCIETY EST. 1865

arXiv:2506.14319

Motivation

Enriching the structure skein-modules...

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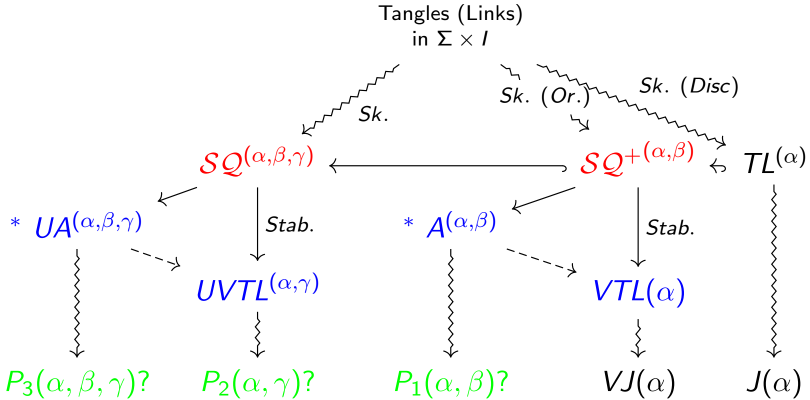
In particular, we consider **nested** $(0, 1, 2)$ - "cobordism categories".


Tangles (Links) in $\Sigma \times I$
Sk.
Sk. (Or.)
Sk. (Disc)
$\mathcal{SQ}^{(\alpha,\beta,\gamma)}$
$\mathcal{SQ}^{+(\alpha,\beta)}$
$TL^{(\alpha)}$
* $UA^{(\alpha,\beta,\gamma)}$
Stab.
* $A^{(\alpha,\beta)}$
Stab.
$UVTL^{(\alpha,\gamma)}$
$VTL(\alpha)$
$P_3(\alpha,\beta,\gamma)$?
$P_2(\alpha,\gamma)$?
$P_1(\alpha,\beta)$?
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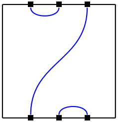
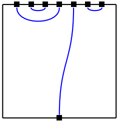
 $\in \mathrm{Hom}(3,3)$,  $\in \mathrm{Hom}(1,7)$.

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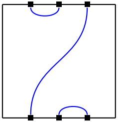
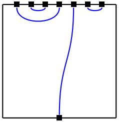
 $\in$ Hom$(3, 3)$,  $\in$ Hom$(1, 7)$.

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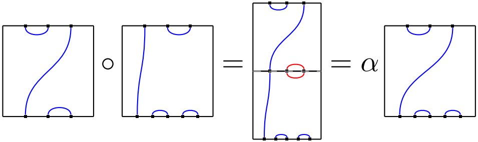
$D_2 \circ D_1 =$  $= \alpha\, D_2 \# D_1$

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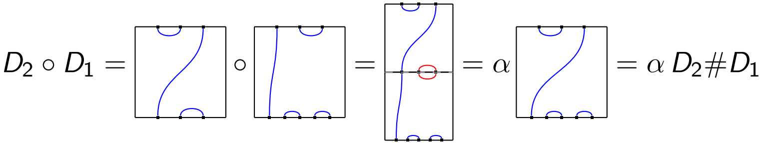

Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.

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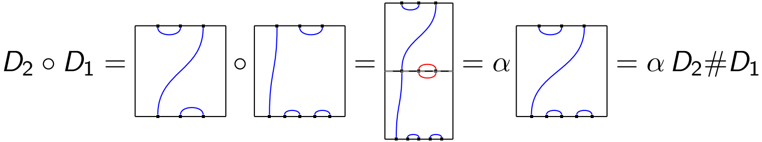

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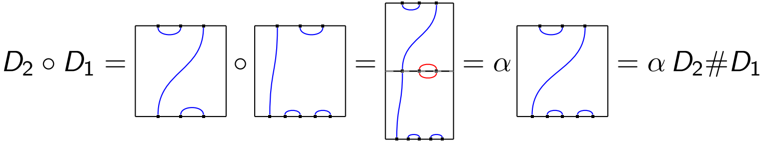

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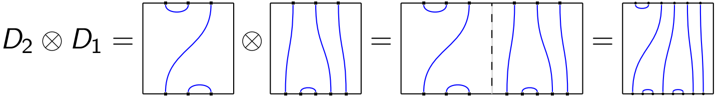

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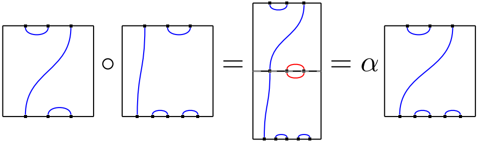
$D_2 \circ D_1 =$  $= \alpha\, D_2 \# D_1$

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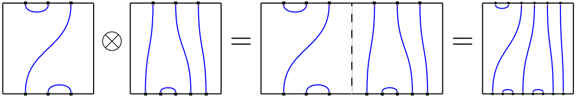
$D_2 \otimes D_1 =$ 

$(n_1 \otimes n_2 = n_1 + n_2)$.

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TL-category: a nested $(0,1,2)$ "cobordism category" with

- 0-manifolds: points $\sqcup_{\text{finite}} *$.
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we will restrict to surface types Σ with one boundary component.

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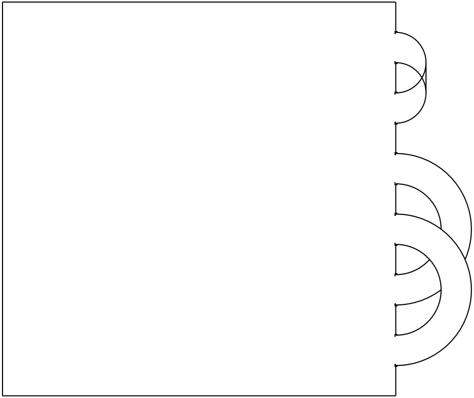

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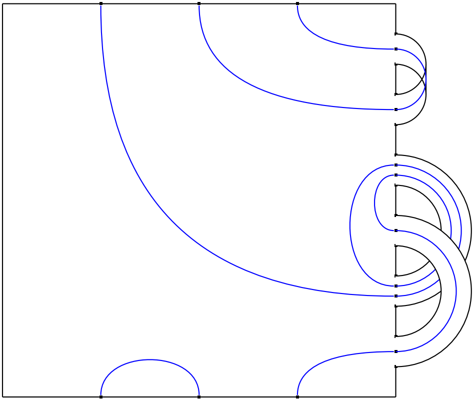

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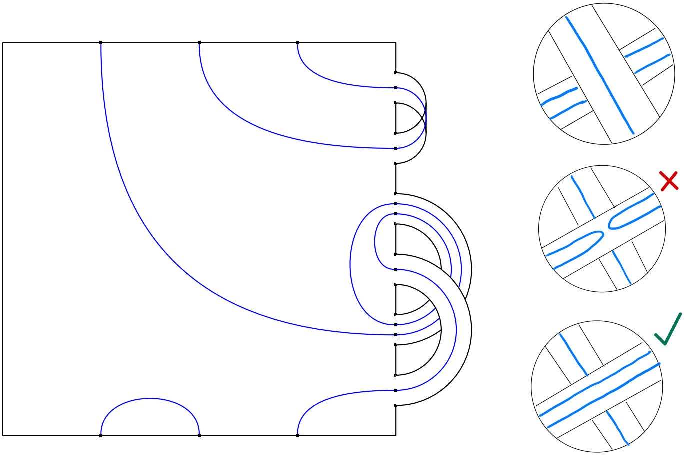

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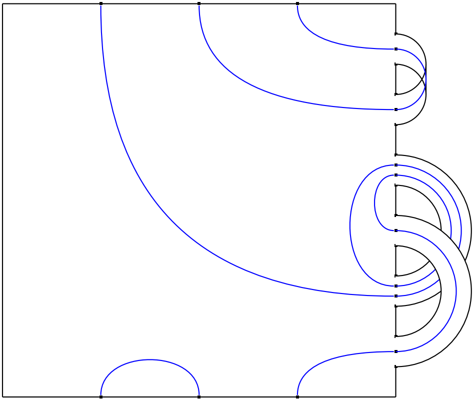

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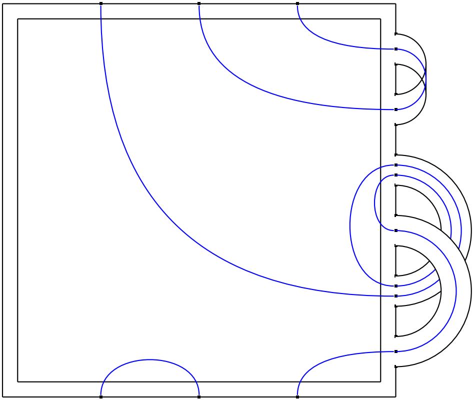

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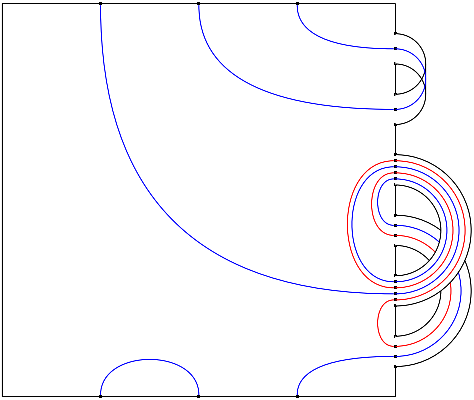
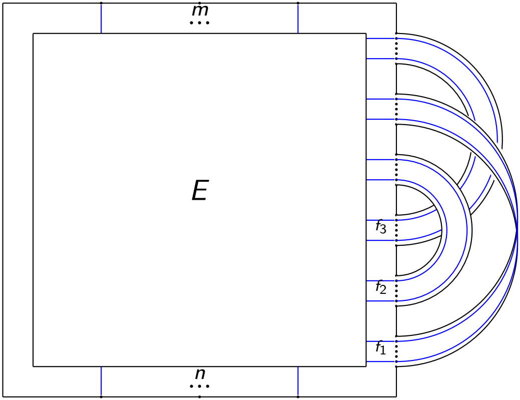
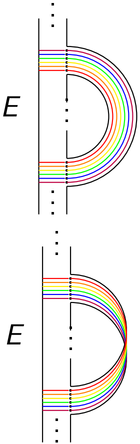


SWB diagrams

Square with bands (SWB) diagram encoded by $\Theta = (P, s, f, E)$ (type n, m)





SWB diagrams - Isotopy

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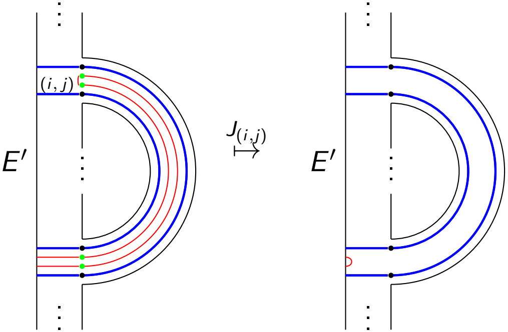
Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove "turnbacks" by "pull-throughs"



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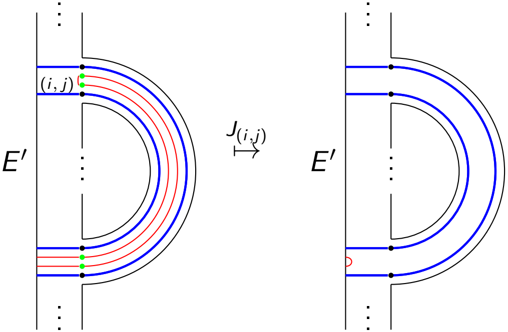

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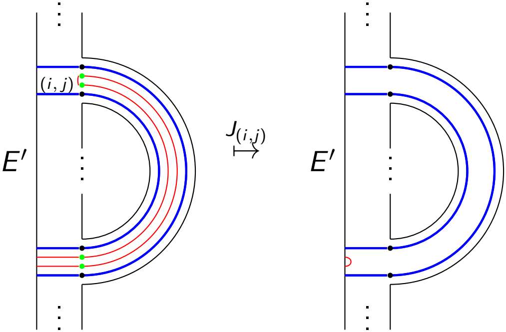

$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', o(E''))$$

Generate an equivalence relation with this move.

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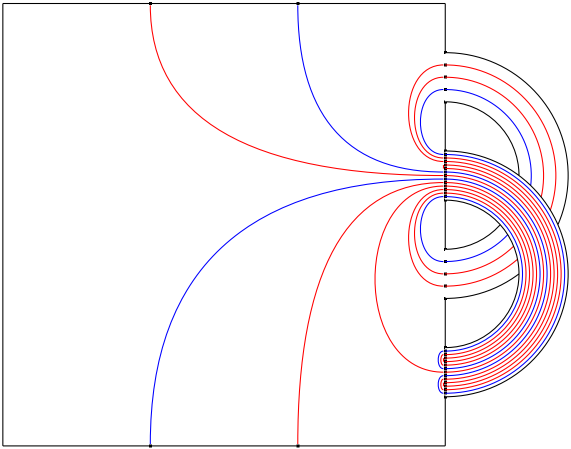

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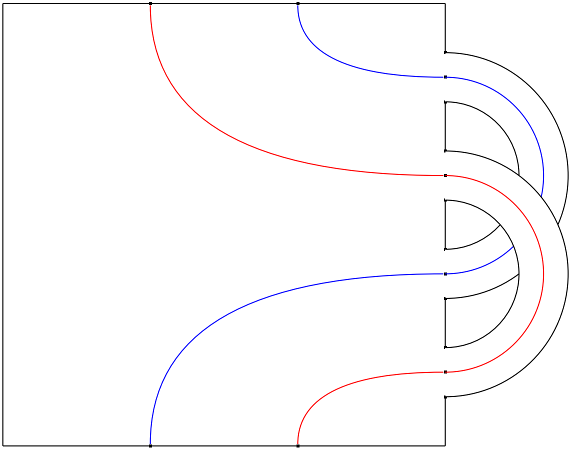

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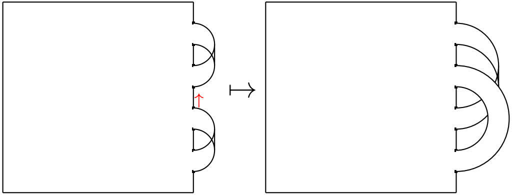

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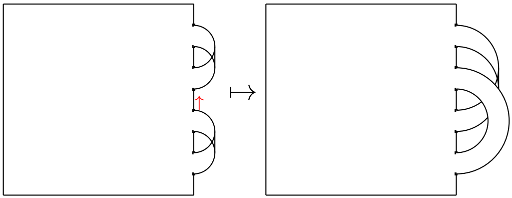

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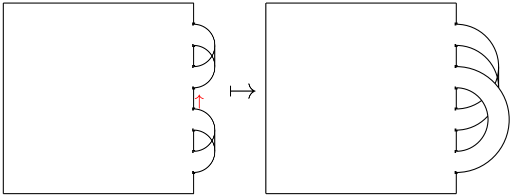

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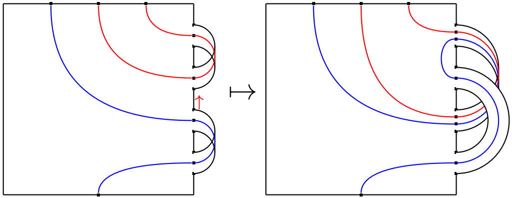

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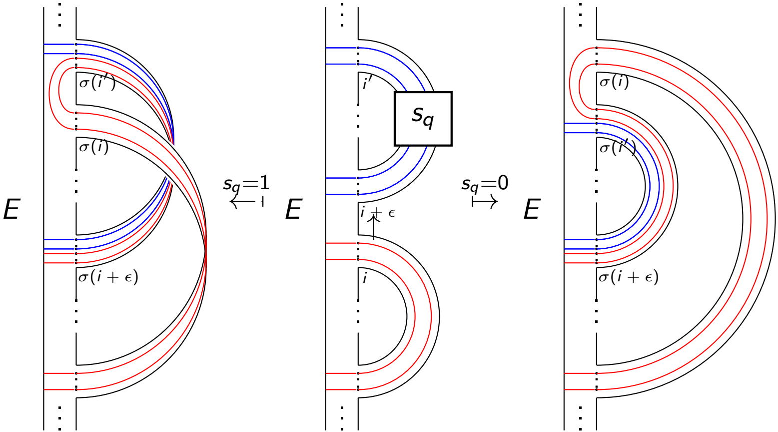

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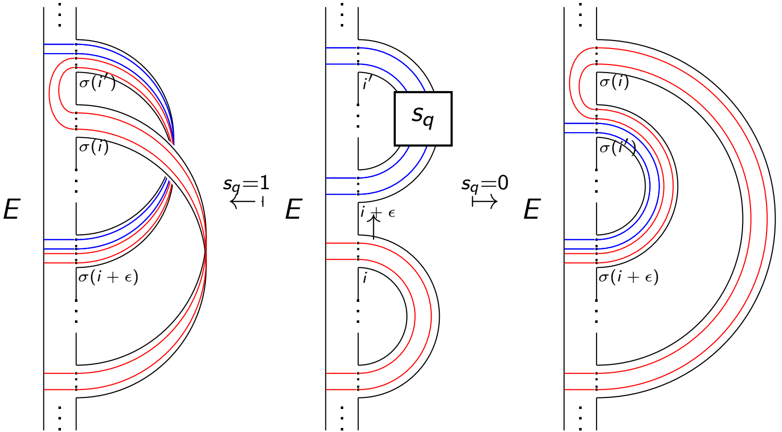

$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{\text{"new red arcs"}\})$$

SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

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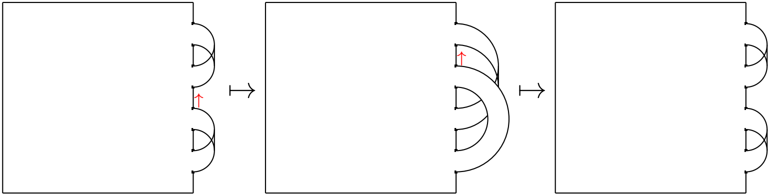

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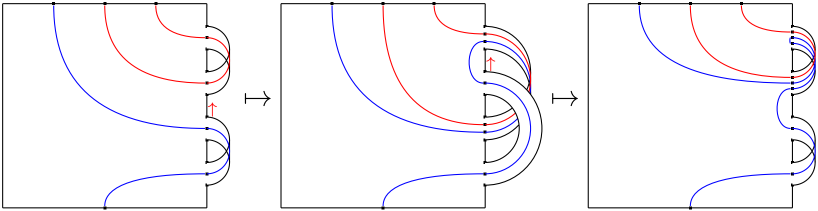

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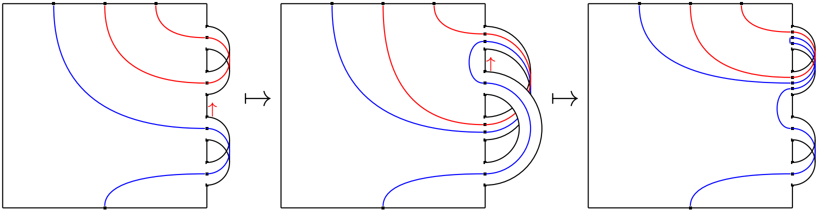
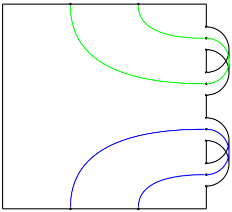


Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

SWB diagrams - Handleslide Equivalence

$\Theta =$ 

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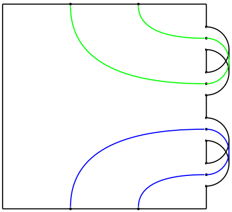
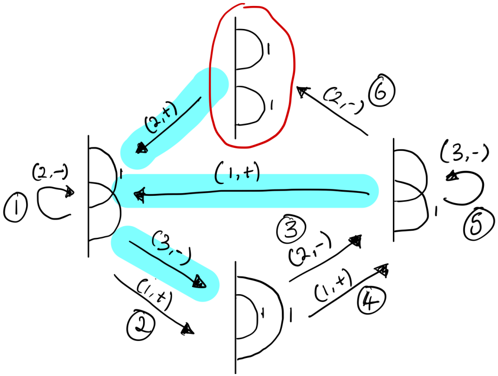
$\Theta =$ 



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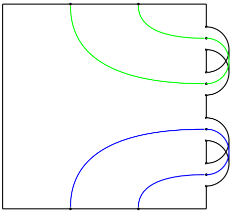
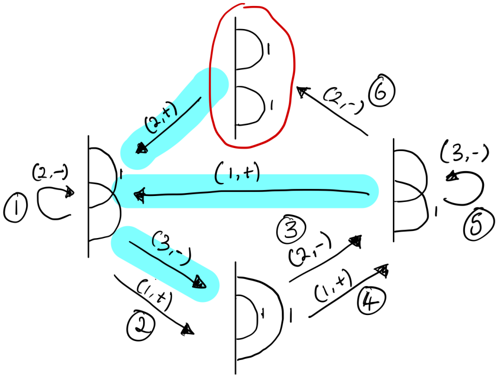
$\Theta =$ 

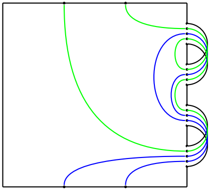


Associate the "reduced" sequence A_i for each edge outside the tree, e.g.

$$A_2 = (3,+) \circ (4,-) \circ (1,+) \circ (2,+)$$

SWB diagrams - Handleslide Equivalence

$$A_1(\Theta) = A_2(\Theta) = A_5(\Theta) =$$



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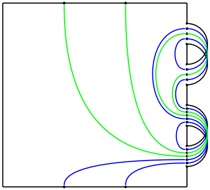
$$A_3(\Theta) =$$



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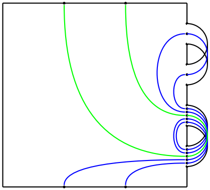
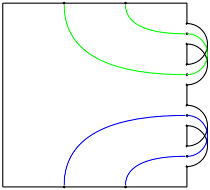
$$A_4(\Theta) =$$



,

$$A_6(\Theta) = \Theta =$$



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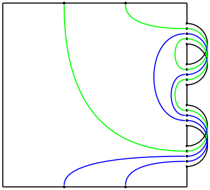

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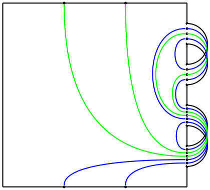

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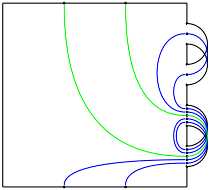

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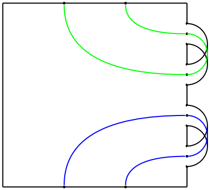

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$$\langle A_2, A_3, A_4 \mid A_3A_2 = A_4,\ A_2A_4 = A_4A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

SWB diagrams - Handleslide Equivalence

$A_1(\Theta) =$
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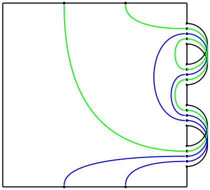

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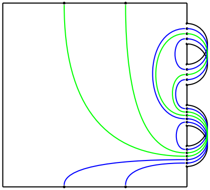

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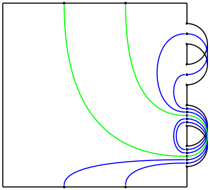

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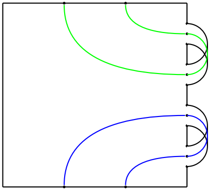

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(Chord Diag. Pres. of Mapping Class Group - Bene 2009)

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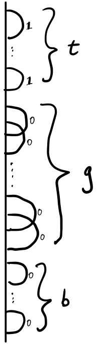
Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$(P, s) \sim$ 

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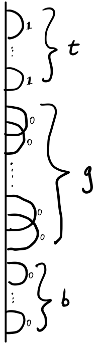
$(P, s) \sim$ 

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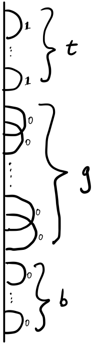
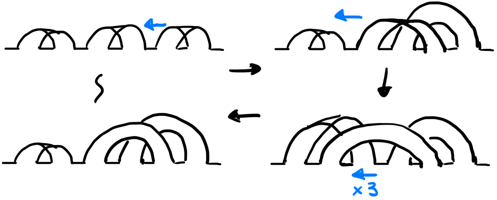
$(P, s) \sim$ 



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The Category $\mathcal{SQ}$

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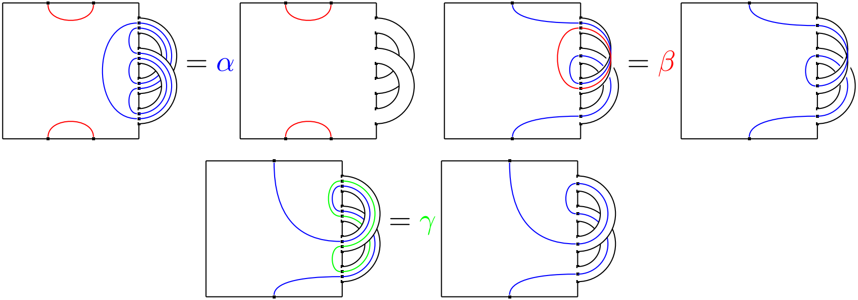

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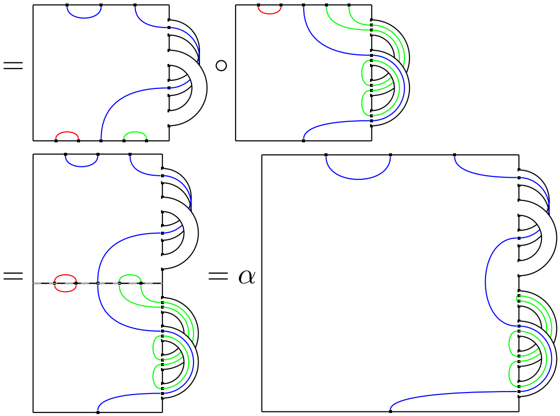

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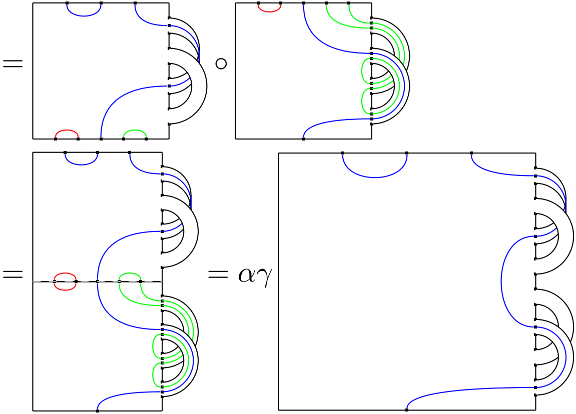

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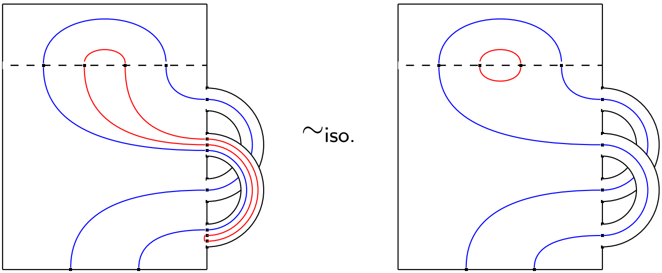

□

The Category $\mathcal{SQ}$ - Basic Facts

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Fact 0: We have two (wide) subcategories:

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Fact 2: For any $\Theta \in Sq(n, m)$, there exist **unique** integers l_s, l_t and l_u, and $\Theta' \in Sq(n, m)$ without closed loops, such that:

$$\overline{\Theta} = \alpha^{l_s} \beta^{l_t} \gamma^{l_u} \, \overline{\Theta'} \in \mathrm{Hom}(n, m).$$

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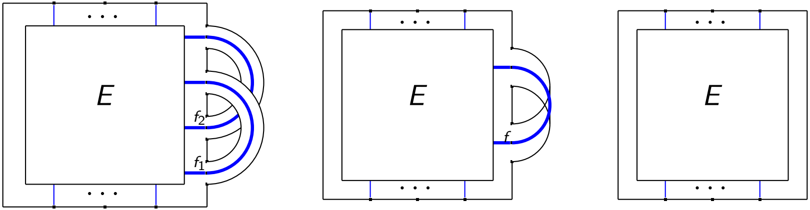

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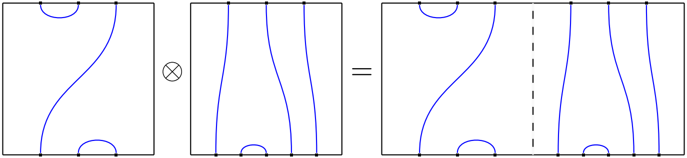

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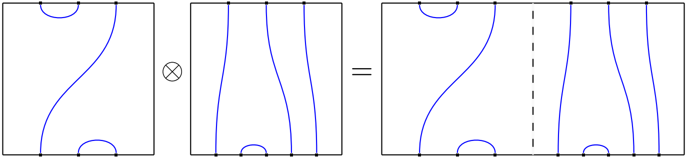

Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects.

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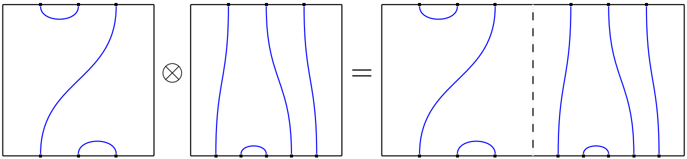

Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

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Indirect answer: Step 1 - Put the identity diagram on the left:

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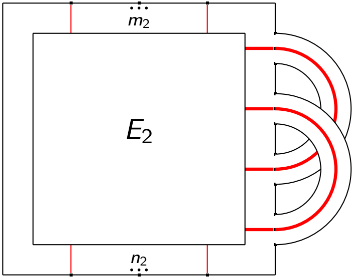
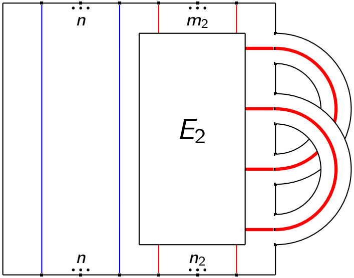
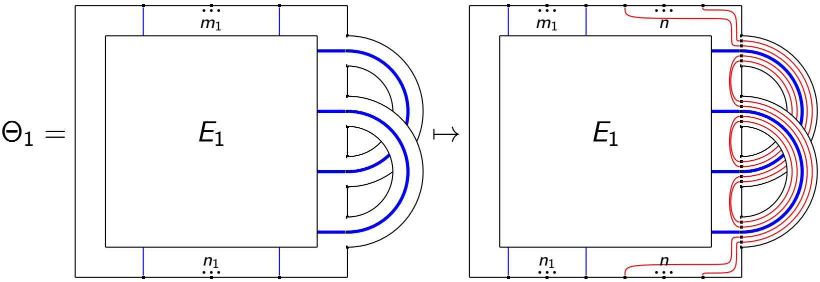
$\Theta_2 =$ 

$\mapsto$ 

$\overset{(?)}{:=} \mathrm{id}_n \otimes \Theta_2$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:



$\stackrel{(?)}{:=} \Theta_1 \otimes \mathrm{id}_n$

The Category $\mathcal{SQ}$ - Tensor Product

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$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\mathrm{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \mathrm{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \mathrm{id}_{m_2})} \circ \overline{(\mathrm{id}_{n_1} \otimes \Theta_2)}$$

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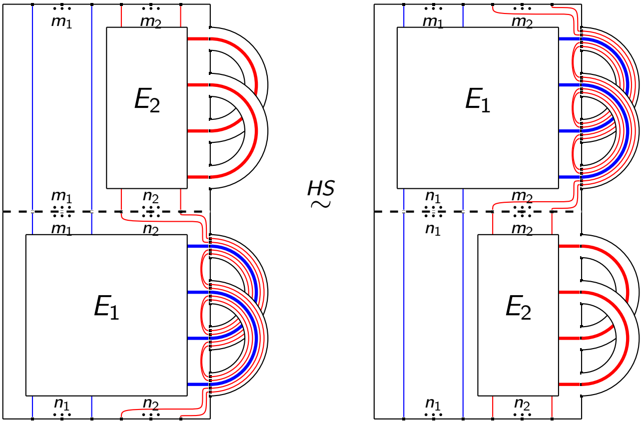

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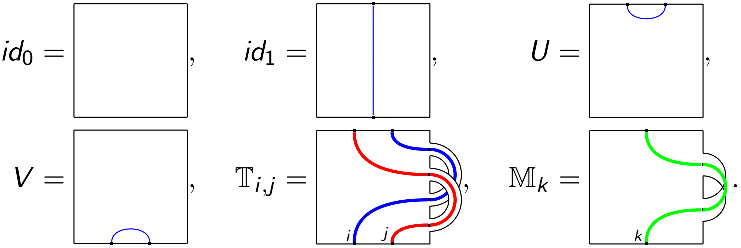
Proposition

The following is a monoidal generating set:



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Fact 4: The tensor product restricts to "horizontal stacking" on the TL subcategory.

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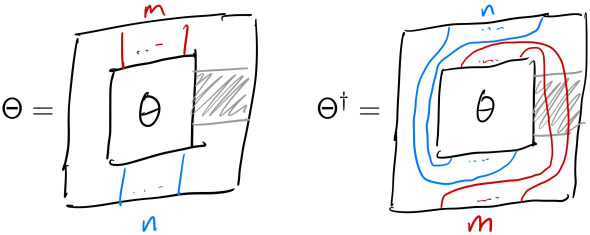
Fact 5: There is a rigid monoidal structure with $n^\dagger = n$ and the usual TL eval. $V_n \in Hom(2n, 0)$ and coeval. $U_n \in Hom(0, 2n)$ diagrams. Write $(\_)^\dagger : \mathcal{SQ} \to \mathcal{SQ}$ for the associated contravariant functor:



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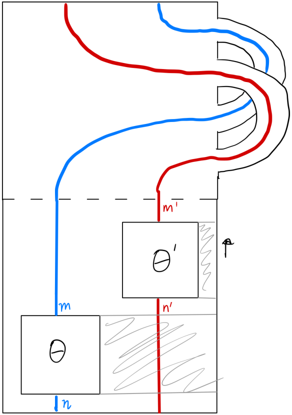
$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') =$ 

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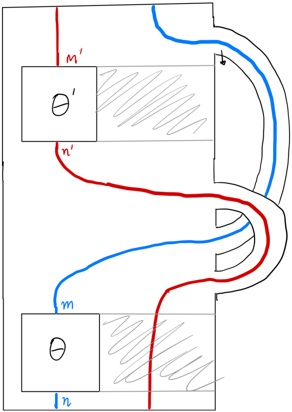

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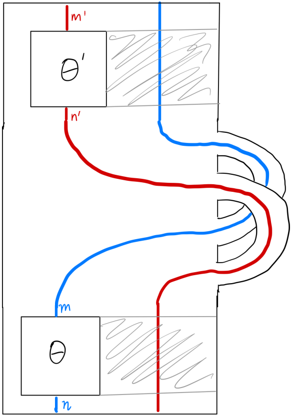

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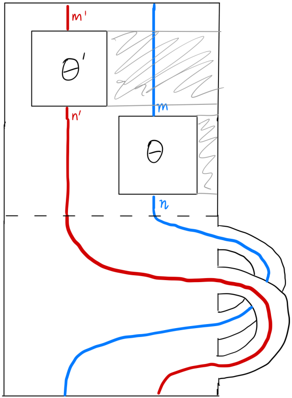

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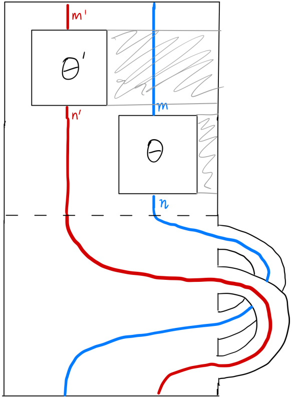
$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$  $= (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$

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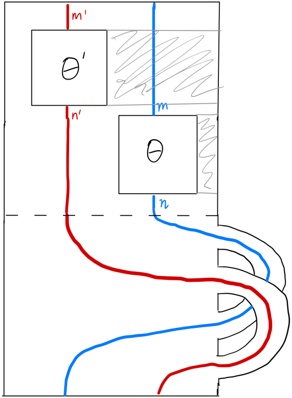

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These define a braiding $\mathbb{T} : \otimes \to \otimes^{\text{op}}$!!

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Fact 6: Consider the $\mathbb{T}_{i,j} \in \mathrm{Hom}(i \otimes j, j \otimes i)$:

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Note: The $\mathbb{T}_{i,j}$ are not invertible, so this doesn't make $\mathcal{SQ}$ braided monoidal. However, they do obey the (categorical) **YBE**:

$$(\mathbb{T}_{j,k} \otimes \mathrm{id}_i) \circ (\mathrm{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \mathrm{id}_k)$$
$$= (\mathrm{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \mathrm{id}_j) \circ (\mathrm{id}_i \otimes \mathbb{T}_{j,k})$$

The Category $\mathcal{SQ}$ - Tensor Product - More Facts!

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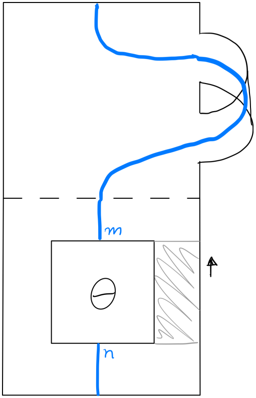

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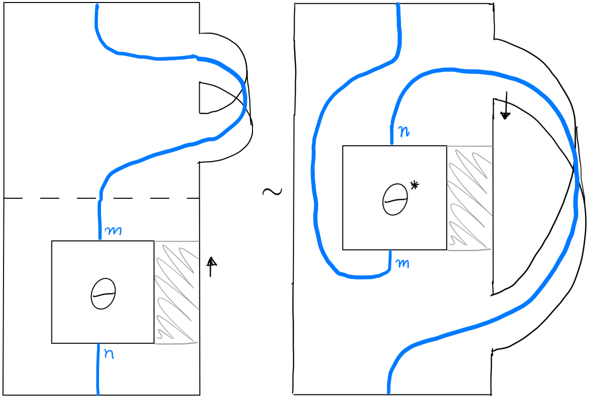

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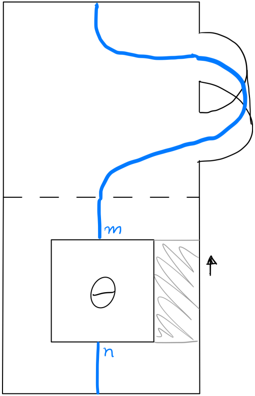
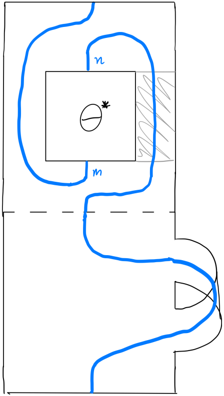
$\mathbb{M}_m \circ \Theta =$  $\sim$ 

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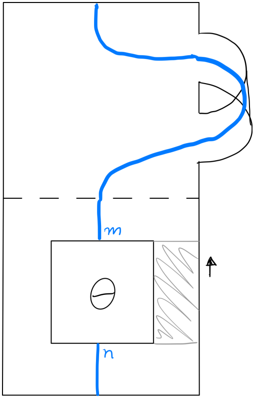
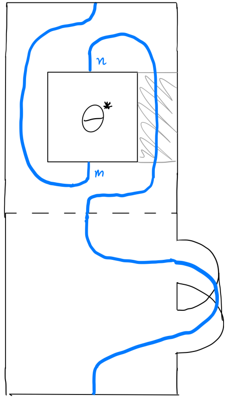
$\mathbb{M}_m \circ \Theta =$  $\sim$  $:= R(\Theta) \circ \mathbb{M}_n$

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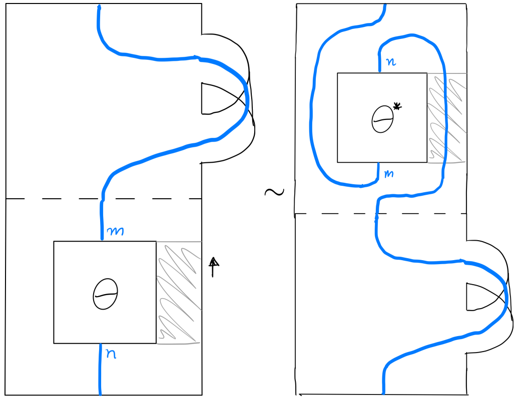
$\mathbb{M}_m \circ \Theta =$  $:= R(\Theta) \circ \mathbb{M}_n$

i.e. $\mathbb{M} : \text{id} \Rightarrow R$, where $R : \mathcal{SQ} \to \mathcal{SQ}$ is $R(\Theta) = (\Theta^*)^\dagger$.

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

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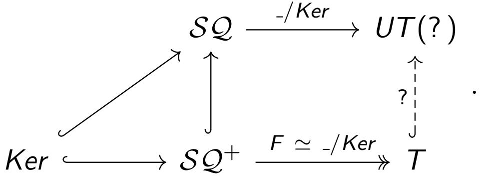
Scheme for "finitising": Find a target category T with a "finitising functor" $F : \mathcal{SQ}^+ \to T$ (full, ess. surj., monoidal).



The Infinity problem

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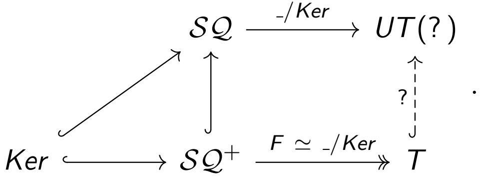

Try to lift, creating an "unorientable extension" of T, UT with $UF : \mathcal{SQ} \to UT$

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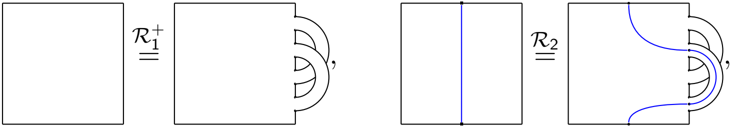

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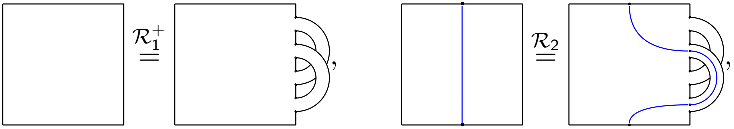

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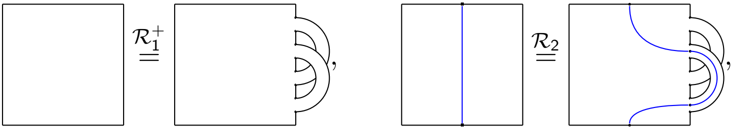

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Proposition

$\mathcal{SQ}^+(\alpha,\alpha)/(\mathcal{R}_2\&\mathcal{R}_1^+) = VTL(\alpha)$ *($\mathcal{R}_2$ forces $\alpha = \gamma$).*

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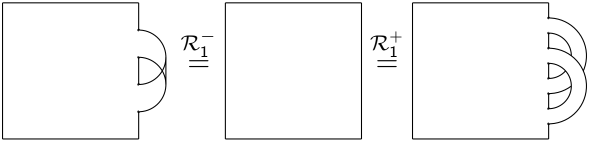

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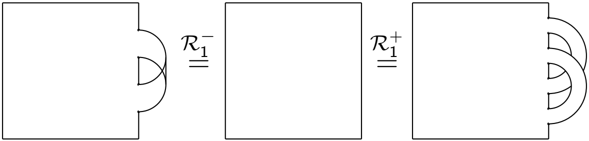

Conjecture

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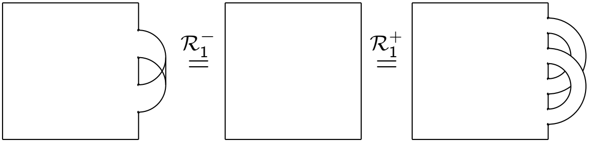

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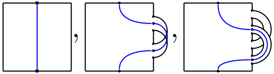
- $\mathrm{Hom}(1,1) = \mathbb{K}\Big\{$  $\Big\} \simeq \mathbb{K}\langle a \mid a^3 = a \rangle$,

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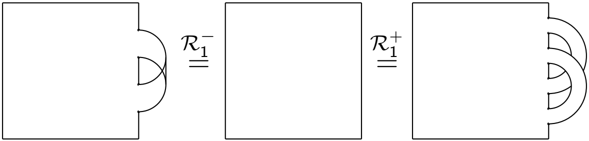

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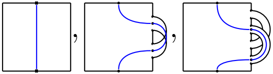
- $\mathrm{Hom}(1,1) = \mathbb{K}\Big\{$  $\Big\} \simeq \mathbb{K}\langle a \mid a^3 = a\rangle$,
- $\dim(\mathrm{Hom}(2,2)) \geq 23$.

Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

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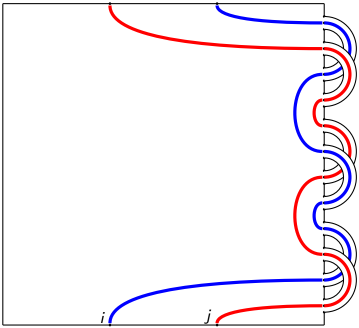

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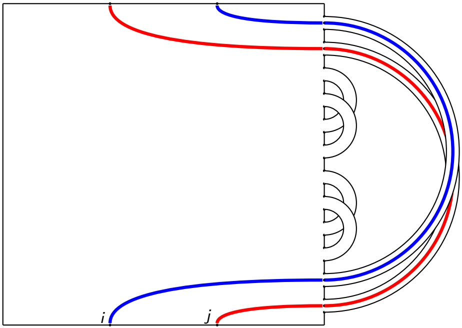

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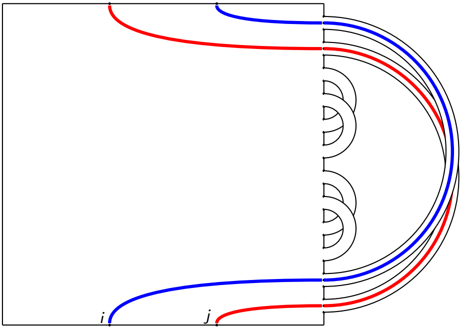

$= \mathbb{T}_{i,j} \mod \mathcal{R}_1$

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- What does the (monoidal) representation theory look like?
- What about non-functorial quotients? e.g. terminate at a finite genus

Thank You!

Questions?