



Temperley-Lieb categories on Non-Orientable Surfaces

Benjamin Morris¹

Joint work with Dionne Ibarra² and Gabriel Montoya-Vega³

¹University of Leeds

²Monash University, Melbourne

³CUNY Graduate Center, NYC

The Yang-Baxter Equation and all that, Bedlewo, June 2025



LONDON
MATHEMATICAL
SOCIETY
EST. 1865



Motivation

Enriching the structure skein-modules...



Motivation

Enriching the structure skein-modules...

Construct interesting low-dim “cobordism categories” amenable to representation theory study:

- ▶ Combinatorial Description
- ▶ Finite Dimensional Hom-spaces
- ▶ More structure? (tensor product, duals, braidings ... etc)



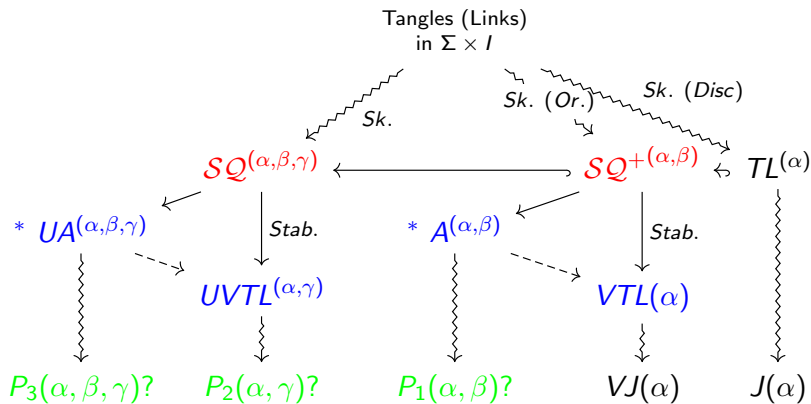
Motivation

Enriching the structure skein-modules...

Construct interesting low-dim “cobordism categories” amenable to representation theory study:

- ▶ Combinatorial Description
- ▶ Finite Dimensional Hom-spaces
- ▶ More structure? (tensor product, duals, braidings ... etc)

In particular, we consider **nested** $(0, 1, 2)$ - “cobordism categories”.





Example: Temperley-Lieb Category



Example: Temperley-Lieb Category

Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:



Example: Temperley-Lieb Category

Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:

- **Objects:** $(0, 1)$ part



Example: Temperley-Lieb Category

Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:

- **Objects:** $(0, 1)$ part - points in $[0, 1]$ (skeletally \mathbb{N})



Example: Temperley-Lieb Category

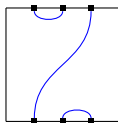
Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:

- **Objects:** $(0, 1)$ part - points in $[0, 1]$ (skeletally \mathbb{N})
- **Morphisms:** $(1, 2)$ part

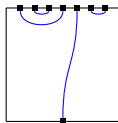
Example: Temperley-Lieb Category

Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:

- **Objects:** $(0, 1)$ part - points in $[0, 1]$ (skeletally \mathbb{N})
- **Morphisms:** $(1, 2)$ part - $\text{Hom}(n, m)$ is \mathbb{K} -linear combinations of type n, m “TL-diagrams”, (embedded intervals in $[0, 1]^2$):



$\in \text{Hom}(3, 3),$



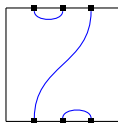
$\in \text{Hom}(1, 7).$

up to homeomorphisms of $[0, 1]^2$ (ambient isotopy).

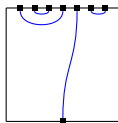
Example: Temperley-Lieb Category

Fix \mathbb{K} . For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0, 1, 2)$ - “cobordism category” where:

- **Objects:** $(0, 1)$ part - points in $[0, 1]$ (skeletally \mathbb{N})
- **Morphisms:** $(1, 2)$ part - $\text{Hom}(n, m)$ is \mathbb{K} -linear combinations of type n, m “TL-diagrams”, (embedded intervals in $[0, 1]^2$):



$\in \text{Hom}(3, 3),$



$\in \text{Hom}(1, 7).$

up to homeomorphisms of $[0, 1]^2$ (ambient isotopy).

$\{ \text{classes of diagrams} \} \leftrightarrow \{ \text{xless pair ptns of } V(n, m) \}$



Example: Temperley-Lieb Category



Example: Temperley-Lieb Category

Composition: “defined” on diagrams by vertically stacking

$$((\phi, \psi) \mapsto \psi \circ \phi):$$

Example: Temperley-Lieb Category

Composition: “defined” on diagrams by vertically stacking

$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \alpha \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} = \alpha D_2 \# D_1$$

The diagrams are square boxes with four vertices. Diagram 1 has a blue strand from the bottom-left to the top-right, with a blue arc on the top edge and a blue arc on the bottom edge. Diagram 2 has a blue strand from the bottom-left to the top-right, with a blue arc on the top edge and two blue arcs on the bottom edge. Diagram 3 is the vertical stack of Diagram 1 and Diagram 2, with a red circle at the intersection of the two strands.

Example: Temperley-Lieb Category

Composition: “defined” on diagrams by vertically stacking

$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \alpha \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} = \alpha D_2 \# D_1$$

The diagrams are square boxes with four vertices. Diagram 1 has a blue strand from the bottom-left to the top-right, with a blue arc on the top edge and a blue arc on the bottom edge. Diagram 2 has a blue strand from the bottom-left to the top-right, with a blue arc on the top edge and a blue arc on the bottom edge. Diagram 3 is the vertical stack of Diagram 1 and Diagram 2, with a red circle at the intersection of the two strands.

Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.

Example: Temperley-Lieb Category

Composition: “defined” on diagrams by vertically stacking

$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \alpha \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} = \alpha D_2 \# D_1$$

The diagram shows the composition of two Temperley-Lieb diagrams, D_2 and D_1 , by vertically stacking them. The first diagram, D_2 , has a blue strand entering from the bottom, curving to the right, and exiting from the top. The second diagram, D_1 , has a blue strand entering from the bottom, curving to the left, and exiting from the top. The composition $D_2 \circ D_1$ is shown as the vertical stack of these two diagrams. This is equal to a single diagram where the two strands cross, with a red circle indicating a crossing. This is then equal to α times the diagram where the two strands are parallel, representing the twist α .

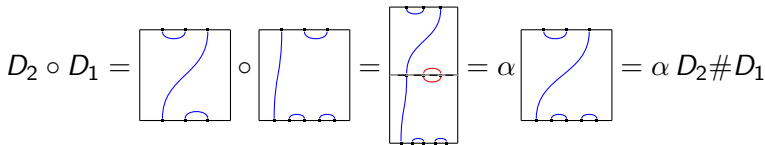
Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.

Tensor Product: “defined” on diagrams by horizontally stacking:

Example: Temperley-Lieb Category

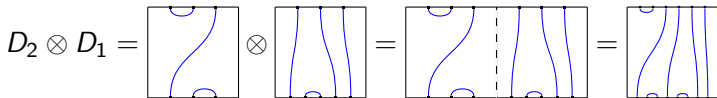
Composition: “defined” on diagrams by vertically stacking

$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \alpha \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array} = \alpha D_2 \# D_1$$


Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.

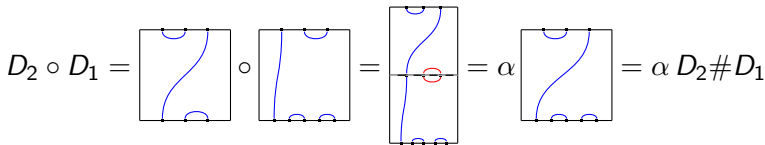
Tensor Product: “defined” on diagrams by horizontally stacking:

$$D_2 \otimes D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array}$$


Example: Temperley-Lieb Category

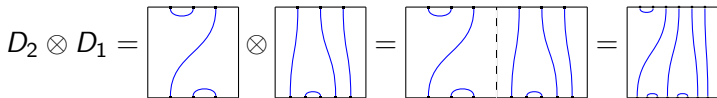
Composition: “defined” on diagrams by vertically stacking

$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \alpha \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array} = \alpha D_2 \# D_1$$


Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.

Tensor Product: “defined” on diagrams by horizontally stacking:

$$D_2 \otimes D_1 = \begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array}$$


$(n_1 \otimes n_2 = n_1 + n_2)$.



Our Aim



Our Aim

TL-category: a nested $(0, 1, 2)$ “cobordism category” with

- ▶ 0-manifolds: points \sqcup_{finite}^* .
- ▶ 1-manifolds: interval $[0, 1]$.
- ▶ 2-manifolds: square $[0, 1]^2$.

Our Aim

TL-category: a nested $(0, 1, 2)$ “cobordism category” with

- ▶ 0-manifolds: points $\sqcup_{\text{finite}} *$.
- ▶ 1-manifolds: interval $[0, 1]$.
- ▶ 2-manifolds: square $[0, 1]^2$.

We essentially will consider the question of when the “2” can have different surface type (especially **unorientable**) i.e.

$$[0, 1]^2 \longrightarrow \Sigma.$$

Our Aim

TL-category: a nested $(0, 1, 2)$ “cobordism category” with

- ▶ 0-manifolds: points $\sqcup_{\text{finite}} *$.
- ▶ 1-manifolds: interval $[0, 1]$.
- ▶ 2-manifolds: square $[0, 1]^2$.

We essentially will consider the question of when the “2” can have different surface type (especially **unorientable**) i.e.

$$[0, 1]^2 \longrightarrow \Sigma.$$

we will restrict to surface types Σ with one boundary component.

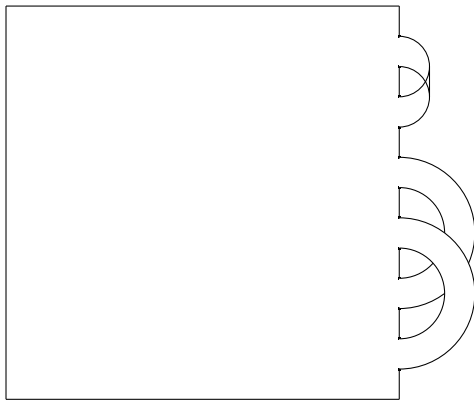


How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame

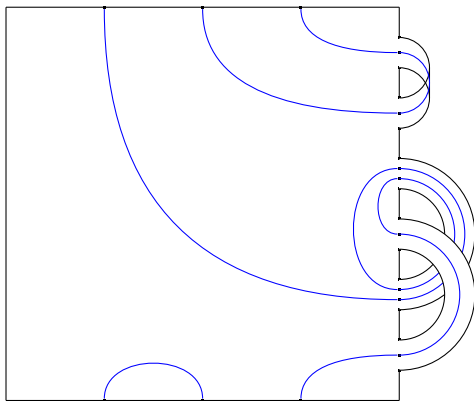
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



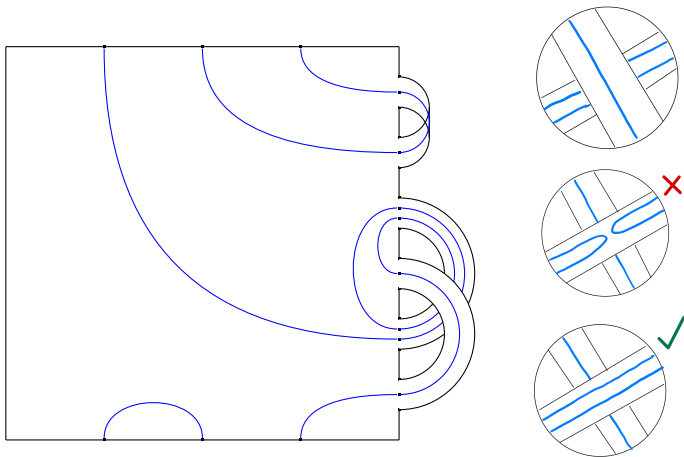
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



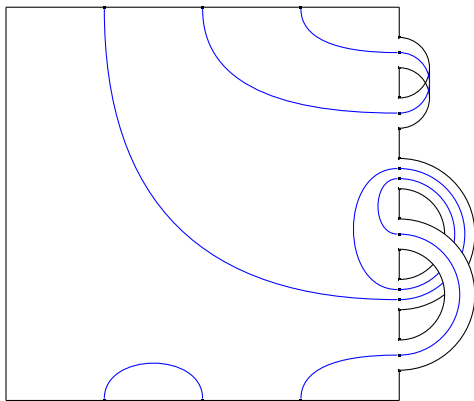
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



How to draw TL diagrams on other surfaces?

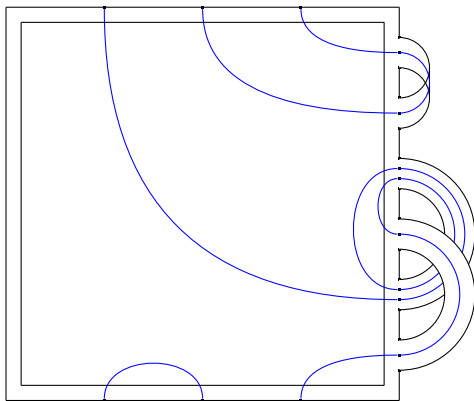
Proceed concretely; attach “handles” to our square frame



described by a quadruple (P, s, f, E) .

How to draw TL diagrams on other surfaces?

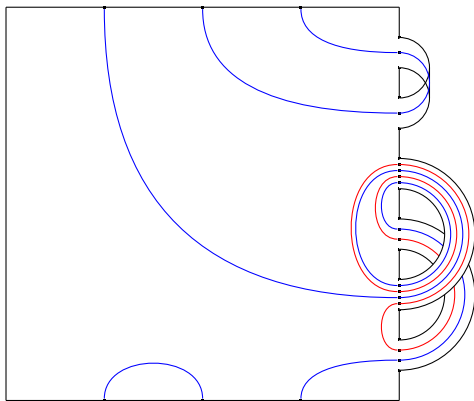
Proceed concretely; attach “handles” to our square frame



described by a quadruple (P, s, f, E) .

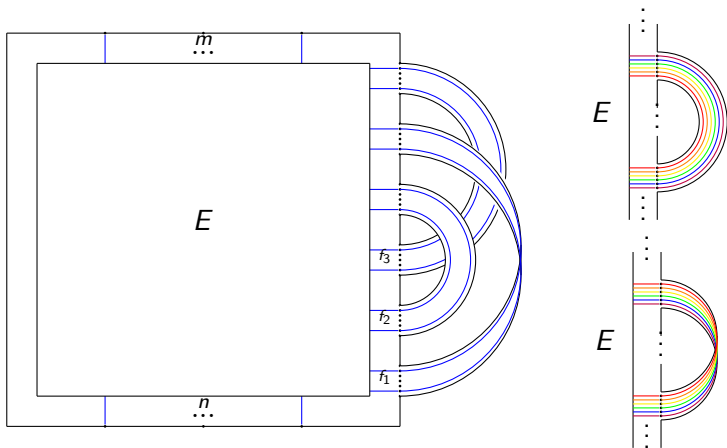
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



SWB diagrams

Square with bands (SWB) diagram encoded by $\Theta = (P, s, f, E)$
(type n, m)

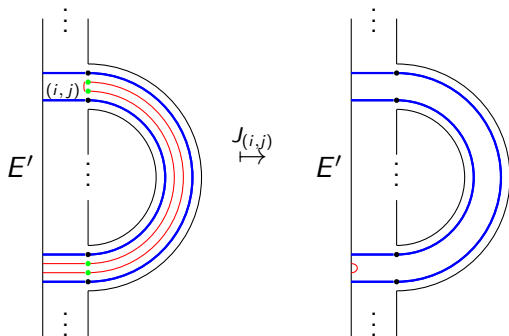




SWB diagrams - Isotopy

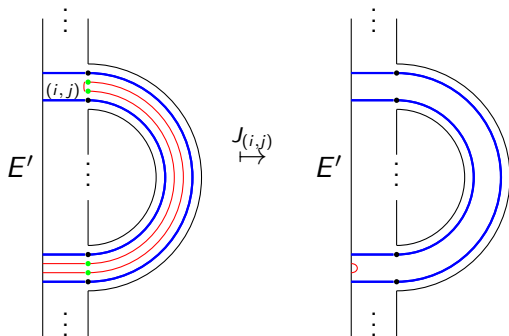
SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove “turnbacks” by “pull-throughs”



SWB diagrams - Isotopy

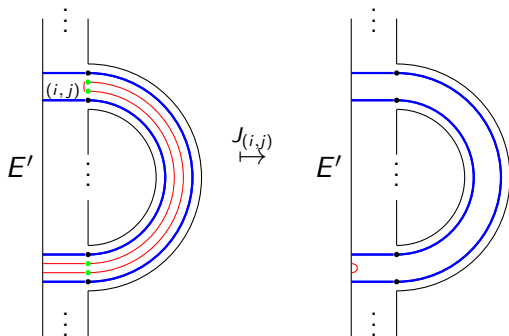
Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove “turnbacks” by “pull-throughs”



$$(P, s, f, E' \sqcup \{\{(i,j), (i,j+1)\}\}) \mapsto (P, s, f', o(E''))$$

SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove “turnbacks” by “pull-throughs”



$$(P, s, f, E' \sqcup \{(i,j), (i,j+1)\}) \mapsto (P, s, f', o(E''))$$

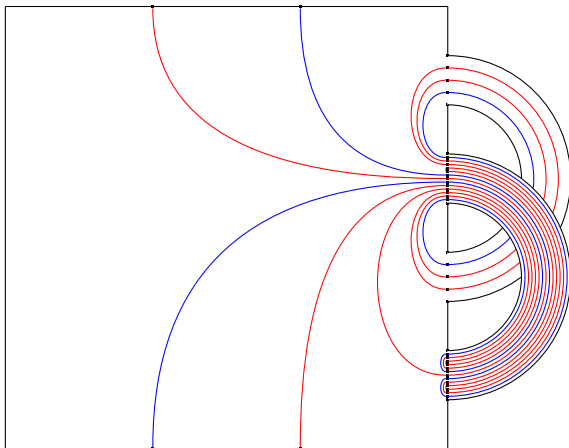
Generate an equivalence relation with this move.

SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks

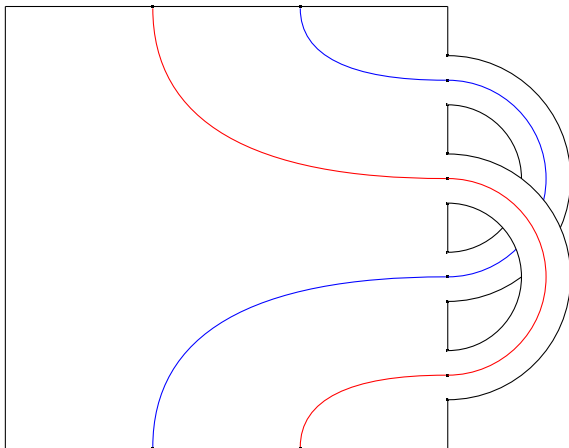
SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks



SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks

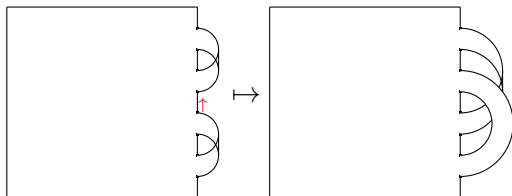


SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:

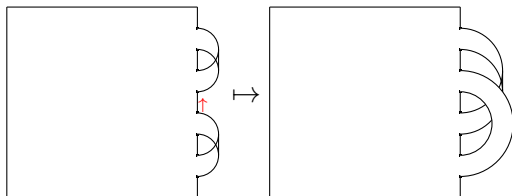
SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:



SWB diagrams - Handlesliding

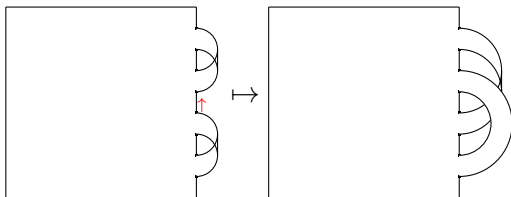
Different realisations of a surface are related by **handleslides**:



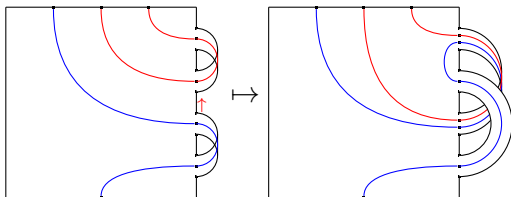
This induces moves on SWB diagrams:

SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:



This induces moves on SWB diagrams:



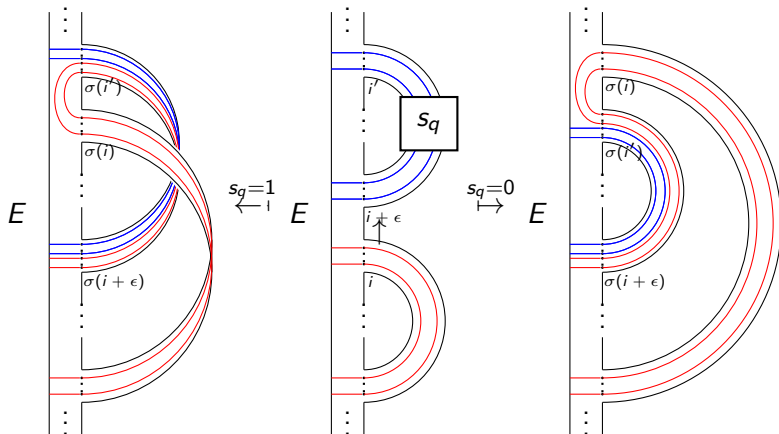


SWB diagrams - Handlesliding

Generically: “Two bands involved”

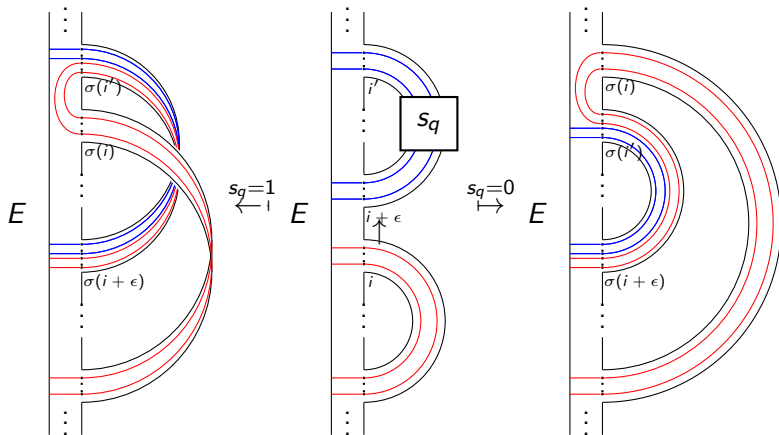
SWB diagrams - Handlesliding

Generically: "Two bands involved"



SWB diagrams - Handlesliding

Generically: “Two bands involved”



$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{ \text{“new red arcs”} \})$$

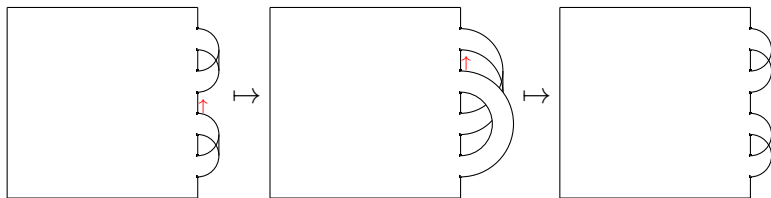
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

.

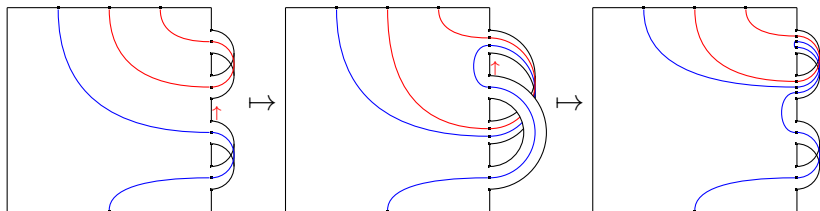
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.



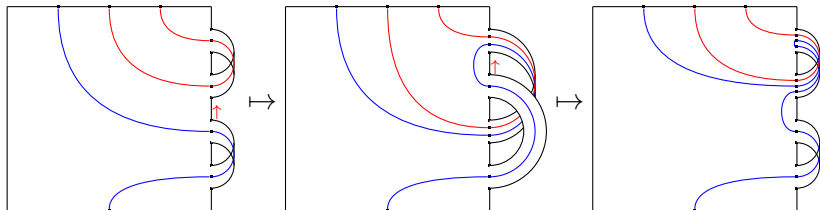
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.



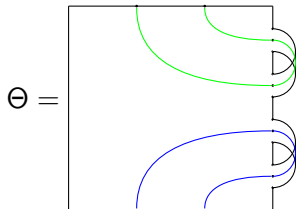
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

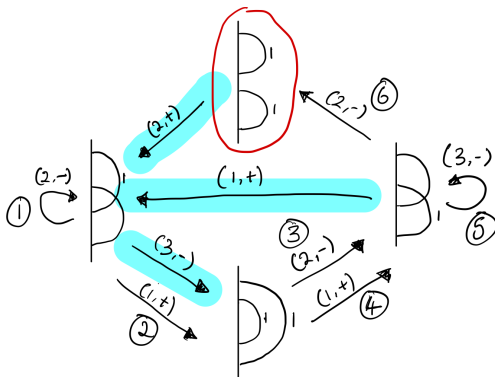
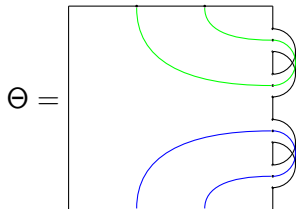


Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

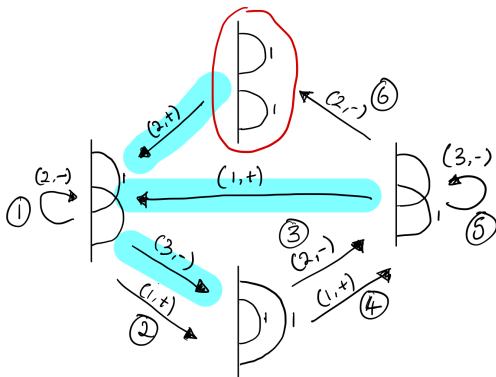
SWB diagrams - Handleslide Equivalence



SWB diagrams - Handleslide Equivalence



The diagram shows a rectangular region representing a surface. On the right side, there are two semi-circular boundary components. Each boundary component has a marked point, indicated by a small dot. A green curve starts from the top marked point and extends to the left. A blue curve starts from the bottom marked point and extends to the left. The text $\Theta =$ is placed to the left of the diagram.


$$A_2 = (3, +) \circ (4, -) \circ (1, +) \circ (2, +)$$

SWB diagrams - Handleslide Equivalence

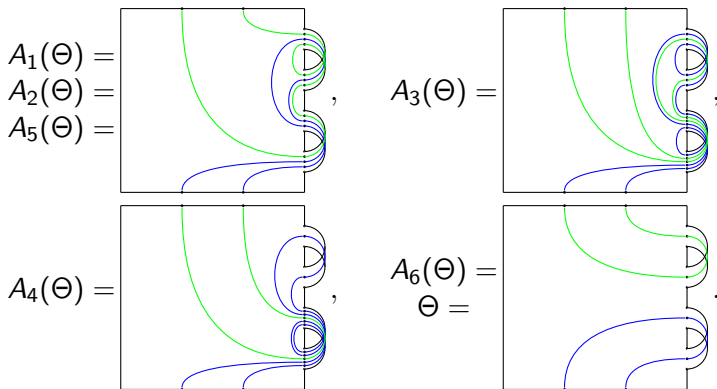
$$\begin{aligned} A_1(\Theta) &= \\ A_2(\Theta) &= \\ A_5(\Theta) &= \end{aligned} \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array},$$

$$A_4(\Theta) = \quad \begin{array}{c} \text{Diagram 4} \end{array},$$

$$A_3(\Theta) = \quad \begin{array}{c} \text{Diagram 5} \end{array},$$

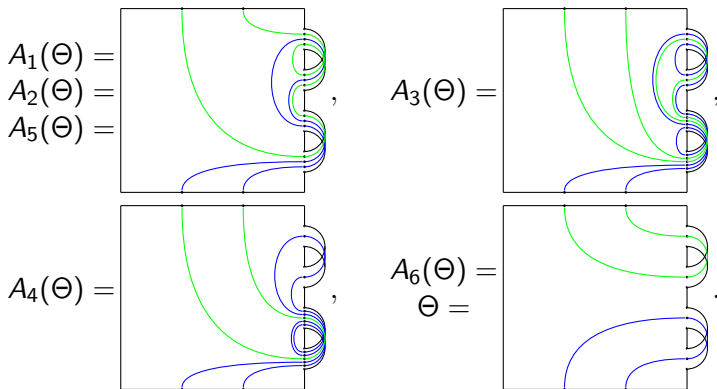
$$\begin{aligned} A_6(\Theta) &= \\ \Theta &= \end{aligned} \quad \begin{array}{c} \text{Diagram 6} \end{array}.$$

SWB diagrams - Handleslide Equivalence



$$\langle A_2, A_3, A_4 \mid A_3 A_2 = A_4, A_2 A_4 = A_4 A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

SWB diagrams - Handleslide Equivalence



$$\langle A_2, A_3, A_4 \mid A_3 A_2 = A_4, A_2 A_4 = A_4 A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

(Chord Diag. Pres. of Mapping Class Group - Bene 2009)

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$$(P, s) \sim \begin{array}{c} \left. \begin{array}{c} D_1 \\ \vdots \\ D_1 \end{array} \right\} t \\ \left. \begin{array}{c} D_0 \\ \vdots \\ D_0 \end{array} \right\} g \\ \left. \begin{array}{c} D_0 \\ \vdots \\ D_0 \end{array} \right\} b \end{array}$$

where $g, b \in \mathbb{Z}_{\geq 0}$,

Handleslide Equivalence - Caravan form

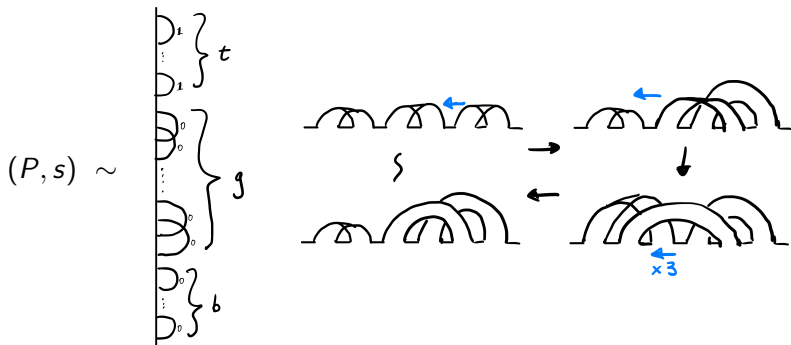
FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$$(P, s) \sim \left. \begin{array}{c} \left. \begin{array}{c} D_1 \\ \vdots \\ D_1 \end{array} \right\} t \\ \left. \begin{array}{c} D_0 \\ \vdots \\ D_0 \end{array} \right\} g \\ \left. \begin{array}{c} D_0 \\ \vdots \\ D_0 \end{array} \right\} b \end{array} \right\}$$

where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:



where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.



The Category \mathcal{SQ}

Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$.

The Category \mathcal{SQ}

Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. Define $\mathcal{SQ} = \mathcal{SQ}(\alpha, \beta, \gamma)$ as the \mathbb{K} -linear category with:

The Category \mathcal{SQ}

Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. Define $\mathcal{SQ} = \mathcal{SQ}(\alpha, \beta, \gamma)$ as the \mathbb{K} -linear category with:

- Objects: non-negative integers \mathbb{N}

The Category \mathcal{SQ}

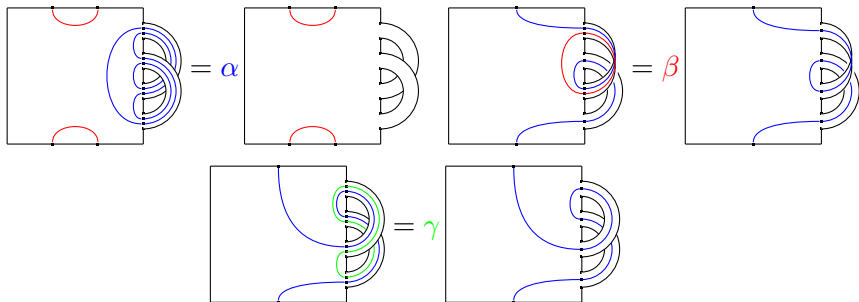
Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. Define $\mathcal{SQ} = \mathcal{SQ}(\alpha, \beta, \gamma)$ as the \mathbb{K} -linear category with:

- Objects: non-negative integers \mathbb{N}
- Morphisms: $\text{Hom}(n, m)$ consists of \mathbb{K} -linear combinations of HS classes of type (n, m) SWB diagrams, $[\Theta]_{\text{HS}}$,

The Category \mathcal{SQ}

Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. Define $\mathcal{SQ} = \mathcal{SQ}(\alpha, \beta, \gamma)$ as the \mathbb{K} -linear category with:

- Objects: non-negative integers \mathbb{N}
- Morphisms: $\text{Hom}(n, m)$ consists of \mathbb{K} -linear combinations of HS classes of type (n, m) SWB diagrams, $[\Theta]_{\text{HS}}$, modulo the **delooping** relations e.g.



The Category \mathcal{SQ}

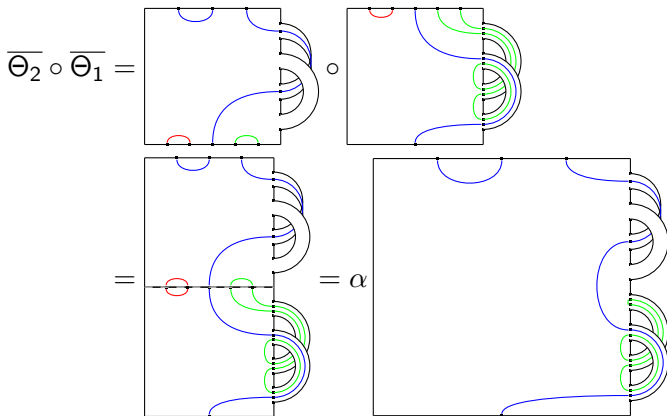
Composition: $\text{Hom}(n, m) \times \text{Hom}(m, l) \rightarrow \text{Hom}(n, l)$ is given by

$$\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}:$$

The Category \mathcal{SQ}

Composition: $\text{Hom}(n, m) \times \text{Hom}(m, l) \rightarrow \text{Hom}(n, l)$ is given by

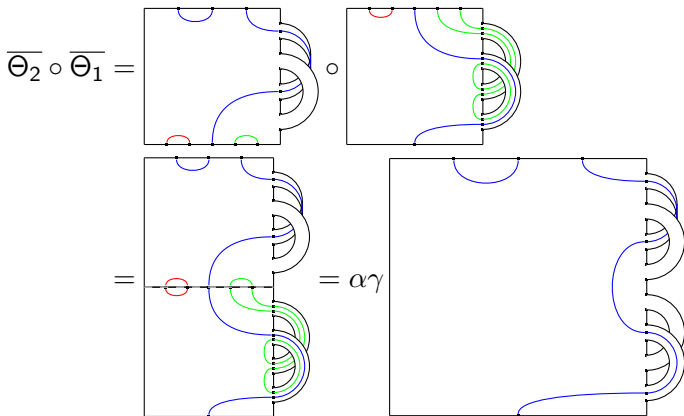
$$\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}:$$



The Category \mathcal{SQ}

Composition: $\text{Hom}(n, m) \times \text{Hom}(m, l) \rightarrow \text{Hom}(n, l)$ is given by

$$\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}:$$



The Category \mathcal{SQ}

Theorem

This defines a \mathbb{K} -linear category

The Category \mathcal{SQ}

Theorem

This defines a \mathbb{K} -linear category

Proof.

Main obstacle is well-definedness of composition:

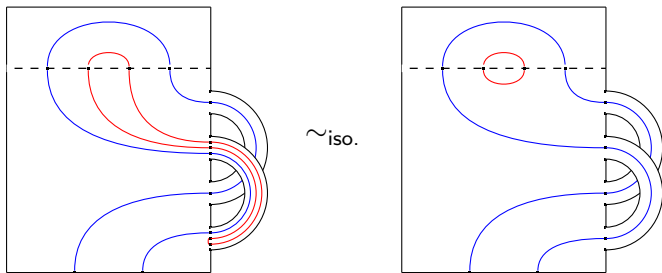
The Category \mathcal{SQ}

Theorem

This defines a \mathbb{K} -linear category

Proof.

Main obstacle is well-definedness of composition:





The Category \mathcal{SQ} - Basic Facts

The Category \mathcal{SQ} - Basic Facts

Fact 0: We have two (wide) subcategories:

- ▶ $\mathcal{SQ}^+ = \mathcal{SQ}^+(\alpha, \beta)$ with diagrams on orientable surfaces,
- ▶ $TL = TL(\alpha)$ with diagrams on squares.

The Category \mathcal{SQ} - Basic Facts

Fact 0: We have two (wide) subcategories:

- ▶ $\mathcal{SQ}^+ = \mathcal{SQ}^+(\alpha, \beta)$ with diagrams on orientable surfaces,
- ▶ $TL = TL(\alpha)$ with diagrams on squares.

Fact 1: We have a contravariant endofunctor $(-)^* : \mathcal{SQ} \rightarrow \mathcal{SQ}$, given by $n^* = n$ and which flips diagrams upside down.

The Category \mathcal{SQ} - Basic Facts

Fact 0: We have two (wide) subcategories:

- ▶ $\mathcal{SQ}^+ = \mathcal{SQ}^+(\alpha, \beta)$ with diagrams on orientable surfaces,
- ▶ $TL = TL(\alpha)$ with diagrams on squares.

Fact 1: We have a contravariant endofunctor $(-)^* : \mathcal{SQ} \rightarrow \mathcal{SQ}$, given by $n^* = n$ and which flips diagrams upside down.

Fact 2: For any $\Theta \in Sq(n, m)$, there exist **unique** integers l_s, l_t and l_u , and $\Theta' \in Sq(n, m)$ without closed loops, such that:

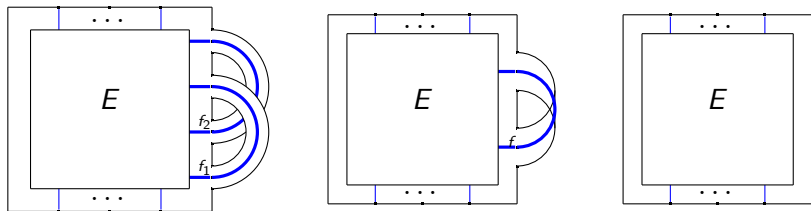
$$\overline{\Theta} = \alpha^{l_s} \beta^{l_t} \gamma^{l_u} \overline{\Theta'} \in \text{Hom}(n, m).$$

The Category \mathcal{SQ} - Basic Facts

Fact 3: Any morphism $\bar{\Theta} \in \text{Hom}(n, m)$ has a factorisation in terms of diagrams of the following form

The Category \mathcal{SQ} - Basic Facts

Fact 3: Any morphism $\bar{\Theta} \in \text{Hom}(n, m)$ has a factorisation in terms of diagrams of the following form (AND)



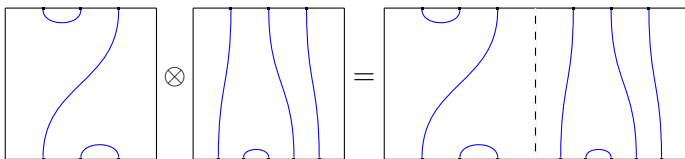


The Category \mathcal{SQ} - Tensor Product

Recall: In TL case we had a tensor product given by “horizontal stacking” of diagrams:

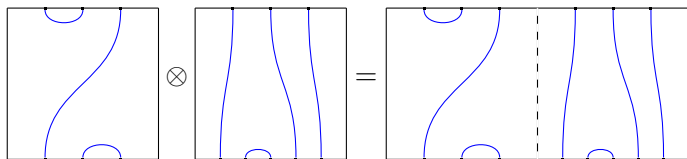
The Category \mathcal{SQ} - Tensor Product

Recall: In TL case we had a tensor product given by “horizontal stacking” of diagrams:



The Category \mathcal{SQ} - Tensor Product

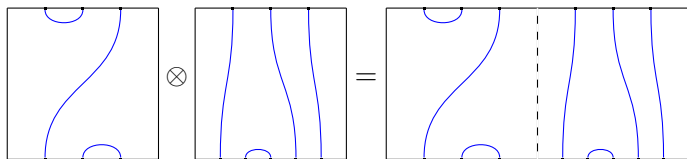
Recall: In TL case we had a tensor product given by “horizontal stacking” of diagrams:



Can we extend this to a tensor product on \mathcal{SQ} which has $n_1 \otimes n_2 = n_1 + n_2$ on objects.

The Category \mathcal{SQ} - Tensor Product

Recall: In TL case we had a tensor product given by “horizontal stacking” of diagrams:



Can we extend this to a tensor product on \mathcal{SQ} which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

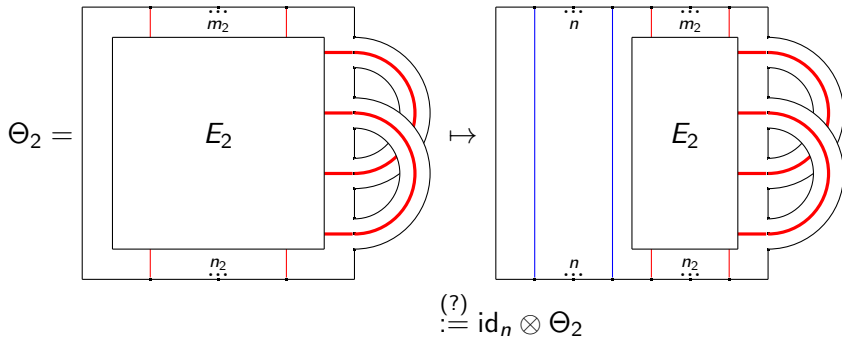


The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:

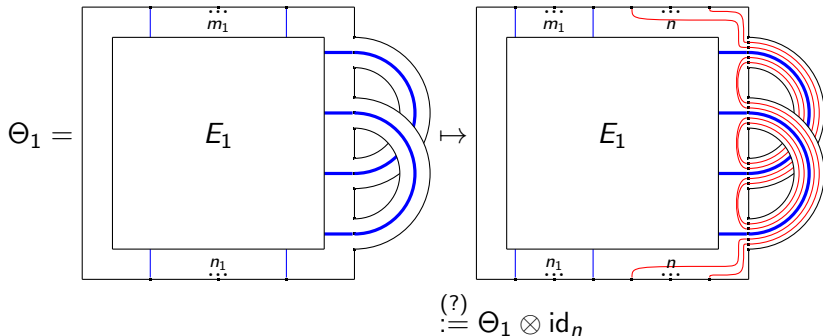
The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:



The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:





The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:

The Category \mathcal{SQ} - Tensor Product

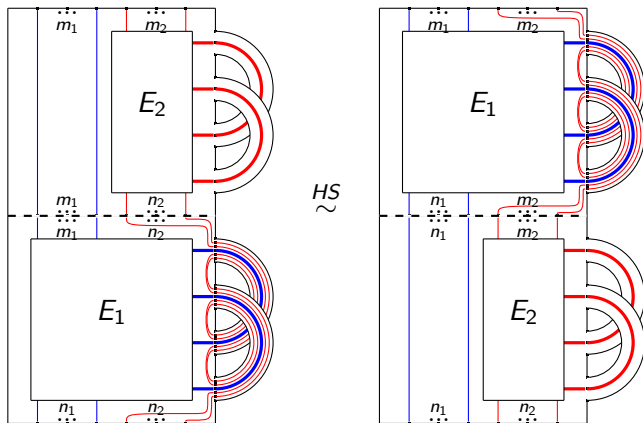
Indirect answer: Step 3 - Insist upon functoriality:

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\text{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \text{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \text{id}_{m_2})} \circ \overline{(\text{id}_{n_1} \otimes \Theta_2)}$$

The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\text{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \text{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \text{id}_{m_2})} \circ \overline{(\text{id}_{n_1} \otimes \Theta_2)}$$





The Category \mathcal{SQ} - Tensor Product

Theorem

This defines a tensor product on \mathcal{SQ} .



The Category \mathcal{SQ} - Tensor Product

Theorem

This defines a tensor product on \mathcal{SQ} .

and

The Category \mathcal{SQ} - Tensor Product

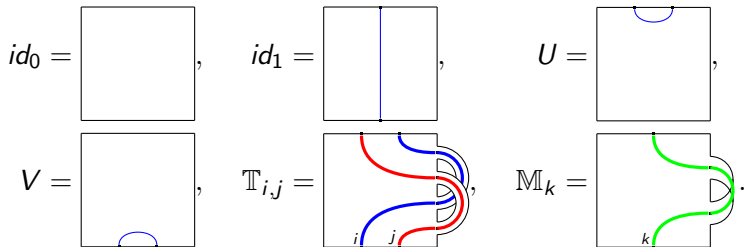
Theorem

This defines a tensor product on \mathcal{SQ} .

and

Proposition

The following is a monoidal generating set:





The Category \mathcal{SQ} - Tensor Product - More Facts!



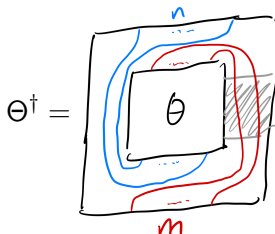
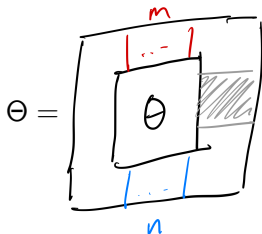
The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 4: The tensor product restricts to “horizontal stacking” on the TL subcategory.

The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 4: The tensor product restricts to “horizontal stacking” on the TL subcategory.

Fact 5: There is a rigid monoidal structure with $n^\dagger = n$ and the usual TL eval. $V_n \in \text{Hom}(2n, 0)$ and coeval. $U_n \in \text{Hom}(0, 2n)$ diagrams. Write $(-)^\dagger : \mathcal{SQ} \rightarrow \mathcal{SQ}$ for the associated contravariant functor:



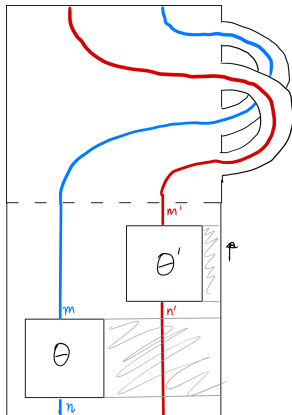
The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

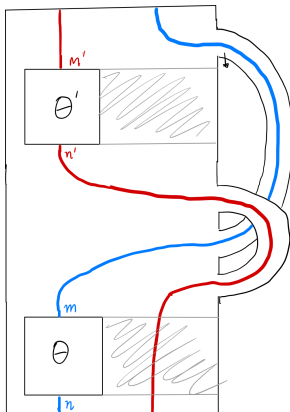
$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') =$$



The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

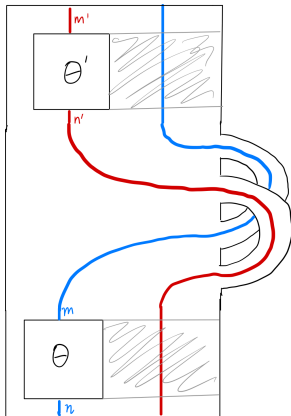
$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$$



The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

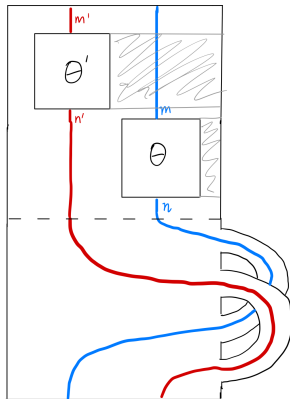
$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$$



The Category \mathcal{SQ} - Tensor Product - More Facts!

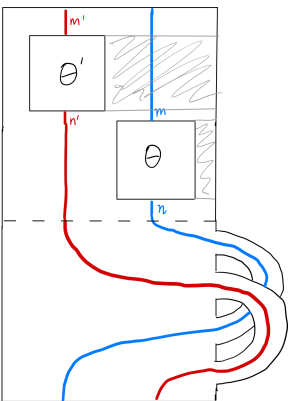
Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$$



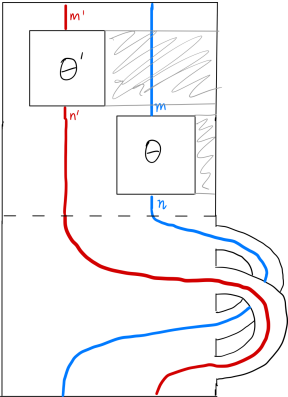
The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim \text{Diagram} = (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$$


The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim \quad \quad \quad = (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$$


These define a braiding $\mathbb{T} : \otimes \rightarrow \otimes^{\text{op}}$!!

The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 6: Consider the $\mathbb{T}_{i,j} \in \text{Hom}(i \otimes j, j \otimes i)$:

$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') = (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$$

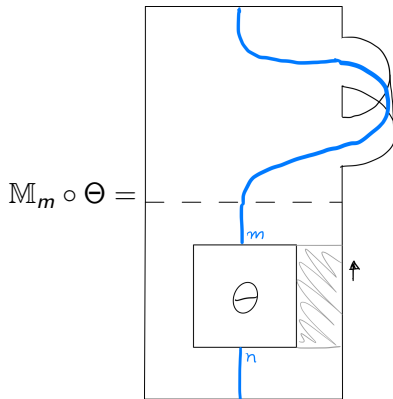
These define a braiding $\mathbb{T} : \otimes \rightarrow \otimes^{\text{op}}$!!

Note: The $\mathbb{T}_{i,j}$ are not invertible, so this doesn't make \mathcal{SQ} braided monoidal. However, they do obey the (categorical) **YBE**:

$$\begin{aligned} & (\mathbb{T}_{j,k} \otimes \text{id}_i) \circ (\text{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \text{id}_k) \\ &= (\text{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \text{id}_j) \circ (\text{id}_i \otimes \mathbb{T}_{j,k}) \end{aligned}$$

The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 7: Consider the $\mathbb{M}_n \in \text{Hom}(m, m)$:

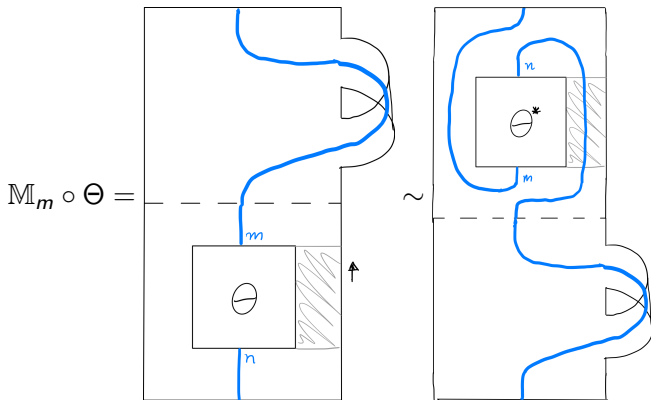


Fact 7: Consider the $\mathbb{M}_n \in \text{Hom}(m, m)$:



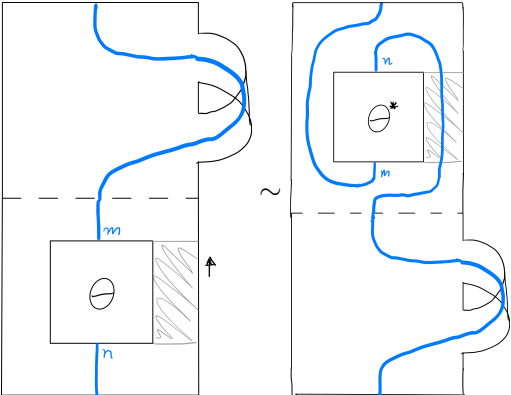
The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 7: Consider the $\mathbb{M}_n \in \text{Hom}(m, m)$:



The Category \mathcal{SQ} - Tensor Product - More Facts!

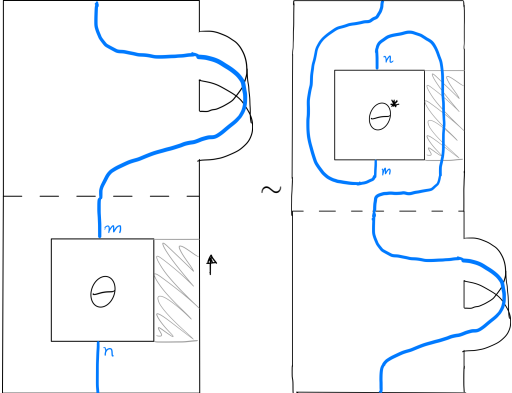
Fact 7: Consider the $\mathbb{M}_n \in \text{Hom}(m, m)$:

$$\mathbb{M}_m \circ \Theta = \text{Diagram 1} \sim \text{Diagram 2} := R(\Theta) \circ \mathbb{M}_n$$


The diagrammatic equation illustrates the relationship between the composition of the multiplication \mathbb{M}_m and the comultiplication Θ , and the composition of the comultiplication $R(\Theta)$ and the multiplication \mathbb{M}_n . The left diagram shows a blue strand entering from the bottom, passing through a box labeled Θ , and then continuing upwards. The right diagram shows a blue strand entering from the bottom, passing through a box labeled Θ^* , and then continuing upwards. The two diagrams are connected by a tilde symbol, indicating they are equivalent.

The Category \mathcal{SQ} - Tensor Product - More Facts!

Fact 7: Consider the $\mathbb{M}_n \in \text{Hom}(m, m)$:

$$\mathbb{M}_m \circ \Theta = \sim \quad := R(\Theta) \circ \mathbb{M}_n$$


i.e. $\mathbb{M} : \text{id} \Rightarrow R$, where $R : \mathcal{SQ} \rightarrow \mathcal{SQ}$ is $R(\Theta) = (\Theta^*)^\dagger$.



The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

Scheme for “finitising”: Find a target category T with a “finitising functor” $F : SQ^+ \rightarrow T$ (full, ess. surj., monoidal).

$$\begin{array}{ccccc}
 & & SQ & \xrightarrow{-/Ker} & UT(?) \\
 & \nearrow & \uparrow & & \uparrow \text{?} \\
 Ker & \hookrightarrow & SQ^+ & \xrightarrow{F \simeq -/Ker} \twoheadrightarrow & T
 \end{array}
 \cdot$$

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

Scheme for “finitising”: Find a target category T with a “finitising functor” $F : SQ^+ \rightarrow T$ (full, ess. surj., monoidal).

$$\begin{array}{ccccc}
 & & SQ & \xrightarrow{-/Ker} & UT(?) \\
 & \nearrow & \uparrow & & \uparrow \text{?} \\
 Ker & \hookrightarrow & SQ^+ & \xrightarrow{F \simeq -/Ker} \twoheadrightarrow & T
 \end{array}
 \cdot$$

Try to lift, creating an “unorientable extension” of T , UT with $UF : SQ \rightarrow UT$



The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

If you're like me, it bothers you that \mathcal{SQ} is not braided...

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

If you're like me, it bothers you that \mathcal{SQ} is not braided...

IDEA: lets force it to be!

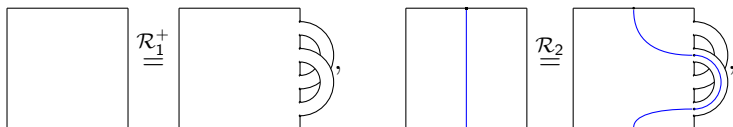
The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

If you're like me, it bothers you that \mathcal{SQ} is not braided...

IDEA: lets force it to be!

The smallest such quotient of \mathcal{SQ} is obtained by imposing relations \mathcal{R}_1^+ and \mathcal{R}_2 :



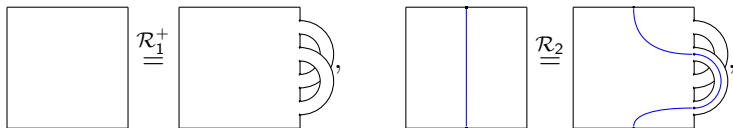
The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

If you're like me, it bothers you that \mathcal{SQ} is not braided...

IDEA: lets force it to be!

The smallest such quotient of \mathcal{SQ} is obtained by imposing relations \mathcal{R}_1^+ and \mathcal{R}_2 :



This is stabilisation! *only need \mathcal{R}_1^+ if α non-invertible.

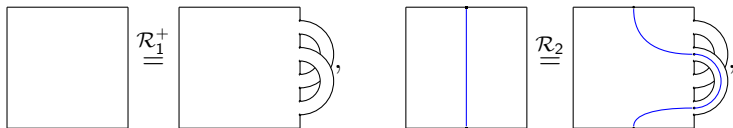
The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

If you're like me, it bothers you that \mathcal{SQ} is not braided...

IDEA: lets force it to be!

The smallest such quotient of \mathcal{SQ} is obtained by imposing relations \mathcal{R}_1^+ and \mathcal{R}_2 :



This is stabilisation! *only need \mathcal{R}_1^+ if α non-invertible.

Proposition

$\mathcal{SQ}^+(\alpha, \alpha)/(\mathcal{R}_2 \& \mathcal{R}_1^+) = VTL(\alpha)$ (\mathcal{R}_2 forces $\alpha = \gamma$).



The Infinity problem

Another very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$.

The Infinity problem

Another very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$. This requires imposing the relation \mathcal{R}_1 :



The Infinity problem

Another very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$. This requires imposing the relation \mathcal{R}_1 :



Conjecture

$\mathcal{SQ}/\mathcal{R}_1$ has finite dimensional hom-spaces.

The Infinity problem

Another very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$. This requires imposing the relation \mathcal{R}_1 :



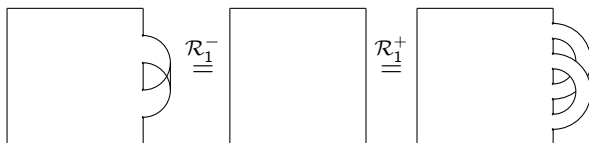
Conjecture

$\mathcal{SQ}/\mathcal{R}_1$ has finite dimensional hom-spaces.

$$\blacktriangleright \text{Hom}(1, 1) = \mathbb{K} \left\{ \begin{array}{c} \text{[Square with 1 vertical band]} \\ \text{[Square with 2 nested bands]} \\ \text{[Square with 3 nested bands]} \end{array} \right\} \simeq \mathbb{K} \langle a \mid a^3 = a \rangle,$$

The Infinity problem

Another very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$. This requires imposing the relation \mathcal{R}_1 :



Conjecture

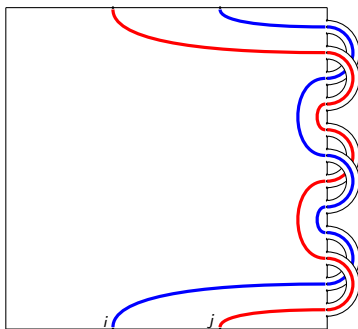
$\mathcal{SQ}/\mathcal{R}_1$ has finite dimensional hom-spaces.

- ▶ $\text{Hom}(1, 1) = \mathbb{K} \left\{ \begin{array}{c} \text{[Diagram 1: Square with a vertical blue line]} \\ \text{[Diagram 2: Square with a blue band on the right]} \\ \text{[Diagram 3: Square with three nested blue bands on the right]} \end{array} \right\} \simeq \mathbb{K} \langle a \mid a^3 = a \rangle,$
- ▶ $\dim(\text{Hom}(2, 2)) \geq 23.$

Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

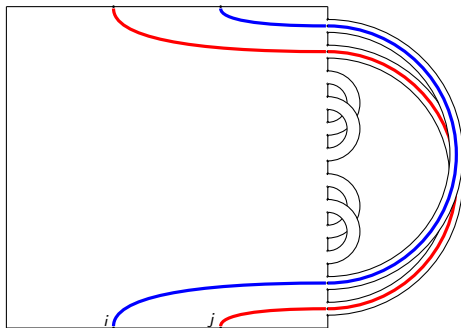
$$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$$



Example in $\mathcal{SQ}/\mathcal{R}_1$

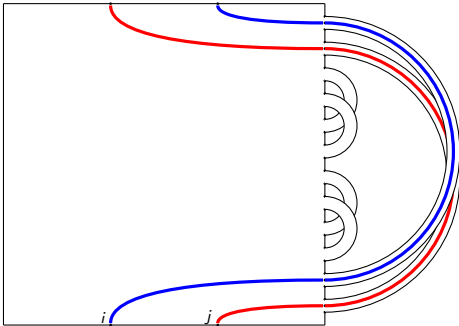
Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$$



Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$$


$$= \mathbb{T}_{i,j} \mod \mathcal{R}_1$$



Outlook



Outlook

- Can we prove this conjecture? (Combinatorially?)



Outlook

- ▶ Can we prove this conjecture? (Combinatorially?)
- ▶ Can we give a more universal description?



Outlook

- ▶ Can we prove this conjecture? (Combinatorially?)
- ▶ Can we give a more universal description?
- ▶ Can we find presentations for SQ and its quotients?



Outlook

- ▶ Can we prove this conjecture? (Combinatorially?)
- ▶ Can we give a more universal description?
- ▶ Can we find presentations for \mathcal{SQ} and its quotients?

$$(\mathrm{id}_j \otimes \mathbb{M}_i) \circ (\mathbb{M}_j \otimes \mathrm{id}_i) \circ \mathbb{M}_{i+j} = \mathbb{T}_{i,j} \circ (\mathrm{id}_{i+j} \otimes \mathbb{M}_0)$$



Outlook

- ▶ Can we prove this conjecture? (Combinatorially?)
- ▶ Can we give a more universal description?
- ▶ Can we find presentations for \mathcal{SQ} and its quotients?

$$(\mathrm{id}_j \otimes \mathbb{M}_i) \circ (\mathbb{M}_j \otimes \mathrm{id}_i) \circ \mathbb{M}_{i+j} = \mathbb{T}_{i,j} \circ (\mathrm{id}_{i+j} \otimes \mathbb{M}_0)$$

- ▶ What does the (monoidal) representation theory look like?



Outlook

- ▶ Can we prove this conjecture? (Combinatorially?)
- ▶ Can we give a more universal description?
- ▶ Can we find presentations for \mathcal{SQ} and its quotients?

$$(\mathrm{id}_j \otimes \mathbb{M}_i) \circ (\mathbb{M}_j \otimes \mathrm{id}_i) \circ \mathbb{M}_{i+j} = \mathbb{T}_{i,j} \circ (\mathrm{id}_{i+j} \otimes \mathbb{M}_0)$$

- ▶ What does the (monoidal) representation theory look like?
- ▶ What about non-functorial quotients? e.g. terminate at a finite genus



Thank You!

Questions?