Temperley-Lieb Categories on Non-Orientable Surfaces

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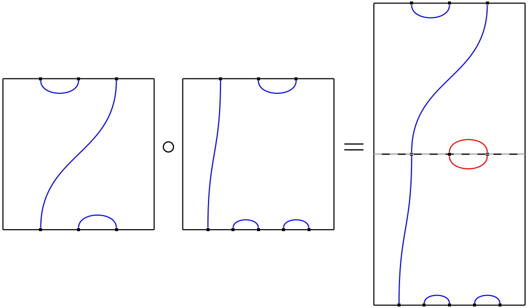
Temperley-Lieb Category

A $(0, 1, 2)$-nested cobordism category.



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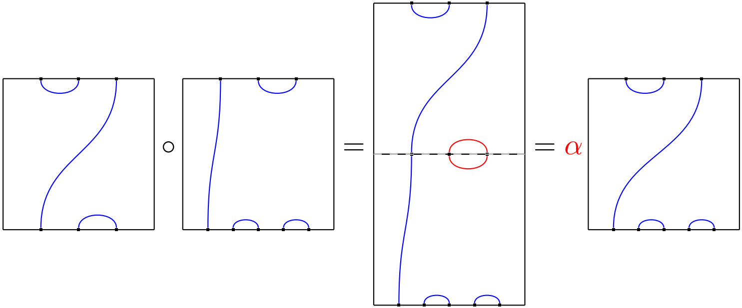

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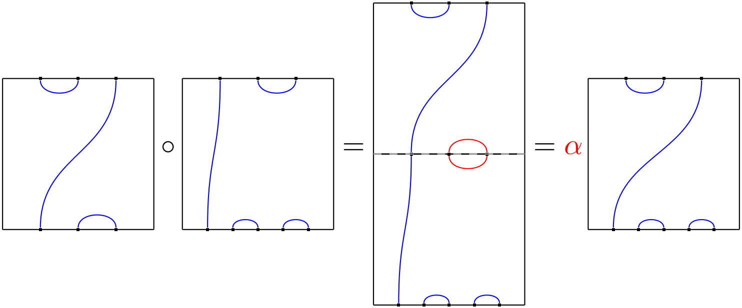
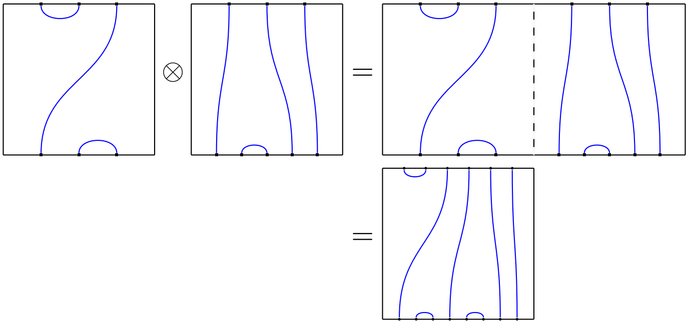


An R-linear category defined by a parameter $\alpha \in R$, write $TL(R, \alpha)$ or $TL(\alpha)$.

$TL(\alpha)$ has a tensor product $\otimes$:



Temperley-Lieb Category - Tensor Product

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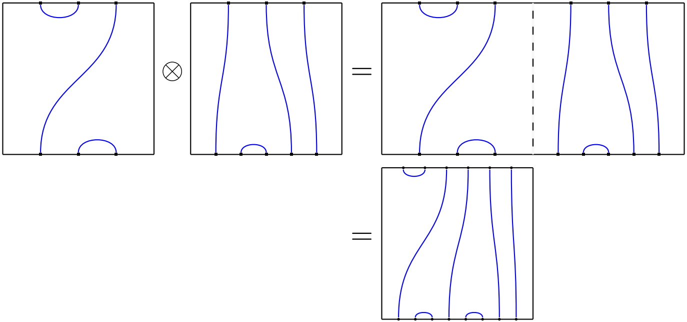

On objects, $n \otimes m = n + m$.

Square with Bands Category - Diagrams

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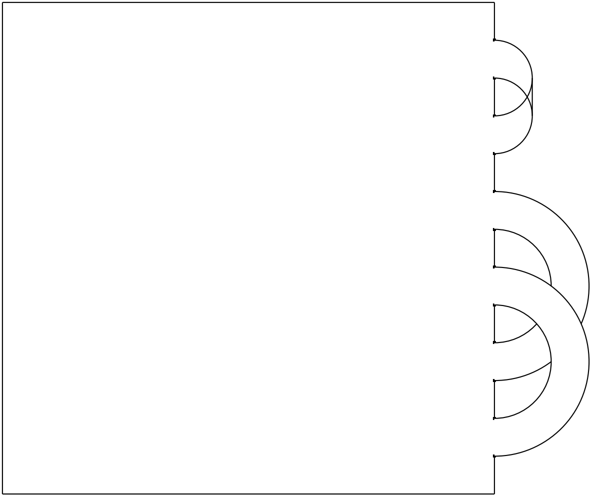
Change the ambient surface type by adding handles or "bands":



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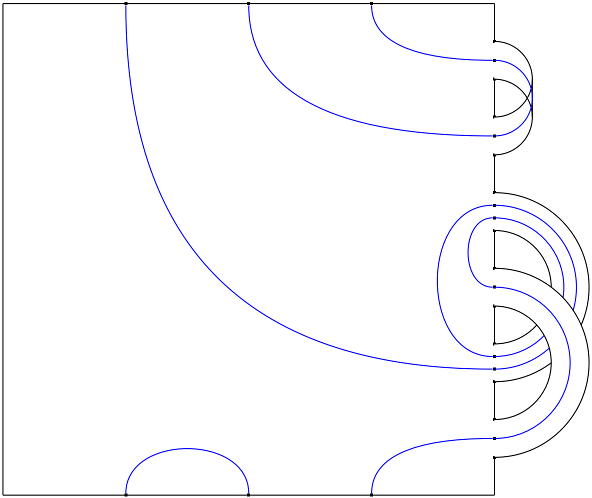

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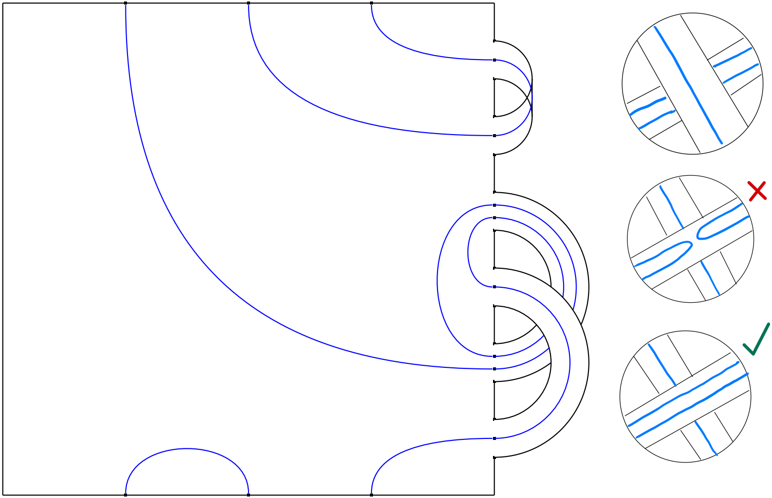

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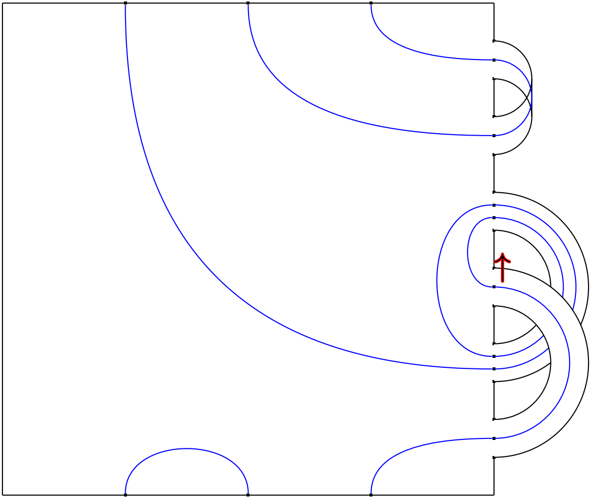

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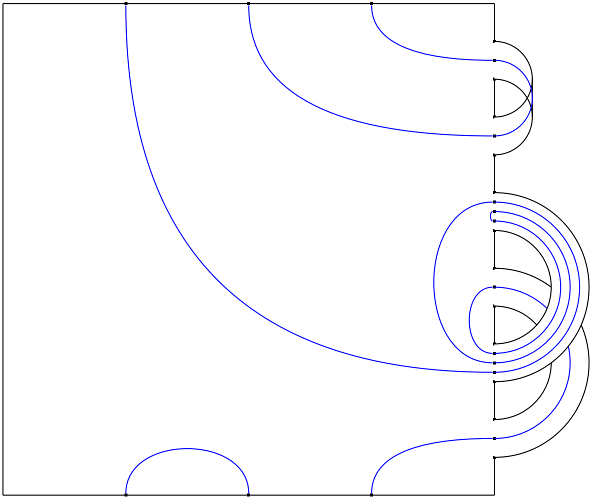

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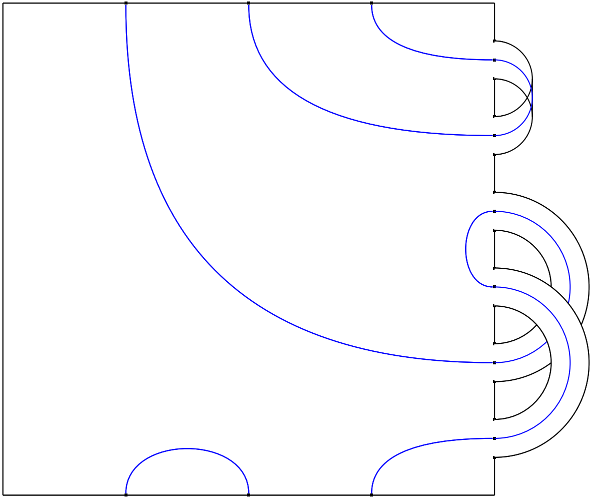
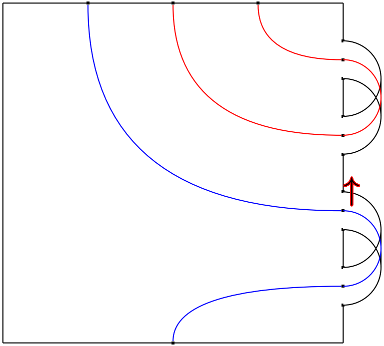
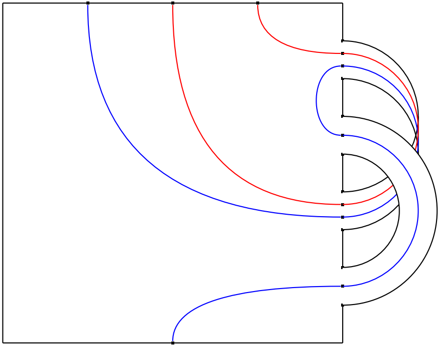


Square with Bands Category - Diagrams

Another Example:



$\mapsto$



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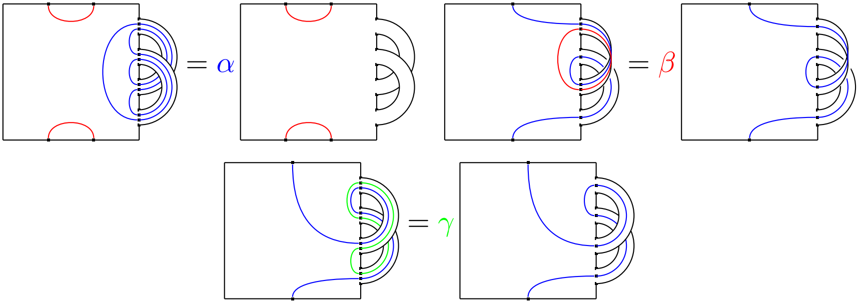

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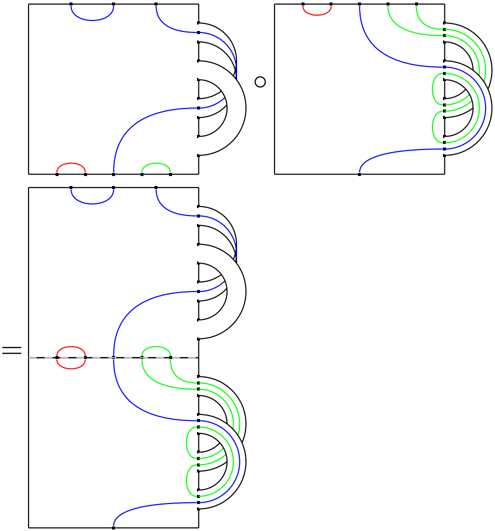

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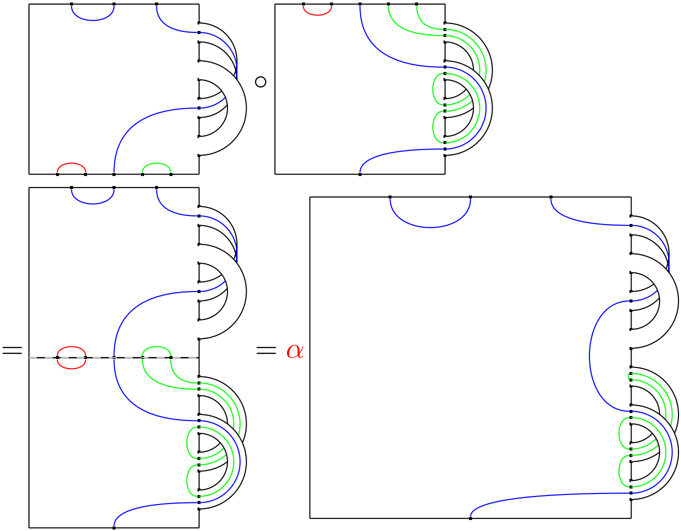

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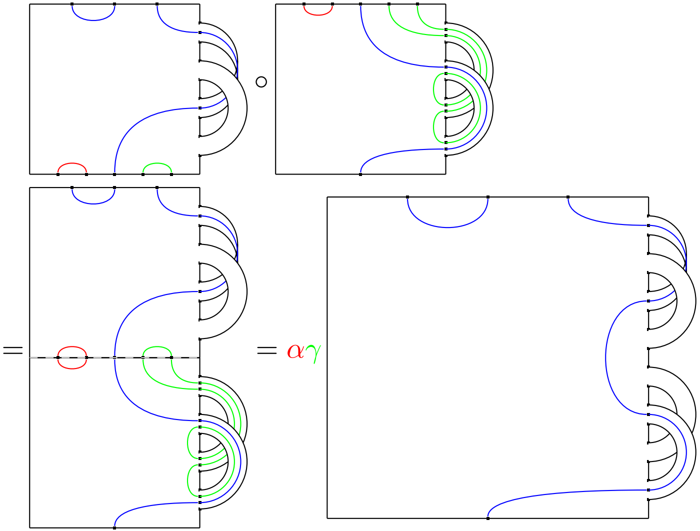
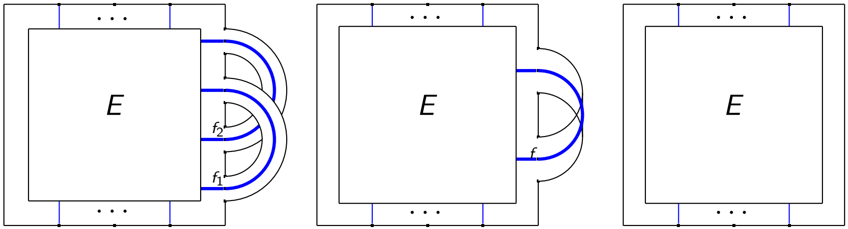


Square with Bands Category - Factorisation

Proposition

Any (homogeneous) morphism has a factorisation in terms of diagrams of the form



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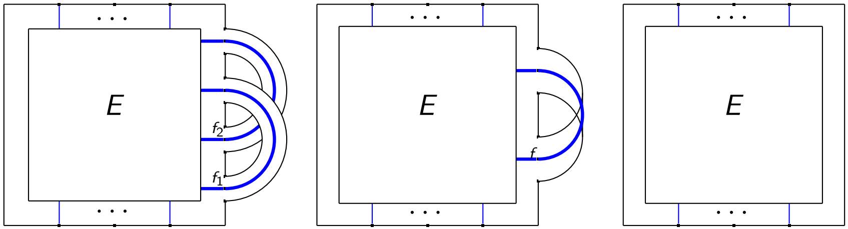

Furthermore, at most two factors of the second type are required.

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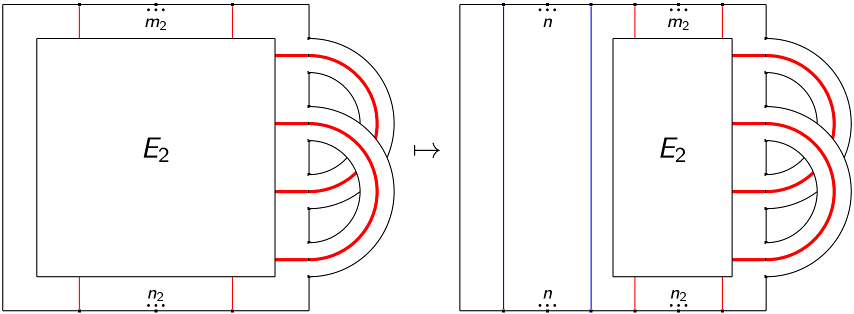

$$\overset{(?)}{:=} \mathrm{id}_n \otimes \Theta_2$$

Square with Bands Category - Tensor Product

Step 2: Put the identity diagram on the right:

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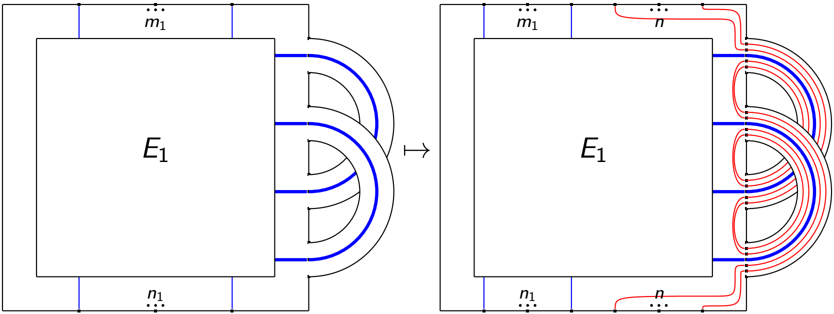

$$\overset{(?)}{:=} \Theta_1 \otimes \mathrm{id}_n$$

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Step 3: Functoriality

$$\overline{\Theta_1} \otimes \overline{\Theta_2} = \overline{(\mathrm{id}_{m_1} \otimes \Theta_2)} \circ \overline{(\Theta_1 \otimes \mathrm{id}_{n_2})} \stackrel{?}{=} \overline{(\Theta_1 \otimes \mathrm{id}_{m_2})} \circ \overline{(\mathrm{id}_{n_1} \otimes \Theta_2)}$$

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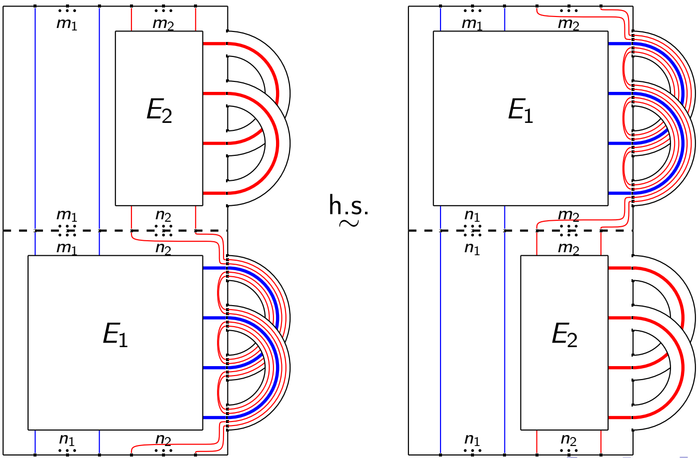
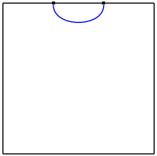
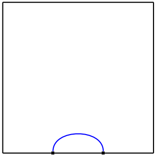


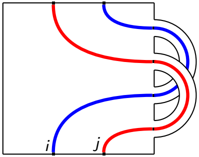
Square with Bands Category - Tensor Product

Theorem

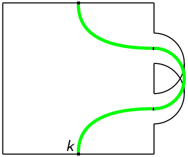
This defines a tensor product on $\mathcal{SQ}$. The following is a monoidal generating set:

$U =$  ,

$V =$  ,

$\mathbb{T}_{i,j} =$ 

,

$\mathbb{M}_k =$ 

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