UNIVERSITY OF LEEDS

A Diagram Category for Non-Orientable Surfaces

Benjamin Morris[1]
Joint work with Dionne Ibarra[2], Gabriel Montoya-Vega[3], and Paul Martin[1] (supervisor)

[1]University of Leeds
[2]Monash University, Melbourne
[3]CUNY Graduate Center, NYC

ITMAIA 2025

Motivation

Construct interesting low-dim “cobordism categories” amenable to rep. th. study:

- Linear
- Combinatorial
- Finite Dimensional Hom-spaces
- More structure? (monoidal... etc)

In particular, we consider **nested** $(0, 1, 2)$ - “cobordism categories”.

Example: Temperley-Lieb Category

Example: Temperley-Lieb Category

Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, $\mathit{TL}(\alpha)$ is a $(0,1,2)$ - "cobordism category" where:

Example: Temperley-Lieb Category

Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0,1,2)$ - “cobordism category” where:

- **Objects:** $(0,1)$ part

Example: Temperley-Lieb Category

Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, *TL*(α) is a $(0,1,2)$ - "cobordism category" where:

- **Objects:** $(0,1)$ part - points in $[0,1]$ (skeletally $\mathbb{N}$)

Example: Temperley-Lieb Category

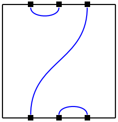
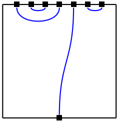
Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0,1,2)$ - "cobordism category" where:

- **Objects:** $(0,1)$ part - points in $[0,1]$ (skeletally $\mathbb{N}$)
- **Morphisms:** $(1,2)$ part

Example: Temperley-Lieb Category

Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0,1,2)$ - "cobordism category" where:

- **Objects:** $(0,1)$ part - points in $[0,1]$ (skeletally $\mathbb{N}$)
- **Morphisms:** $(1,2)$ part - $\mathrm{Hom}(n,m)$ is $\mathbb{K}$-linear combinations of type n,m "TL-diagrams", (embedded intervals in $[0,1]^2$):

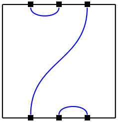
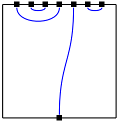
 $\in \mathrm{Hom}(3,3)$,  $\in \mathrm{Hom}(1,7)$.

up to homeomorphisms of $[0,1]^2$ (ambient isotopy).

Example: Temperley-Lieb Category

Fix $\mathbb{K}$. For $\alpha \in \mathbb{K}$, $TL(\alpha)$ is a $(0,1,2)$ - "cobordism category" where:

- **Objects:** $(0,1)$ part - points in $[0,1]$ (skeletally $\mathbb{N}$)
- **Morphisms:** $(1,2)$ part - $\mathrm{Hom}(n,m)$ is $\mathbb{K}$-linear combinations of type n, m "TL-diagrams", (embedded intervals in $[0,1]^2$):

 $\in \mathrm{Hom}(3,3)$,  $\in \mathrm{Hom}(1,7)$.

up to homeomorphisms of $[0,1]^2$ (ambient isotopy).
$\{$ classes of diagrams $\} \leftrightarrow \{$ xless pair ptns of $V(n,m)$ $\}$

Example: Temperley-Lieb Category

UNIVERSITY OF LEEDS

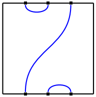
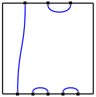
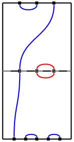
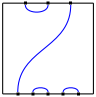
Example: Temperley-Lieb Category

Composition: "defined" on diagrams by vertically stacking $((\phi, \psi) \mapsto \psi \circ \phi)$:

UNIVERSITY OF LEEDS

Example: Temperley-Lieb Category

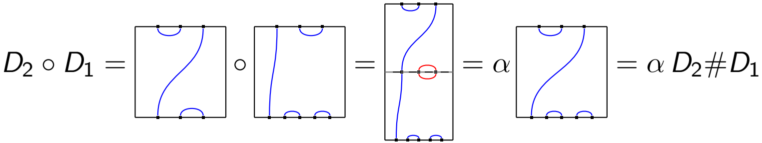
Composition: "defined" on diagrams by vertically stacking $((\phi, \psi) \mapsto \psi \circ \phi)$:

$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1$

UNIVERSITY OF LEEDS

Example: Temperley-Lieb Category

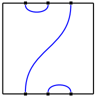
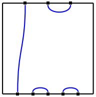
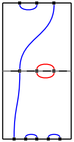
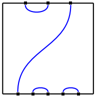
<u>Composition:</u> "defined" on diagrams by vertically stacking $((\phi, \psi) \mapsto \psi \circ \phi)$:



Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)}\, D_2 \# D_1$.

Example: Temperley-Lieb Category

Composition: "defined" on diagrams by vertically stacking $((\phi, \psi) \mapsto \psi \circ \phi)$:

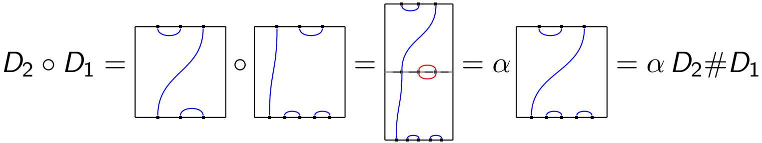
$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1$

Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)}\, D_2 \# D_1$.

Tensor Product: "defined" on diagrams by horizontally stacking:

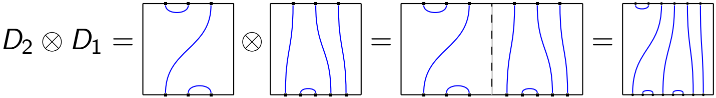
Example: Temperley-Lieb Category

<u>Composition:</u> "defined" on diagrams by vertically stacking $((\phi, \psi) \mapsto \psi \circ \phi)$:



Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} \, D_2 \# D_1$.

<u>Tensor Product:</u> "defined" on diagrams by horizontally stacking:

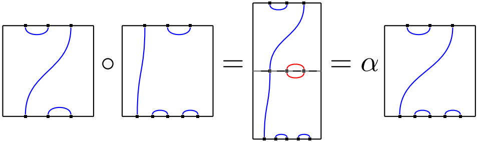


UNIVERSITY OF LEEDS

Example: Temperley-Lieb Category

Composition: "defined" on diagrams by vertically stacking $((\phi,\psi)\mapsto\psi\circ\phi)$:

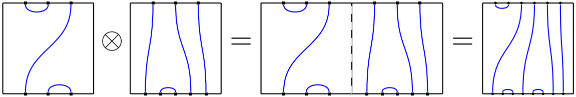
$D_2 \circ D_1 =$ … $= \alpha\, D_2\# D_1$



Generically $D_2 \circ D_1 = \alpha^{L(D_1,D_2)}\, D_2\# D_1$.

Tensor Product: "defined" on diagrams by horizontally stacking:

$D_2 \otimes D_1 =$ …



$(n_1 \otimes n_2 = n_1 + n_2)$.

UNIVERSITY OF LEEDS

Our Aim

Our Aim

TL-category: a nested $(0, 1, 2)$ "cobordism category" with

- 0-manifolds: points $\sqcup_{\text{finite}} *$.
- 1-manifolds: interval $[0, 1]$.
- 2-manifolds: square $[0, 1]^2$.

Our Aim

TL-category: a nested $(0, 1, 2)$ "cobordism category" with

- 0-manifolds: points $\sqcup_{\text{finite}} *$.
- 1-manifolds: interval $[0, 1]$.
- 2-manifolds: square $[0, 1]^2$.

We essentially will consider the question of when the "2" can have different surface type (especially **unorientable**) i.e.

$$[0, 1]^2 \longrightarrow \Sigma.$$

Our Aim

TL-category: a nested $(0,1,2)$ "cobordism category" with

- 0-manifolds: points $\sqcup_{\text{finite}} *$.
- 1-manifolds: interval $[0,1]$.
- 2-manifolds: square $[0,1]^2$.

We essentially will consider the question of when the "2" can have different surface type (especially **unorientable**) i.e.

$$[0,1]^2 \longrightarrow \Sigma.$$

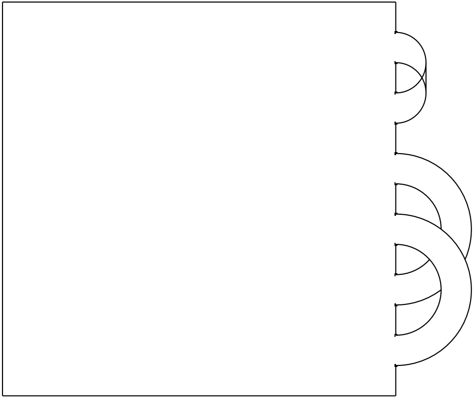
we will restrict to surface types Σ with one boundary component.

How to draw TL diagrams on other surfaces?

Proceed concretely; attach "handles" to our square frame

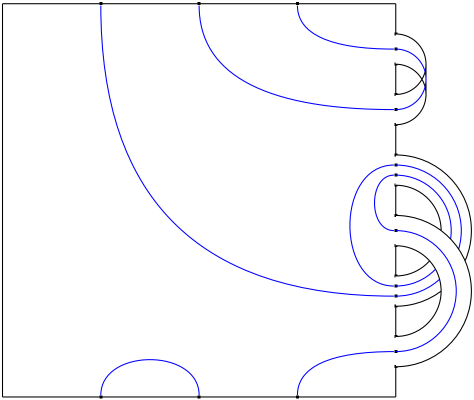
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



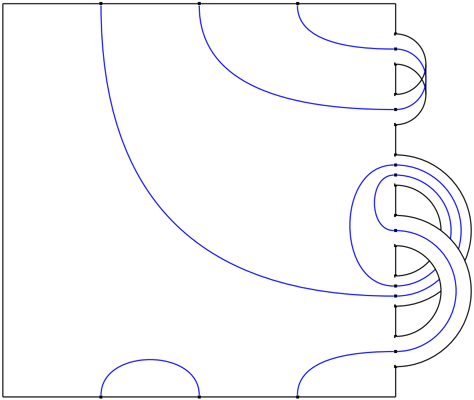
How to draw TL diagrams on other surfaces?

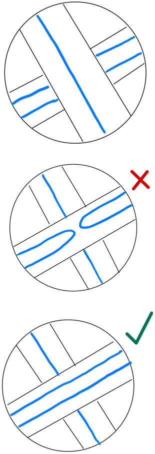
Proceed concretely; attach "handles" to our square frame



How to draw TL diagrams on other surfaces?

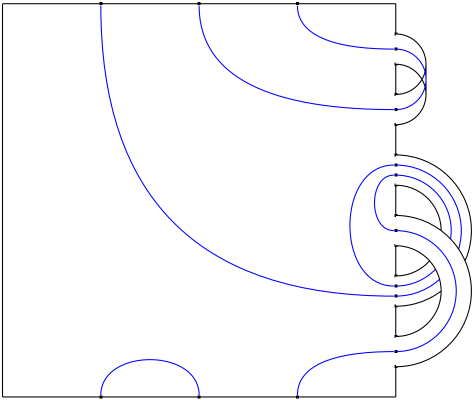
Proceed concretely; attach "handles" to our square frame





How to draw TL diagrams on other surfaces?

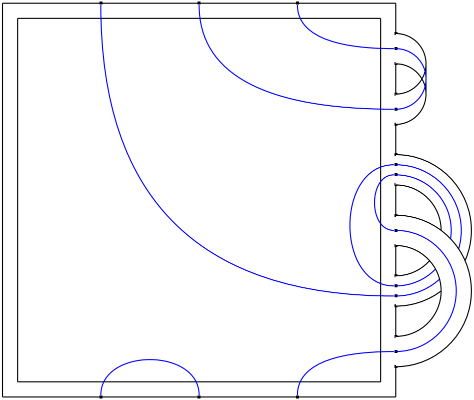
Proceed concretely; attach "handles" to our square frame



described by a quadruple (P, s, f, E).

How to draw TL diagrams on other surfaces?

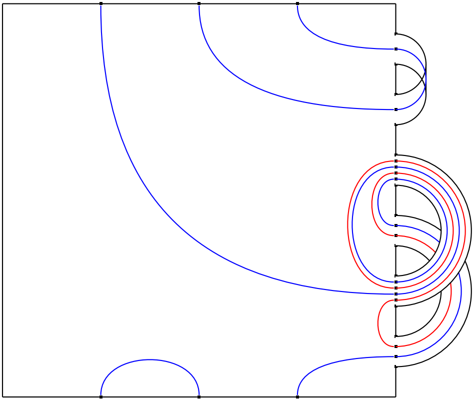
Proceed concretely; attach “handles” to our square frame



described by a quadruple (P, s, f, E).

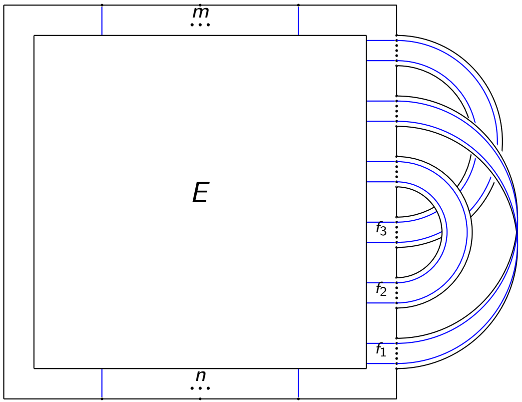
How to draw TL diagrams on other surfaces?

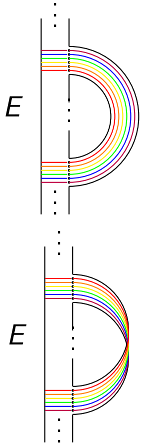
Proceed concretely; attach "handles" to our square frame



SWB diagrams

Square with bands (SWB) diagram encoded by $\Theta = (P, s, f, E)$ (type n, m)

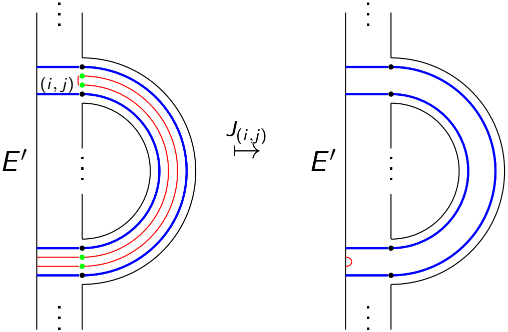




SWB diagrams - Isotopy

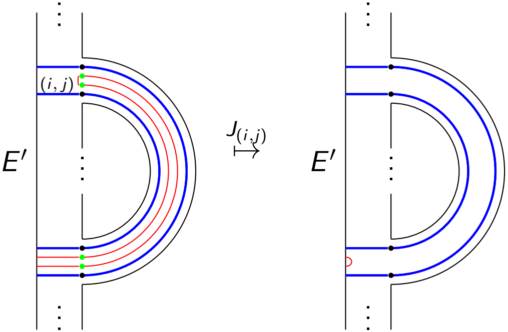
SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove "turnbacks" by "pull-throughs"



SWB diagrams - Isotopy

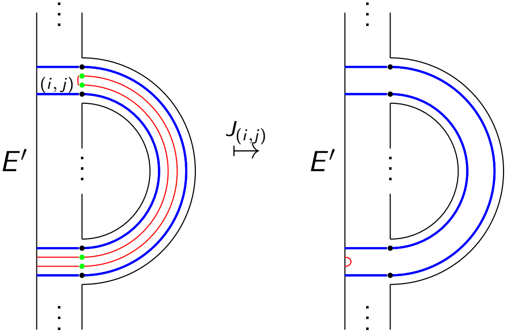
Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove "turnbacks" by "pull-throughs"



$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', o(E''))$$

SWB diagrams - Isotopy

Unlike the TL-case, there is a non-trivial isotopy move on diagrams: We can remove "turnbacks" by "pull-throughs"



$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', o(E''))$$

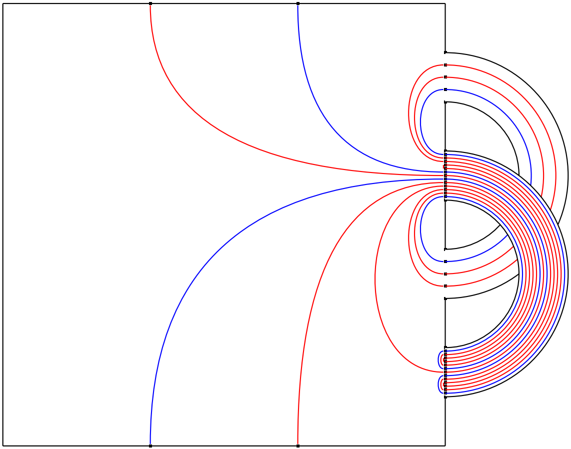
Generate an equivalence relation with this move.

SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks

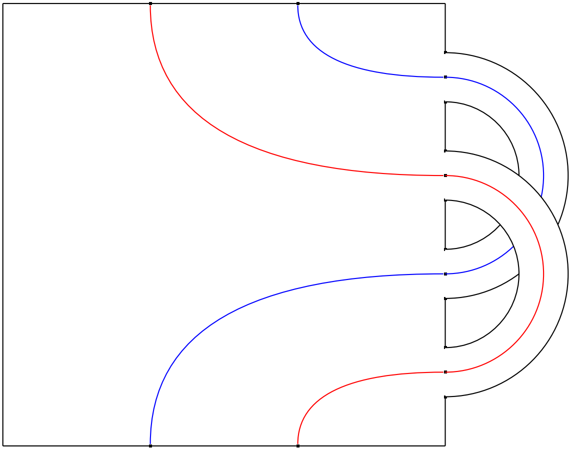
SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks



SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks

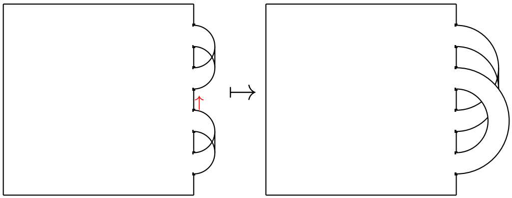


SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:

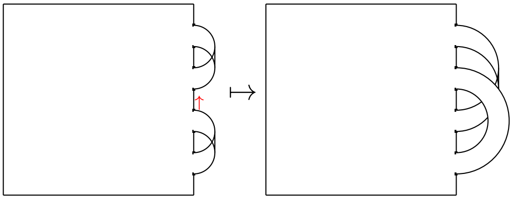
SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:



SWB diagrams - Handlesliding

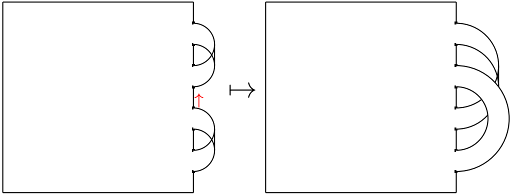
Different realisations of a surface are related by **handleslides**:



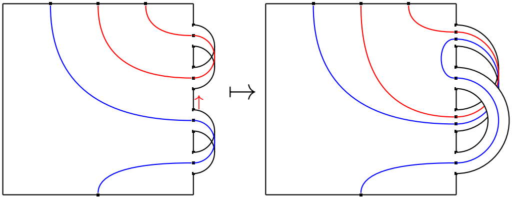
This induces moves on SWB diagrams:

SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:



This induces moves on SWB diagrams:

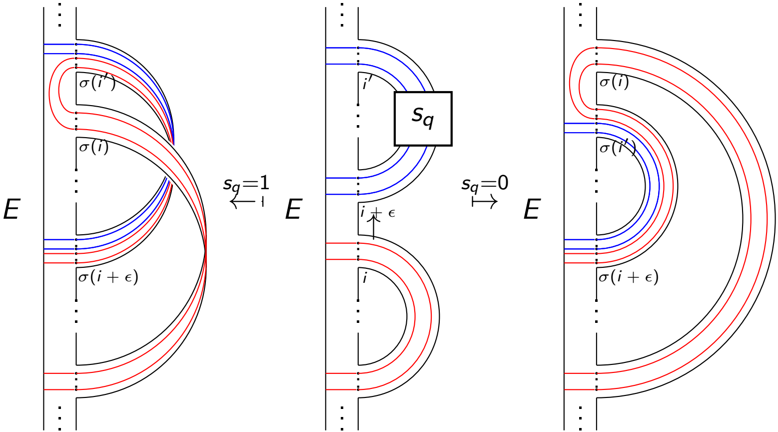


SWB diagrams - Handlesliding

Generically: "Two bands involved"

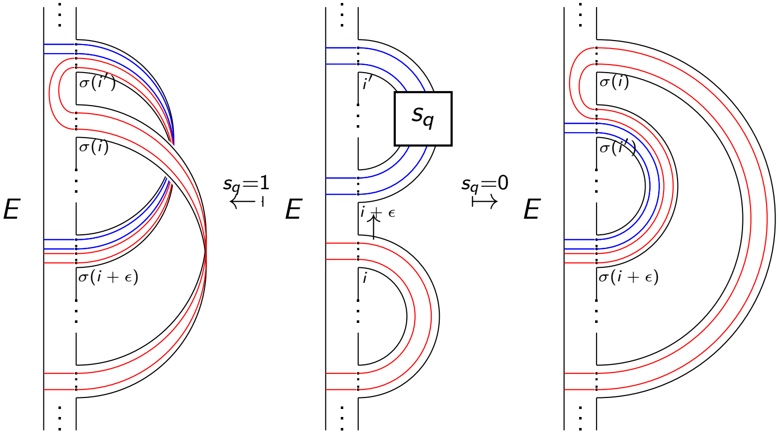
SWB diagrams - Handlesliding

Generically: “Two bands involved”



SWB diagrams - Handlesliding

Generically: "Two bands involved"



$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{\text{"new red arcs"}\})$$

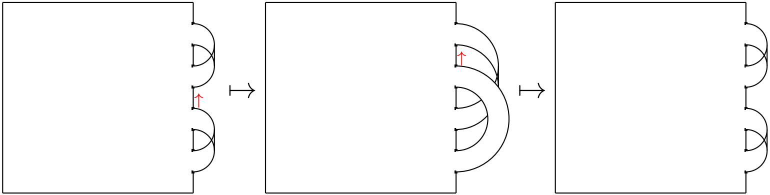
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

.

SWB diagrams - Handleslide Equivalence

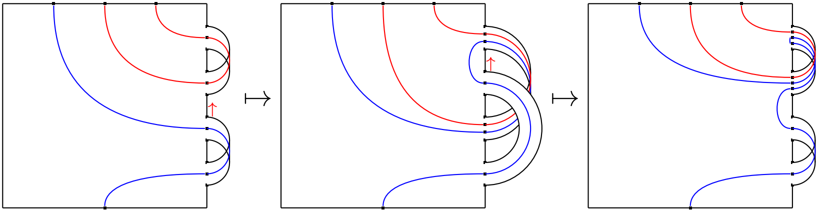
On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.



.

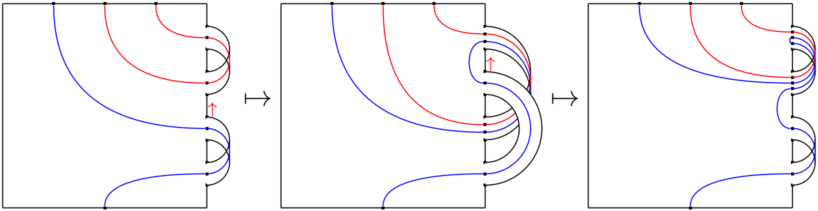
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.



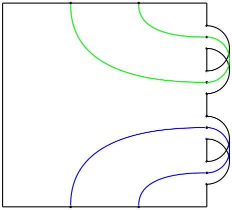
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.

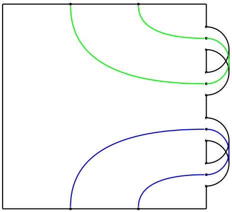


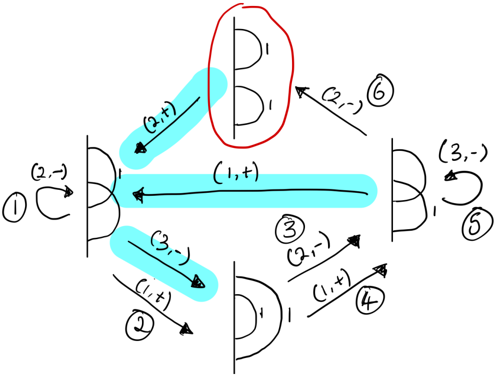
Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

SWB diagrams - Handleslide Equivalence

$\Theta =$ 

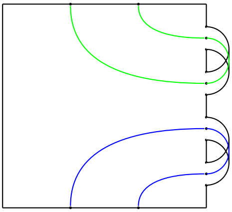
SWB diagrams - Handleslide Equivalence

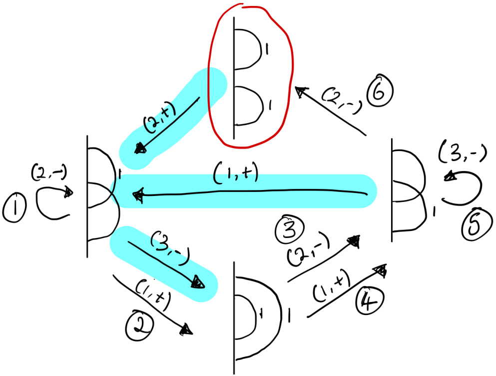
$\Theta =$ 



SWB diagrams - Handleslide Equivalence

$\Theta =$



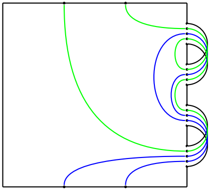


Associate the "reduced" sequence A_i for each edge outside the tree, e.g.

$$A_2 = (3,+) \circ (4,-) \circ (1,+) \circ (2,+)$$

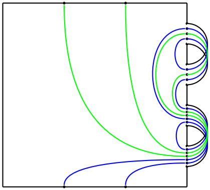
SWB diagrams - Handleslide Equivalence

$A_1(\Theta) =$
$A_2(\Theta) =$
$A_5(\Theta) =$



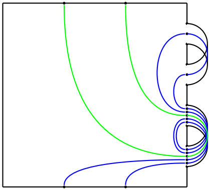
,

$A_3(\Theta) =$



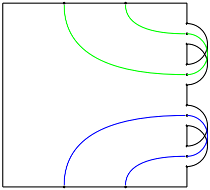
,

$A_4(\Theta) =$



,

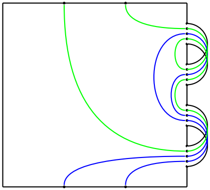
$A_6(\Theta) =$
$\Theta =$



.

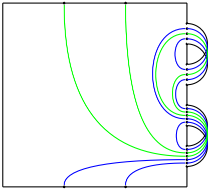
SWB diagrams - Handleslide Equivalence

$$A_1(\Theta) = \quad A_2(\Theta) = \quad A_5(\Theta) =$$



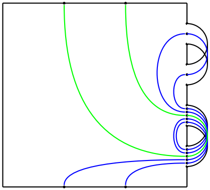
,

$$A_3(\Theta) =$$



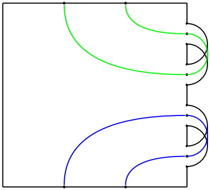
,

$$A_4(\Theta) =$$



,

$$A_6(\Theta) = \quad \Theta =$$



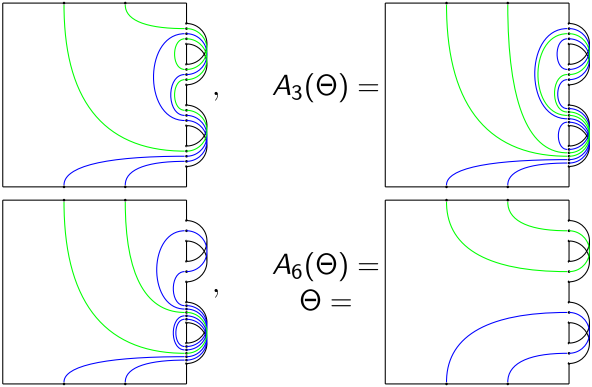
.

$$\langle A_2, A_3, A_4 \mid A_3A_2 = A_4,\ A_2A_4 = A_4A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

SWB diagrams - Handleslide Equivalence

$A_1(\Theta) = A_2(\Theta) = A_5(\Theta) =$, $A_3(\Theta) =$,

$A_4(\Theta) =$, $A_6(\Theta) = \Theta =$.



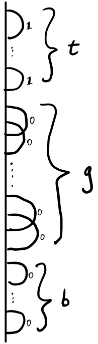
$$\langle A_2, A_3, A_4 \mid A_3A_2 = A_4,\ A_2A_4 = A_4A_2^{-1}\rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

(Chord Diag. Pres. of Mapping Class Group - Bene 2009)

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

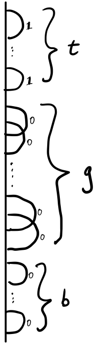
$(P, s) \sim$



where $g, b \in \mathbb{Z}_{\geq 0}$,

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

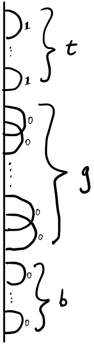
$(P, s) \sim$ 

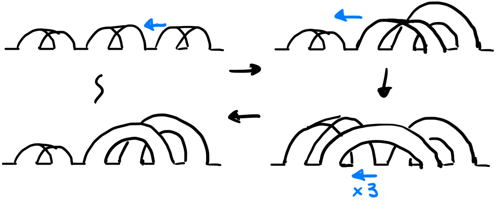
where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.

Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$(P, s) \sim$





where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.

The Category $\mathcal{SQ}$

Fix $\mathbb{K}$ a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$.

The Category $\mathcal{SQ}$

Fix $\mathbb{K}$ a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. The (pre)-category $\mathcal{SQ}(\alpha, \beta, \gamma)$ is defined as the $\mathbb{K}$-linear category with:

The Category $\mathcal{SQ}$

Fix $\mathbb{K}$ a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. The (pre)-category $\mathcal{SQ}(\alpha, \beta, \gamma)$ is defined as the $\mathbb{K}$-linear category with:

- Objects: non-negative integers $\mathbb{N}$

The Category $\mathcal{SQ}$

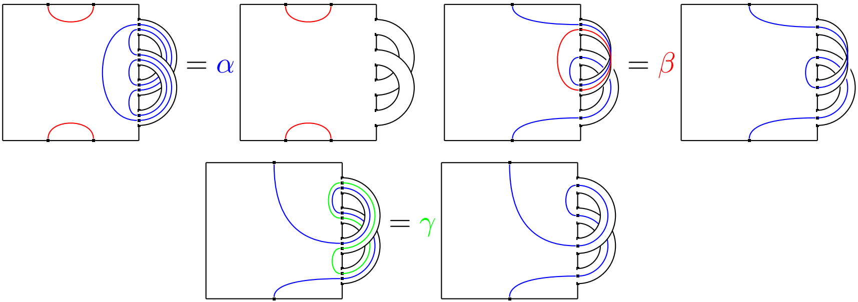
Fix $\mathbb{K}$ a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. The (pre)-category $\mathcal{SQ}(\alpha, \beta, \gamma)$ is defined as the $\mathbb{K}$-linear category with:

- Objects: non-negative integers $\mathbb{N}$
- Morphisms: Hom(n, m) consists of $\mathbb{K}$-linear combinations of HS classes of type (n, m) SWB diagrams, $[\Theta]_{\mathrm{HS}}$,

The Category $\mathcal{SQ}$

Fix $\mathbb{K}$ a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$. The (pre)-category $\mathcal{SQ}(\alpha, \beta, \gamma)$ is defined as the $\mathbb{K}$-linear category with:

- Objects: non-negative integers $\mathbb{N}$
- Morphisms: Hom(n, m) consists of $\mathbb{K}$-linear combinations of HS classes of type (n, m) SWB diagrams, $[\Theta]_{\mathrm{HS}}$, modulo the **delooping** relations *e.g.*

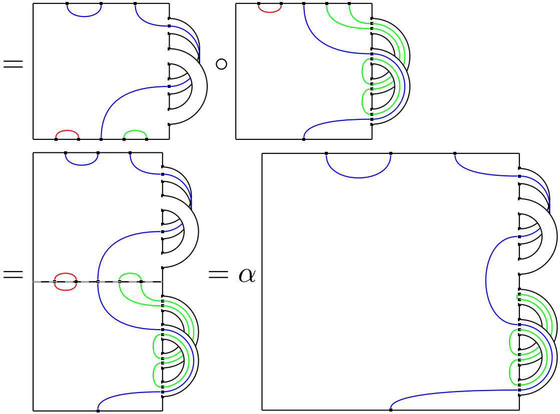


The Category $\mathcal{SQ}$

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is given by
$\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1, \Theta_2)} \overline{\Theta_2 \# \Theta_1}$:

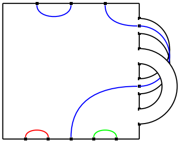
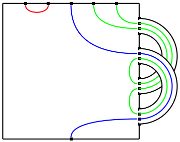
The Category $\mathcal{SQ}$

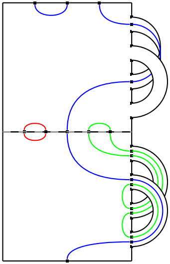
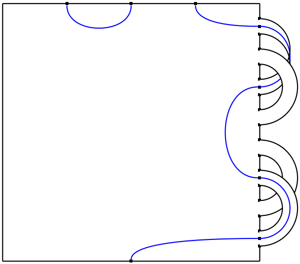
Composition: $\mathrm{Hom}(n,m) \times \mathrm{Hom}(m,l) \to \mathrm{Hom}(n,l)$ is given by $\overline{\overline{\Theta_2} \circ \overline{\Theta_1}} = \alpha^{L(\Theta_1,\Theta_2)}\,\overline{\Theta_2 \# \Theta_1}$:

$\overline{\Theta_2} \circ \overline{\Theta_1} =$ 

The Category $\mathcal{SQ}$

Composition: $\mathrm{Hom}(n, m) \times \mathrm{Hom}(m, l) \to \mathrm{Hom}(n, l)$ is given by $\overline{\Theta_2} \circ \overline{\Theta_1} = \alpha^{L(\Theta_1,\Theta_2)}\, \overline{\Theta_2 \# \Theta_1}$:

$\overline{\Theta_2} \circ \overline{\Theta_1} =$  $\circ$ 

$=$  $= \alpha\gamma$ 

The Category $\mathcal{SQ}$

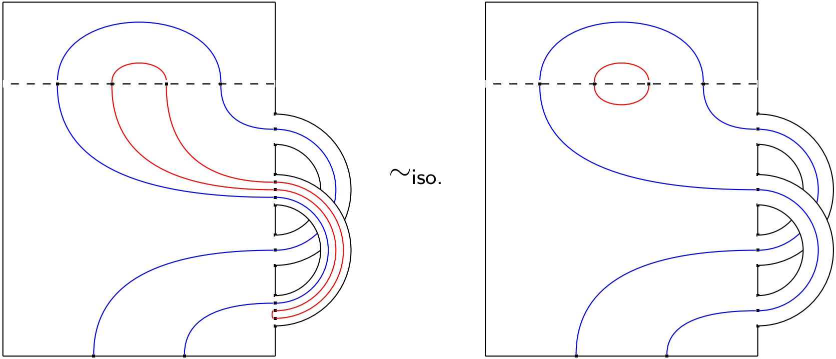
Is composition well defined?

The Category $\mathcal{SQ}$

Is composition well defined? Yes - a subtlety

The Category $\mathcal{SQ}$

Is composition well defined? Yes - a subtlety



The Category $\mathcal{SQ}$ - Basic Facts

The Category $\mathcal{SQ}$ - Basic Facts

Fact 1: For any $\Theta \in Sq(n,m)$, there exist **unique** integers l_s, l_t and l_u such that:

$$\overline{\Theta} = \alpha^{l_s}\beta^{l_t}\gamma^{l_u}\,\overline{\Theta'} \in \mathrm{Hom}(n,m),$$

where $\Theta' \in Sq(n,m)$ has no internal components.

The Category $\mathcal{SQ}$ - Basic Facts

Fact 1: For any $\Theta \in Sq(n, m)$, there exist **unique** integers l_s, l_t and l_u such that:

$$\overline{\Theta} = \alpha^{l_s} \beta^{l_t} \gamma^{l_u} \, \overline{\Theta'} \in \mathrm{Hom}(n, m),$$

where $\Theta' \in Sq(n, m)$ has no internal components.

Fact 2: Any morphism $\overline{\Theta} \in \mathrm{Hom}(n, m)$ has a factorisation in terms of diagrams of the following form

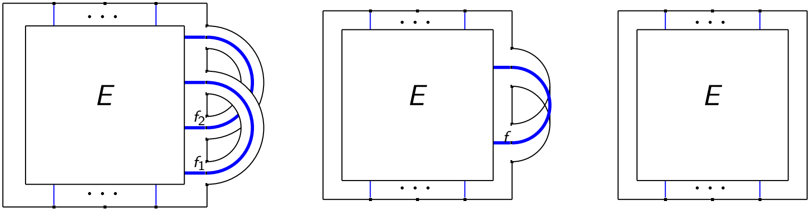
The Category $\mathcal{SQ}$ - Basic Facts

Fact 1: For any $\Theta \in Sq(n, m)$, there exist **unique** integers l_s, l_t and l_u such that:

$$\overline{\Theta} = \alpha^{l_s} \beta^{l_t} \gamma^{l_u} \, \overline{\Theta'} \in \mathrm{Hom}(n, m),$$

where $\Theta' \in Sq(n, m)$ has no internal components.

Fact 2: Any morphism $\overline{\Theta} \in \mathrm{Hom}(n, m)$ has a factorisation in terms of diagrams of the following form (AND)

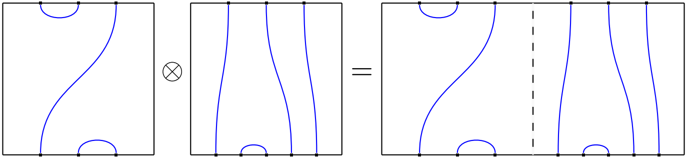


The Category $\mathcal{SQ}$ - Tensor Product

Recall: In TL case we had a tensor product given by "horizontal stacking" of diagrams:

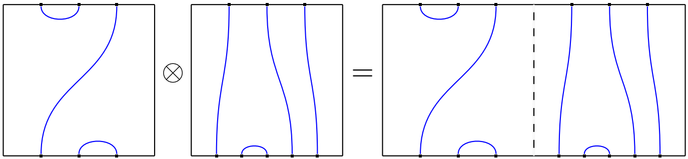
The Category $\mathcal{SQ}$ - Tensor Product

Recall: In TL case we had a tensor product given by "horizontal stacking" of diagrams:



The Category $\mathcal{SQ}$ - Tensor Product

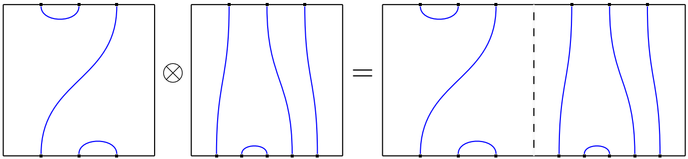
Recall: In TL case we had a tensor product given by "horizontal stacking" of diagrams:



Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects.

The Category $\mathcal{SQ}$ - Tensor Product

Recall: In TL case we had a tensor product given by "horizontal stacking" of diagrams:



Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

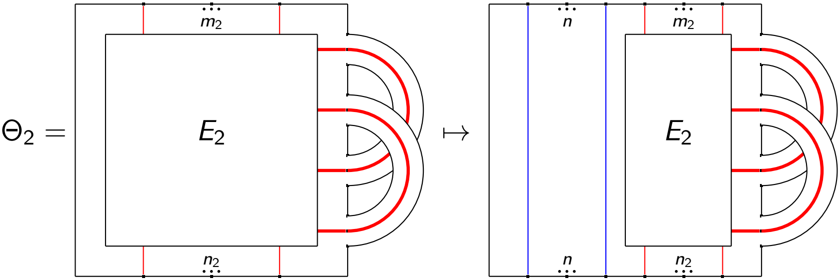
The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:

UNIVERSITY OF LEEDS

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 1 - Put the identity diagram on the left:

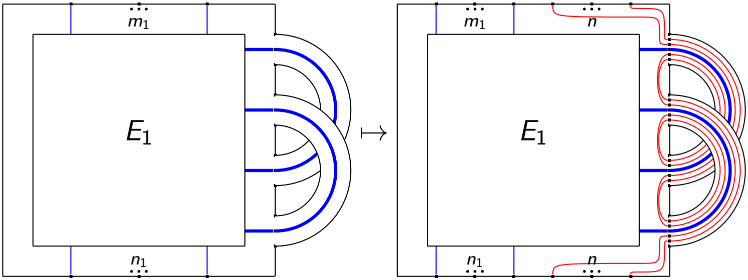


$$\overset{(?)}{:=} \mathrm{id}_n \otimes \Theta_2$$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:

$\Theta_1 =$



$\overset{(?)}{:=} \Theta_1 \otimes \mathrm{id}_n$

The Category $\mathcal{SQ}$ - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:

The Category $\mathcal{SQ}$ - Tensor Product

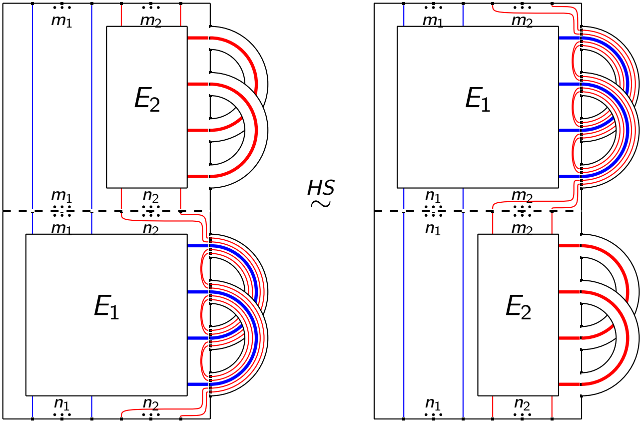
Indirect answer: Step 3 - Insist upon functoriality:

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\text{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \text{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \text{id}_{m_2})} \circ \overline{(\text{id}_{n_1} \otimes \Theta_2)}$$

The Category $\mathcal{SQ}$ - Tensor Product

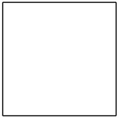
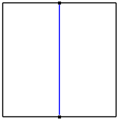
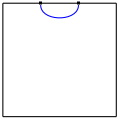
Indirect answer: Step 3 - Insist upon functoriality:

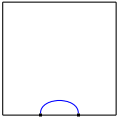
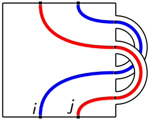
$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\mathrm{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \mathrm{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \mathrm{id}_{m_2})} \circ \overline{(\mathrm{id}_{n_1} \otimes \Theta_2)}$$

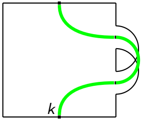


Monoidal Generating Set?

Conjecture: The following is a monoidal generating set for $\mathcal{SQ}(\alpha, \beta, \gamma)$:

$\text{id}_0 =$  , $\text{id}_1 =$  , $U =$  ,

$V =$  , $\mathbb{T}_{i,j} =$ 

, $\mathbb{M}_k =$ 

.

The Infinity problem

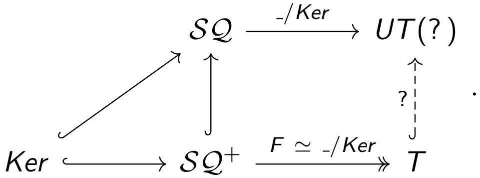
PROBLEM: Hom-sets are infinite dimensional.

UNIVERSITY OF LEEDS

The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

A potential "scheme" for finitising:



The Infinity problem

PROBLEM: Hom-sets are infinite dimensional.

A potential “scheme” for finitising:

$$\begin{array}{ccccc} & & \mathcal{S}\mathcal{Q} & \xrightarrow{\_/Ker} & UT(?) \\ & \nearrow & \uparrow & & \uparrow{?} \\ Ker & \hookrightarrow & \mathcal{S}\mathcal{Q}^+ & \xrightarrow{F \,\simeq\, \_/Ker} & T \end{array} \quad .$$

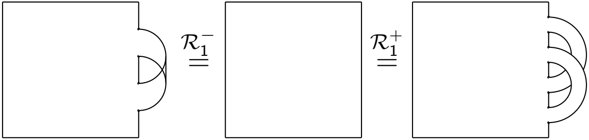
F a full and essentially surjective, monoidal functor, and T a target monoidal $\mathbb{K}$-linear category with f.d. hom spaces. Call F a **finitising functor**.

The Infinity problem

One very natural demand is that $\text{Hom}(0, 0) \simeq \mathbb{K}$.

The Infinity problem

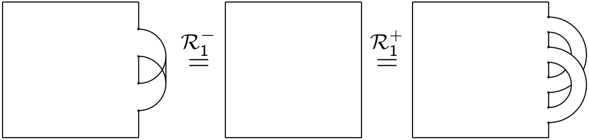
One very natural demand is that $\mathrm{Hom}(0,0) \simeq \mathbb{K}$. This requires imposing the relation $\mathcal{R}_1$:



UNIVERSITY OF LEEDS

The Infinity problem

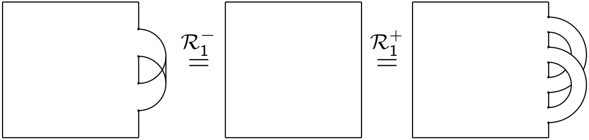
One very natural demand is that $\text{Hom}(0,0) \simeq \mathbb{K}$. This requires imposing the relation $\mathcal{R}_1$:



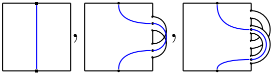
Conjecture: $\mathcal{SQ}/\mathcal{R}_1$ is a $\mathbb{K}$-linear category with finite dimensional hom-spaces:

The Infinity problem

One very natural demand is that $\mathrm{Hom}(0,0) \simeq \mathbb{K}$. This requires imposing the relation $\mathcal{R}_1$:



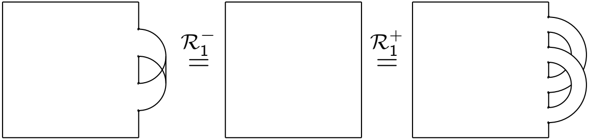
Conjecture: $\mathcal{SQ}/\mathcal{R}_1$ is a $\mathbb{K}$-linear category with finite dimensional hom-spaces:

- $\mathrm{Hom}(1,1) = \mathbb{K}\Big\{$  $\Big\} \simeq \mathbb{K}\langle a \mid a^3 = a \rangle$,

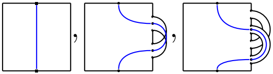
UNIVERSITY OF LEEDS

The Infinity problem

One very natural demand is that $\mathrm{Hom}(0,0) \simeq \mathbb{K}$. This requires imposing the relation $\mathcal{R}_1$:



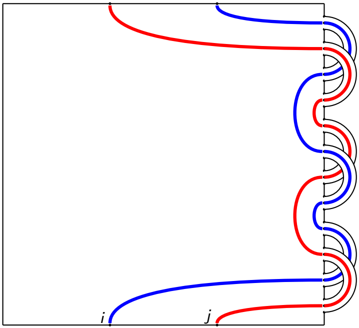
Conjecture: $\mathcal{SQ}/\mathcal{R}_1$ is a $\mathbb{K}$-linear category with finite dimensional hom-spaces:

- $\mathrm{Hom}(1,1) = \mathbb{K}\Bigg\{$  $\Bigg\} \simeq \mathbb{K}\langle a \mid a^3 = a \rangle$,
- $\dim(\mathrm{Hom}(2,2)) \geq 23$ *

Example in $\mathcal{SQ}/\mathcal{R}_1$

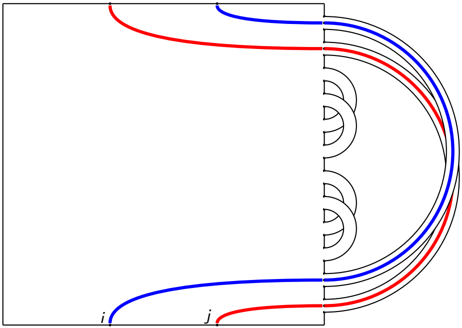
Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$$



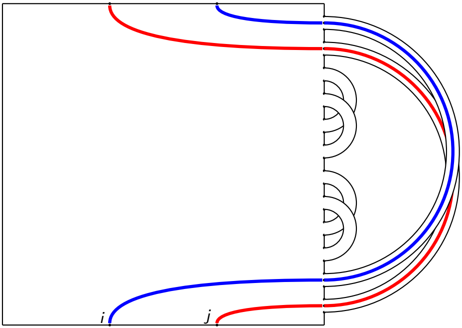
Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$ 

Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$  $= \mathbb{T}_{i,j} \mod \mathcal{R}_1$

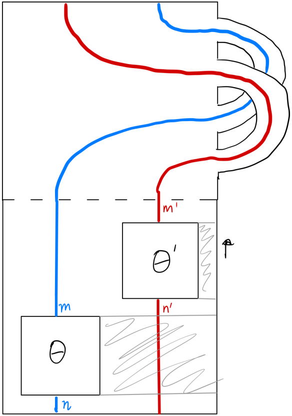
The Infinity problem

Any other candidates for quotients?

The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\text{op}}$:

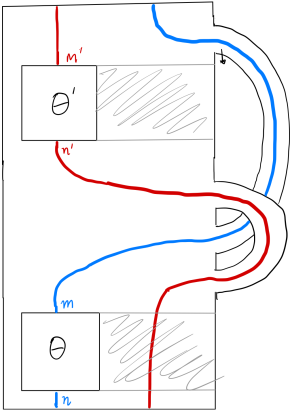
$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') =$



The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\text{op}}$:

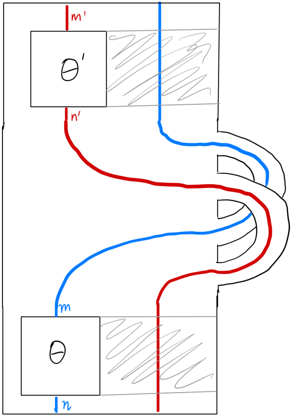
$$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$$



The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{op}$:

$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$

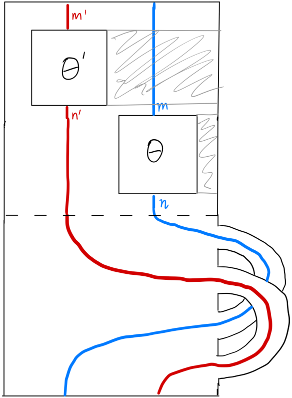


UNIVERSITY OF LEEDS

The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\mathrm{op}}$:

$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$

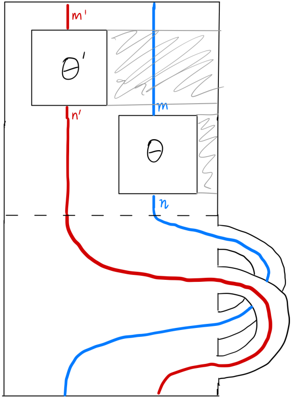


UNIVERSITY OF LEEDS

The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\text{op}}$:

$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') \sim$



$= (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$

The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\mathsf{op}}$:

Can check the $\mathbb{T}_{i,j}$ satisfy the categorical YBE:

$$\begin{aligned}&(\mathbb{T}_{j,k} \otimes \mathsf{id}_i) \circ (\mathsf{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \mathsf{id}_k)\\ &= (\mathsf{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \mathsf{id}_j) \circ (\mathsf{id}_i \otimes \mathbb{T}_{j,k})\end{aligned}$$

The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\mathrm{op}}$:

Can check the $\mathbb{T}_{i,j}$ satisfy the categorical YBE:

$$(\mathbb{T}_{j,k} \otimes \mathrm{id}_i) \circ (\mathrm{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \mathrm{id}_k)$$
$$= (\mathrm{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \mathrm{id}_j) \circ (\mathrm{id}_i \otimes \mathbb{T}_{j,k})$$

However, $\mathcal{SQ}$ is NOT a braided mon.cat.

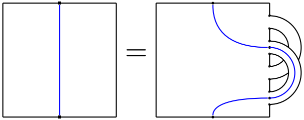
The Infinity problem

Any other candidates for quotients? Note that the $\mathbb{T}_{i,j}$ are components of a braiding $\mathbb{T} : \otimes \to \otimes^{\mathrm{op}}$:

Can check the $\mathbb{T}_{i,j}$ satisfy the categorical YBE:

$$(\mathbb{T}_{j,k} \otimes \mathrm{id}_i) \circ (\mathrm{id}_j \otimes \mathbb{T}_{i,k}) \circ (\mathbb{T}_{i,j} \otimes \mathrm{id}_k)$$
$$= (\mathrm{id}_k \otimes \mathbb{T}_{i,j}) \circ (\mathbb{T}_{i,k} \otimes \mathrm{id}_j) \circ (\mathrm{id}_i \otimes \mathbb{T}_{j,k})$$

However, $\mathcal{SQ}$ is NOT a braided mon.cat. The smallest such quotient is obtained by imposing the relation $\mathcal{R}_2$ (as well as $\mathcal{R}_1^+$ *):

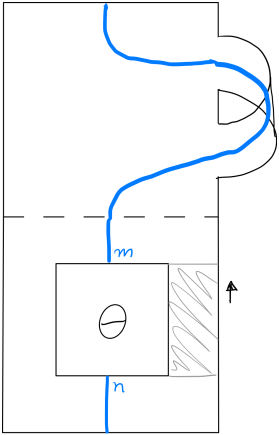


NOTE: This implies $\alpha = \gamma$. * not necessary if α invertible.

A Categorical Interpretation of the $\mathbb{M}_k$

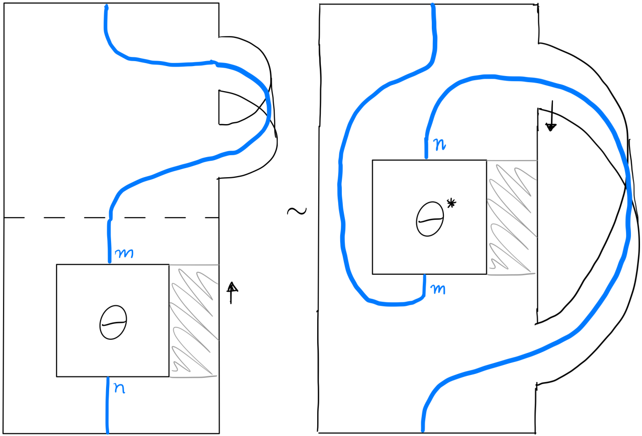
A Categorical Interpretation of the $\mathbb{M}_k$

$\mathbb{M}_m \circ \Theta =$

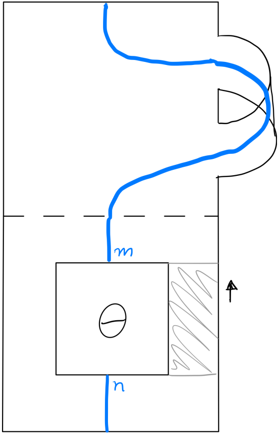
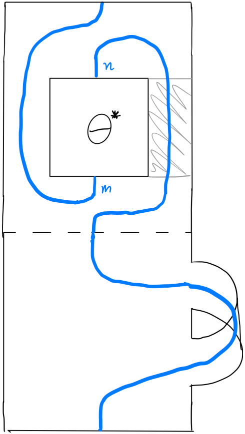


A Categorical Interpretation of the $\mathbb{M}_k$

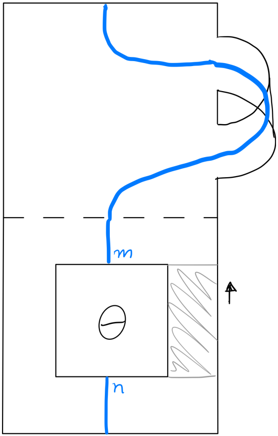
$\mathbb{M}_m \circ \Theta =$

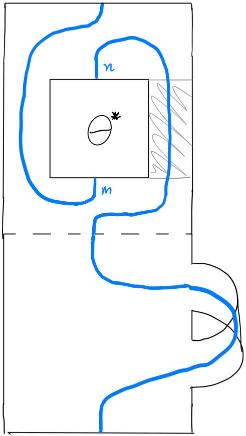


A Categorical Interpretation of the $\mathbb{M}_k$

$\mathbb{M}_m \circ \Theta =$  $\sim$ 

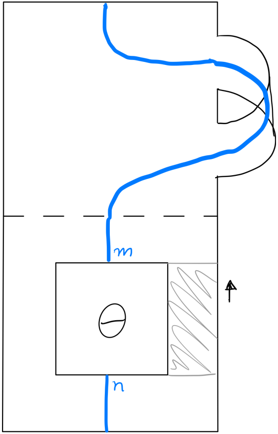
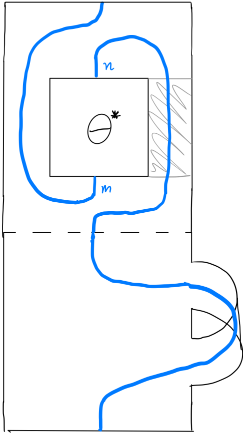
A Categorical Interpretation of the $\mathbb{M}_k$

$\mathbb{M}_m \circ \Theta =$ 

$\sim$ 

$:= F(\Theta) \circ \mathbb{M}_n$

A Categorical Interpretation of the $\mathbb{M}_k$

$\mathbb{M}_m \circ \Theta =$  $\sim$  $:= F(\Theta) \circ \mathbb{M}_n$

where $F^2 = \mathrm{id}$, $F \circ (\Theta \otimes \Theta') = F(\Theta') \otimes F(\Theta)$.

The Infinity problem

Consider the functor $St := \_/(\mathcal{R}_1^+ + \mathcal{R}_2) := \_/\mathcal{R}^+$:

The Infinity problem

Consider the functor $St := \_/(\mathcal{R}_1^+ + \mathcal{R}_2) := \_/\mathcal{R}^+$:

$$\begin{array}{ccccc} & & \mathcal{SQ}(\alpha,\beta,\alpha) & \xrightarrow{\_/\mathcal{R}^+} & \mathcal{SQ}/\mathcal{R}^+ \\ & \nearrow & \uparrow & & \uparrow \\ \mathcal{R}^+ & \hookrightarrow & \mathcal{SQ}^+(\alpha,\beta,\alpha) & \xrightarrow{St\, \simeq\, \_/\mathcal{R}^+} & \mathcal{SQ}^+/\mathcal{R}^+ \begin{smallmatrix} \simeq VTL(\alpha) \\ \simeq Br(\alpha) \end{smallmatrix} \end{array} .$$

The Infinity problem

Consider the functor $St := \_/(\mathcal{R}_1^+ + \mathcal{R}_2) := \_/\mathcal{R}^+$:

$$
\begin{array}{ccccc}
 & & \mathcal{SQ}(\alpha,\beta,\alpha) & \xrightarrow{\ \_/\mathcal{R}^+\ } & \mathcal{SQ}/\mathcal{R}^+ \\
 & \nearrow & \uparrow & & \uparrow \\
\mathcal{R}^+ & \hookrightarrow & \mathcal{SQ}^+(\alpha,\beta,\alpha) & \xrightarrow[\ ]{St\ \simeq\ \_/\mathcal{R}^+} & \mathcal{SQ}^+/\mathcal{R}^+ \; {}^{\simeq VTL(\alpha)}_{\simeq Br(\alpha)}
\end{array}.
$$

$(\mathcal{SQ}/\mathcal{R}^+)/\mathcal{R}_1^- := \mathcal{SQ}/\mathcal{R} \simeq^* dBr(\alpha,\beta)$.

The Infinity problem

Consider the functor $St := \_/(\mathcal{R}_1^+ + \mathcal{R}_2) := \_/\mathcal{R}^+$:

$$\begin{array}{ccccc}
 & & \mathcal{SQ}(\alpha,\beta,\alpha) & \xrightarrow{_/\mathcal{R}^+} & \mathcal{SQ}/\mathcal{R}^+ \\
 & \nearrow & \uparrow & & \uparrow \\
\mathcal{R}^+ & \hookrightarrow & \mathcal{SQ}^+(\alpha,\beta,\alpha) & \xrightarrow{St \,\simeq\, _/\mathcal{R}^+} & \mathcal{SQ}^+/\mathcal{R}^+ \, {}^{\simeq VTL(\alpha)}_{\simeq Br(\alpha)}
\end{array}.$$

$(\mathcal{SQ}/\mathcal{R}^+)/\mathcal{R}_1^- := \mathcal{SQ}/\mathcal{R} \simeq^* dBr(\alpha,\beta)$.

Corollary: Suppose $F : SQ^+(\alpha,\beta,\gamma) \twoheadrightarrow T$ is a f.f. with $F \circ \mathbb{T}$ a braiding in a BMC. Then $\alpha = \gamma$, and F factors through $\mathcal{SQ}^+(\alpha,\beta,\alpha)/\mathcal{R}^+ \simeq VTL(\alpha) \simeq Br(\alpha)$.

Non-orientable extension of TL?

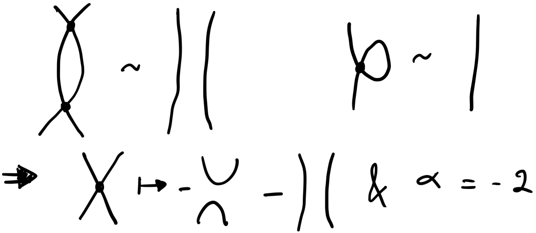
$TL(\alpha)$ is a BMC. Assume a f.f. $F : \mathcal{SQ}(\alpha, \beta, \alpha) \to TL(\alpha)$ sends $\mathbb{T}$ to the braiding in TL.

UNIVERSITY OF LEEDS

Non-orientable extension of TL?

$TL(\alpha)$ is a BMC. Assume a f.f. $F : \mathcal{SQ}(\alpha, \beta, \alpha) \to TL(\alpha)$ sends $\mathbb{T}$ to the braiding in TL.

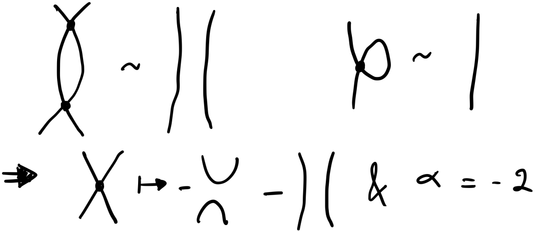
Since this necessarily factors through $VTL(\alpha)$ we are essentially asking to resolve virtual crossings:



Non-orientable extension of TL?

$TL(\alpha)$ is a BMC. Assume a f.f. $F : \mathcal{SQ}(\alpha, \beta, \alpha) \to TL(\alpha)$ sends $\mathbb{T}$ to the braiding in TL.

Since this necessarily factors through $VTL(\alpha)$ we are essentially asking to resolve virtual crossings:



If $2 \neq 0 \in \mathbb{K}$, this quotient on $UVTL(-2, \beta)$ is more severe than hoped...

Thank You!

Questions?