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# A Diagram Category for Non-Orientable Surfaces

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Joint work with Dionne Ibarra[2], Gabriel Montoya-Vega[3], and Paul Martin[1] (supervisor)

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The Interplay Between Skew Braces and Hopf-Galois Theory, Leeds 2025

# Motivation

Construct interesting low-dim "cobordism categories" amenable to rep. th. study:

- Linear
- Combinatorial
- Finite Dimensional Hom-spaces
- More structure? (monoidal... etc)

In particular, we consider **nested** $(0, 1, 2)$ - "cobordism categories".

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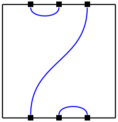
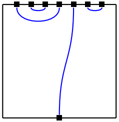
- **Morphisms:** $(1, 2)$ part - $\text{Hom}(n, m)$ is $\mathbb{K}$-linear combinations of type $n, m$ "$TL$-diagrams", (embedded intervals in $[0, 1]^2$):

 $\in \text{Hom}(3, 3)$,  $\in \text{Hom}(1, 7)$.

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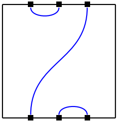
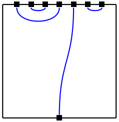
 $\in$ Hom$(3, 3)$,  $\in$ Hom$(1, 7)$.

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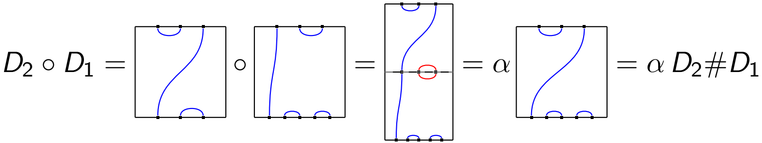

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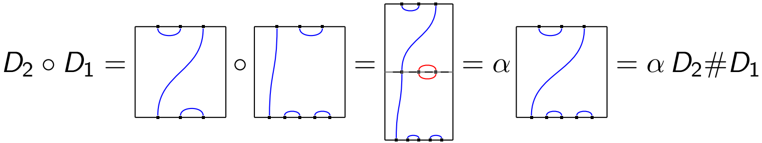


Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)}\, D_2 \# D_1$.

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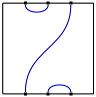
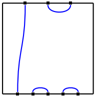
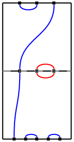
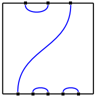
$D_2 \circ D_1 =$  $\circ$  $=$  $= \alpha$  $= \alpha\, D_2 \# D_1$

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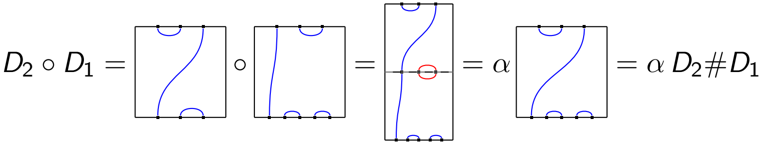


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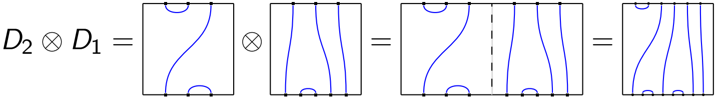

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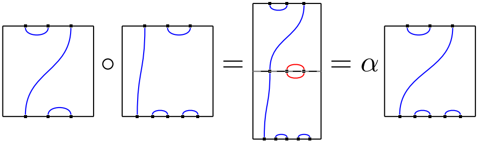
$D_2 \circ D_1 =$ 

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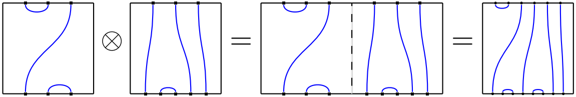
$D_2 \otimes D_1 =$ 


$(n_1 \otimes n_2 = n_1 + n_2)$.

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TL-category: a nested $(0, 1, 2)$ "cobordism category" with

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we will restrict to surface types $\Sigma$ with one boundary component.

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Proceed concretely; attach “handles” to our square frame

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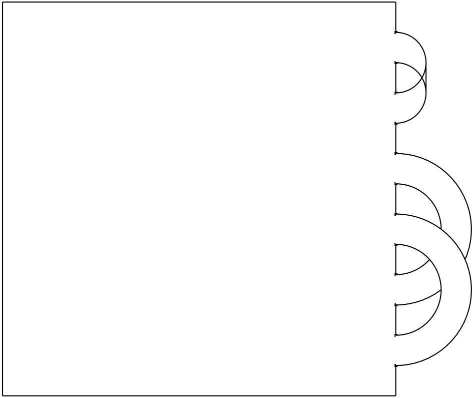

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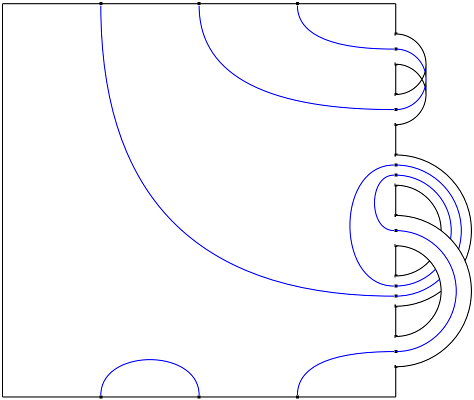

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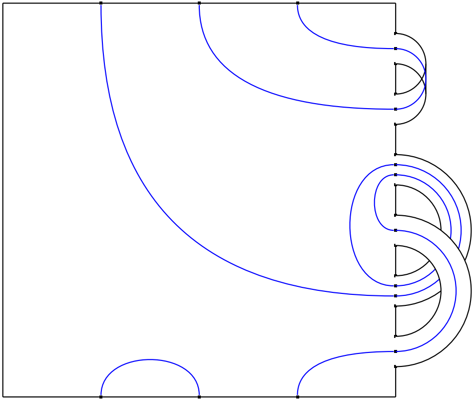

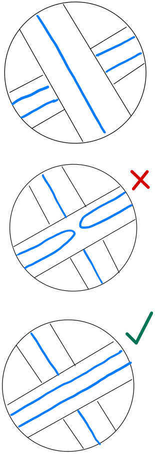

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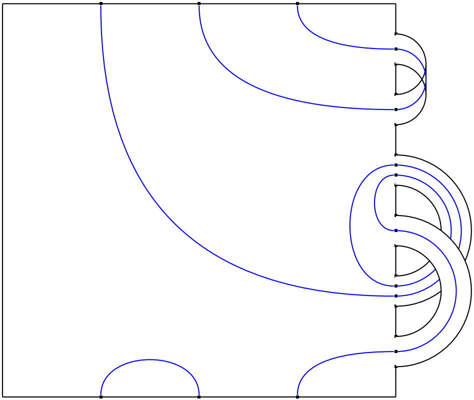

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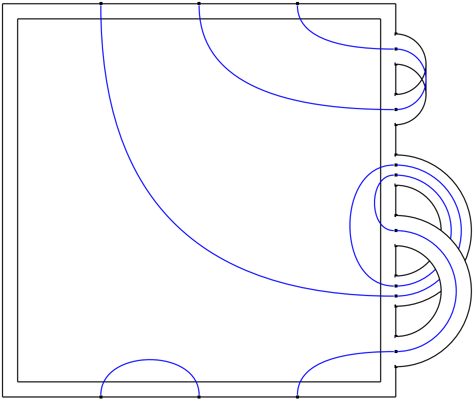

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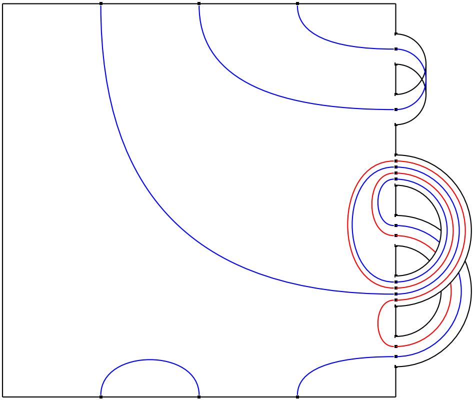
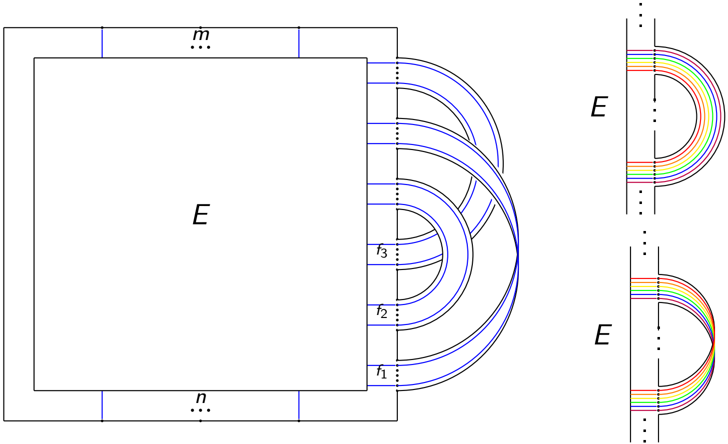


# SWB diagrams

**Square with bands (SWB)** diagram encoded by $\Theta = (P, s, f, E)$ (type $n, m$)



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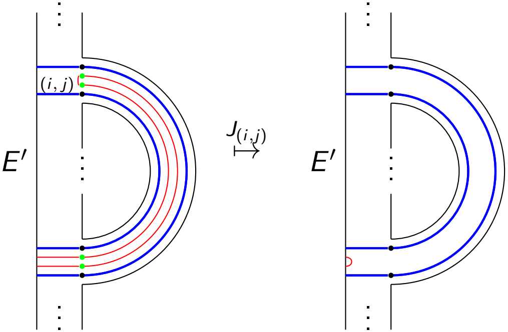

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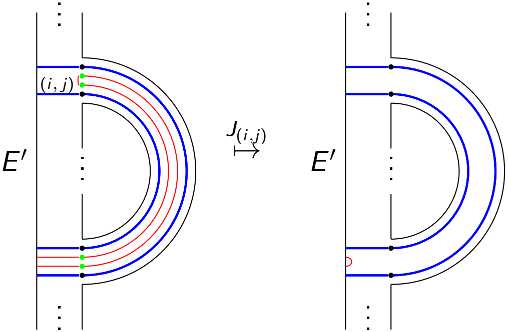


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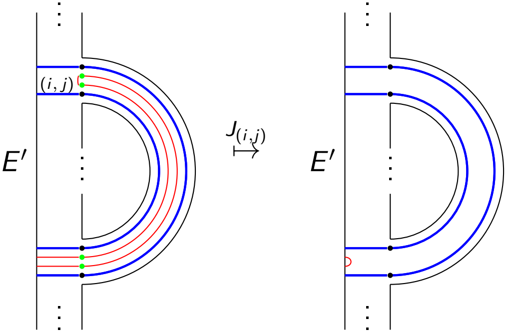


$$(P, s, f, E' \sqcup \{\{(i,j),(i,j+1)\}\}) \mapsto (P, s, f', o(E''))$$

Generate an equivalence relation with this move.

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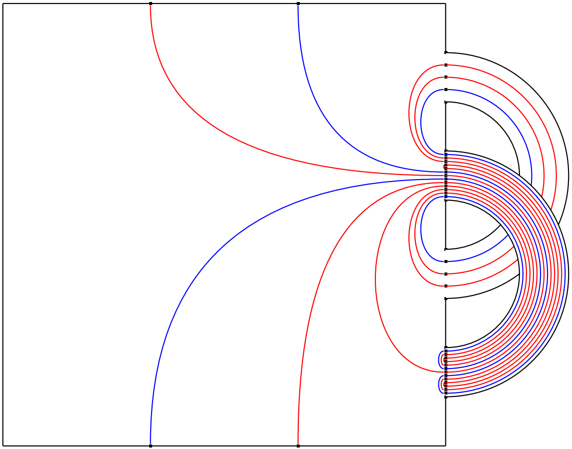

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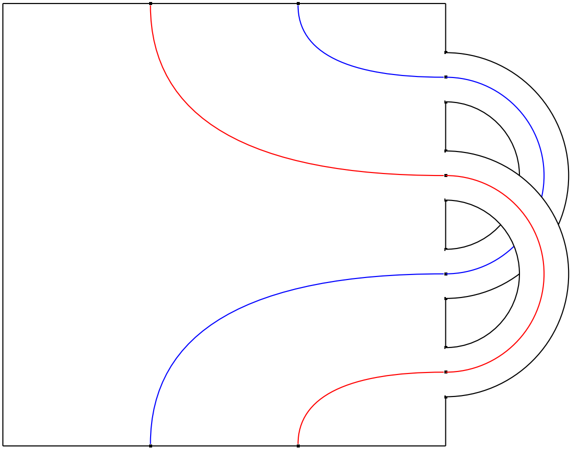

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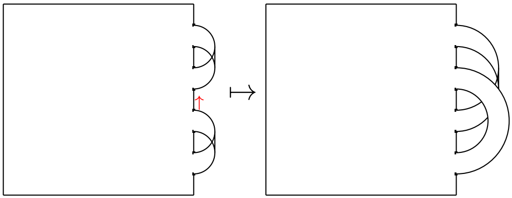

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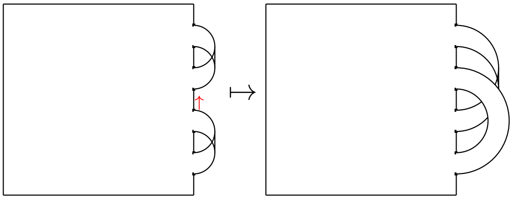

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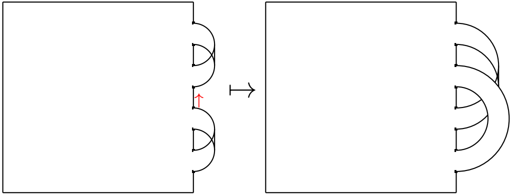

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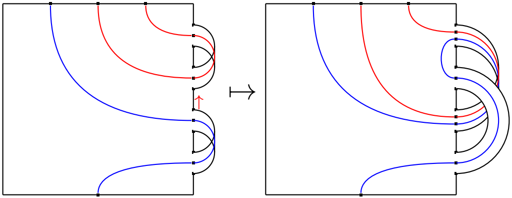

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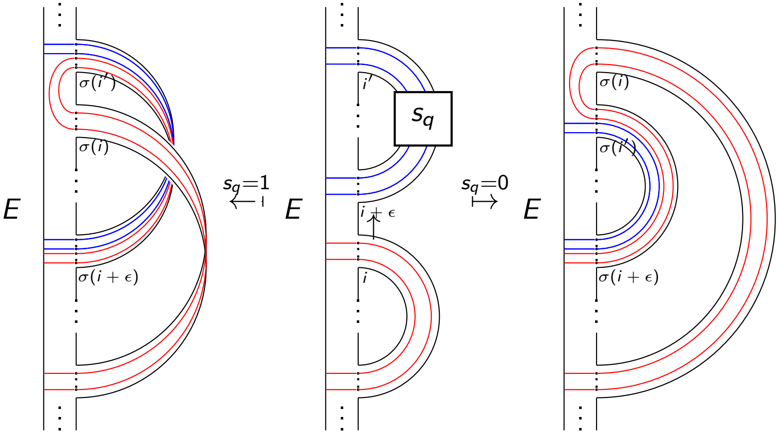

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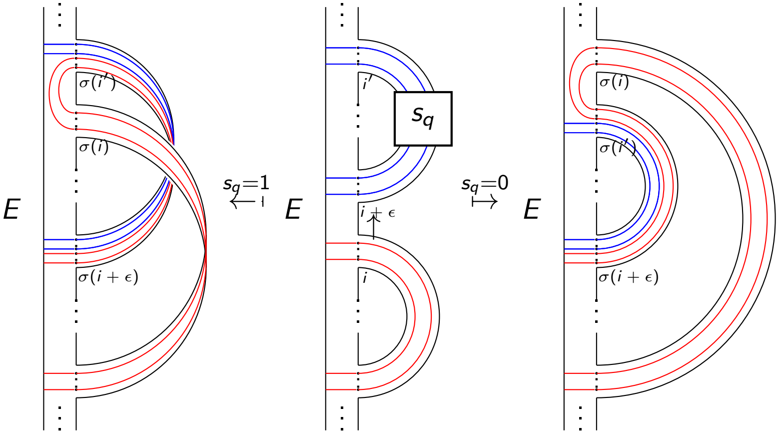


$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{\text{“new red arcs”}\})$$

# SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if $(P', s')$ can be obtained from $(P, s)$ by a finite sequence of handleslides, e.g.

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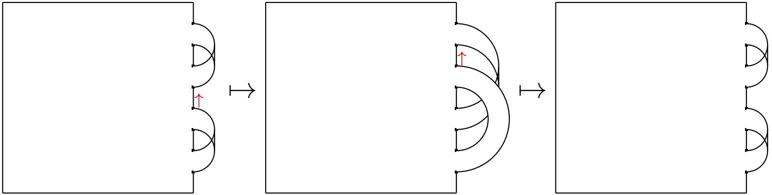

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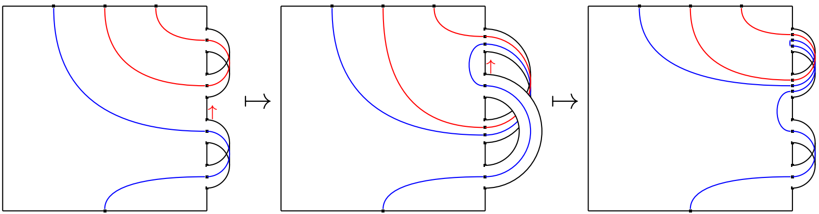

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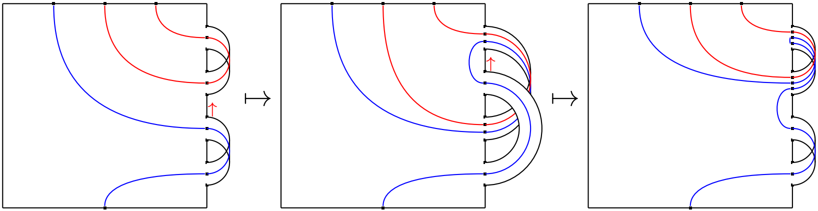

Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

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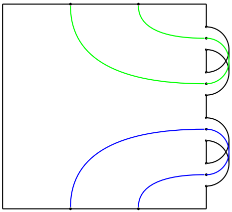
$\Theta =$ 

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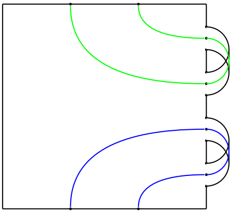
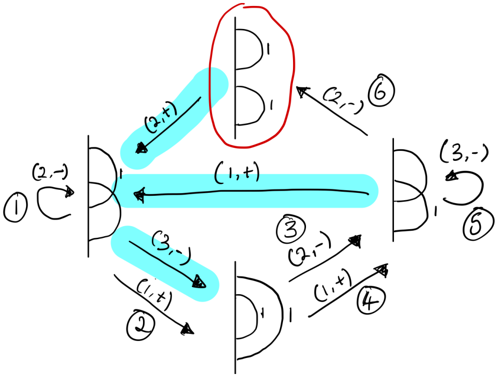
$\Theta =$ 



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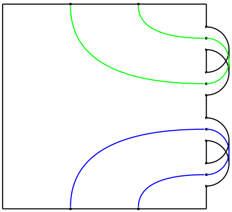
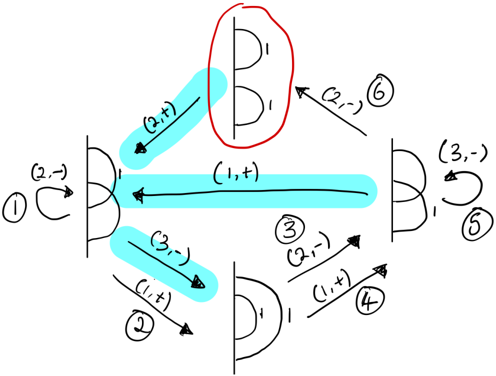
$\Theta =$ 




Associate the "reduced" sequence $A_i$ for each edge outside the tree, e.g.

$$A_2 = (3,+) \circ (4,-) \circ (1,+) \circ (2,+)$$

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$A_2(\Theta) =$
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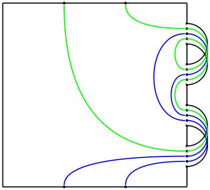
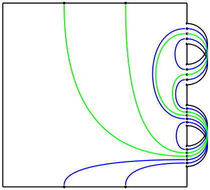


,

$A_3(\Theta) =$



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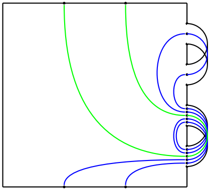

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$A_6(\Theta) =$
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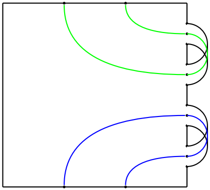

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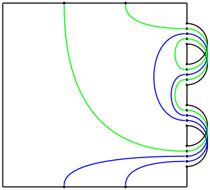

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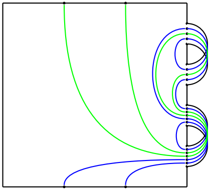

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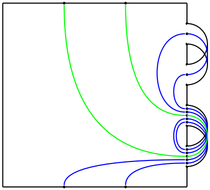

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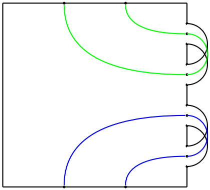

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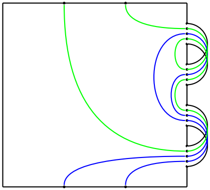
$$\langle A_2, A_3, A_4 \mid A_3 A_2 = A_4,\ A_2 A_4 = A_4 A_2^{-1} \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}.$$

# SWB diagrams - Handleslide Equivalence

$A_1(\Theta) =$
$A_2(\Theta) =$
$A_5(\Theta) =$



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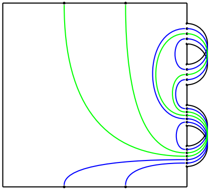

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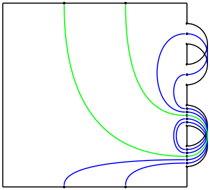

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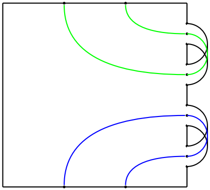

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(Chord Diag. Pres. of Mapping Class Group - Bene 2009)

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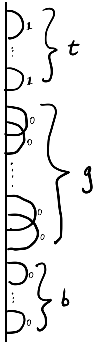
FACT: Any surface $(P, s)$ has a unique representative in the following **caravan form**:

$(P, s) \sim$ 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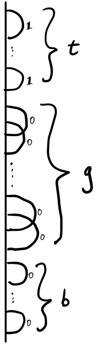
$(P, s) \sim$ 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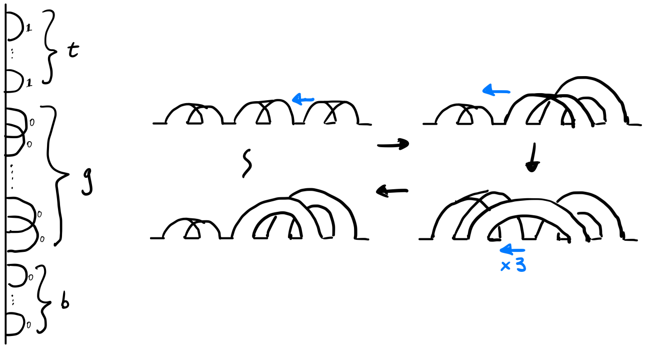


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# The Category $\mathcal{SQ}$

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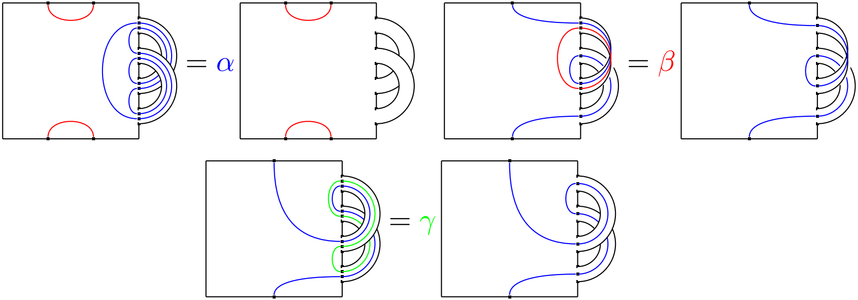

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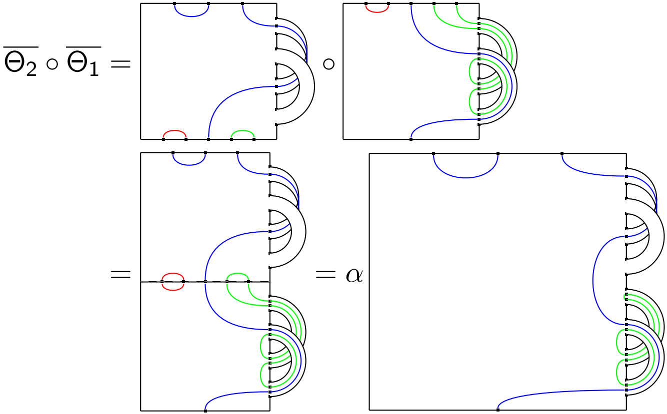

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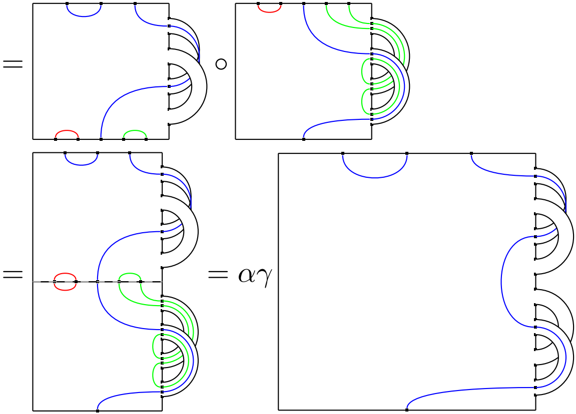

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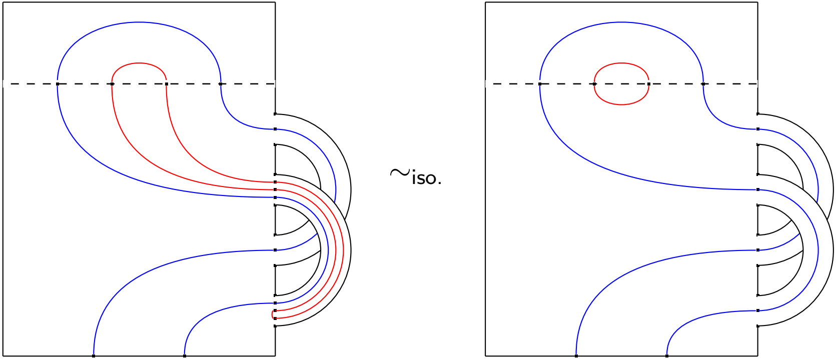

# The Category $\mathcal{SQ}$ - Basic Facts

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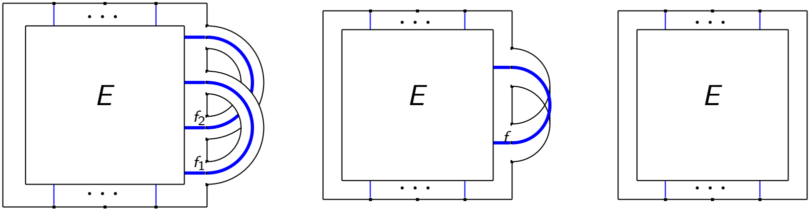

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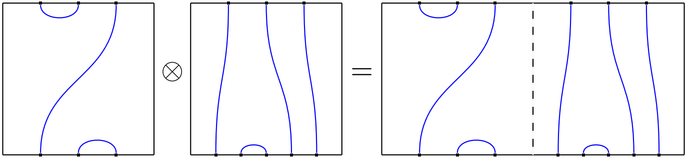

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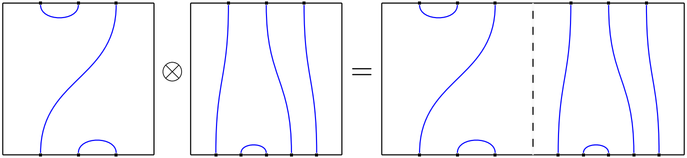

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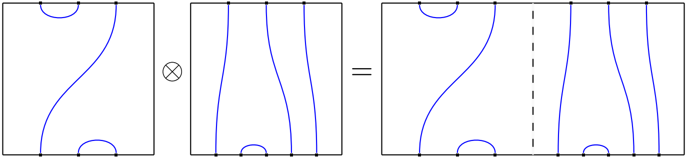

Can we extend this to a tensor product on $\mathcal{SQ}$ which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta'}$ be for SWB diagrams??

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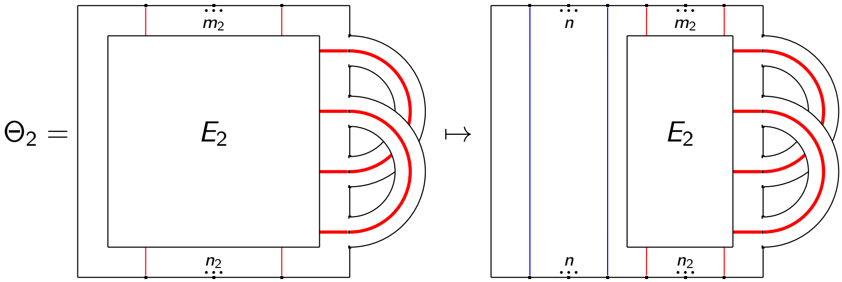
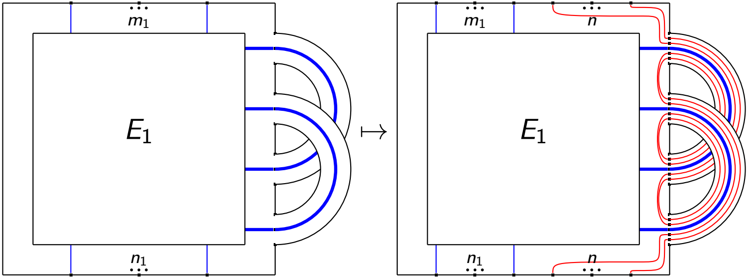



$$\overset{(?)}{:=} \mathrm{id}_n \otimes \Theta_2$$

# The Category $\mathcal{SQ}$ - Tensor Product

**Indirect answer:** Step 2 - Put the identity diagram on the right:

$\Theta_1 =$

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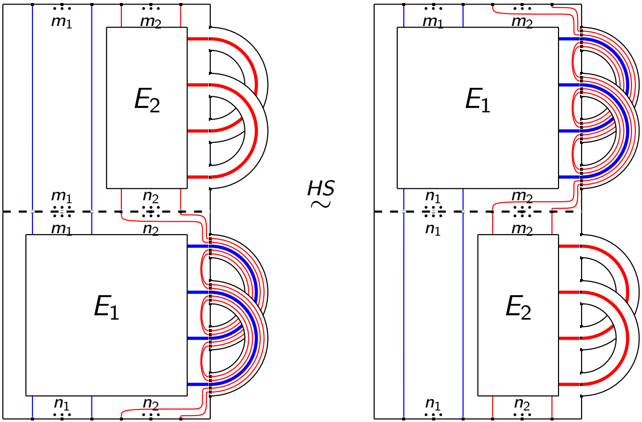
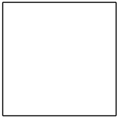
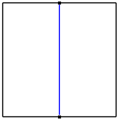
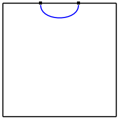


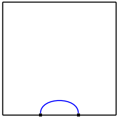
# Monoidal Generating Set?

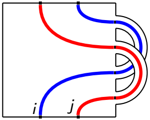
Conjecture: The following is a monoidal generating set for $\mathcal{SQ}(\alpha, \beta, \gamma)$ :

$\text{id}_0 =$  ,

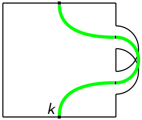
$\text{id}_1 =$  ,

$U =$  ,

$V =$  ,

$\mathbb{T}_{i,j} =$ 

,

$\mathbb{M}_k =$ 

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$F$ a full and essentially surjective, monoidal functor, and $T$ a target monoidal $\mathbb{K}$-linear category with f.d. hom spaces. Call $F$ a **finitising functor**.

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One very natural demand is that $\mathrm{Hom}(0,0) \simeq \mathbb{K}$.

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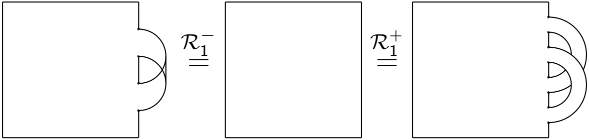

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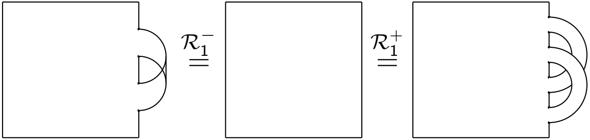


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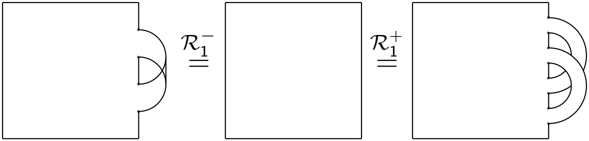


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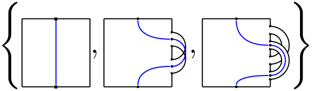
- $\mathrm{Hom}(1,1) = \mathbb{K}$  $\simeq \mathbb{K}\langle a \mid a^3 = a \rangle$,

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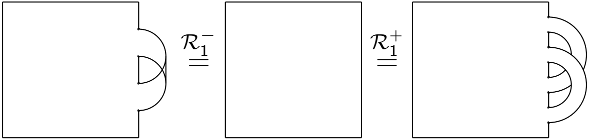


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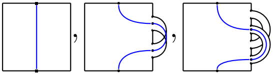
- $\mathrm{Hom}(1,1) = \mathbb{K}\Bigg\{$  $\Bigg\} \simeq \mathbb{K}\langle a \mid a^3 = a\rangle$,
- $\dim(\mathrm{Hom}(2,2)) \geq 23$ *

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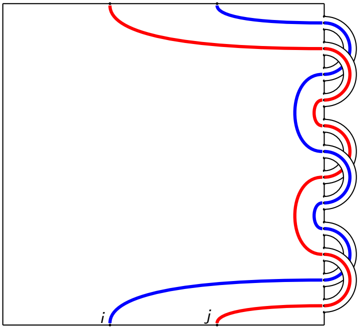
Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$ 

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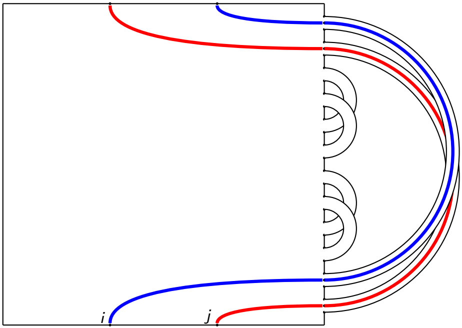

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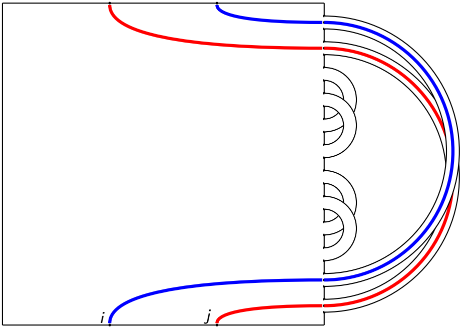
$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$  $= \mathbb{T}_{i,j} \mod \mathcal{R}_1$

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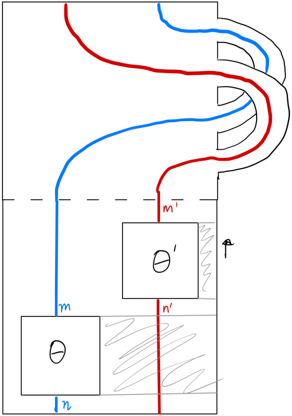
$\mathbb{T}_{m,m'} \circ (\Theta \otimes \Theta') =$ 

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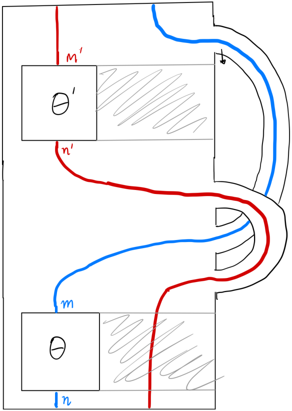

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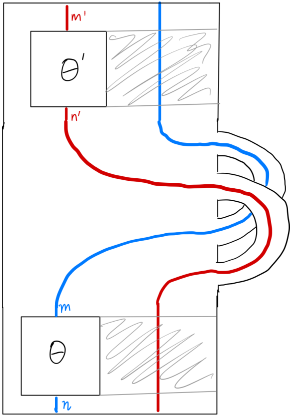

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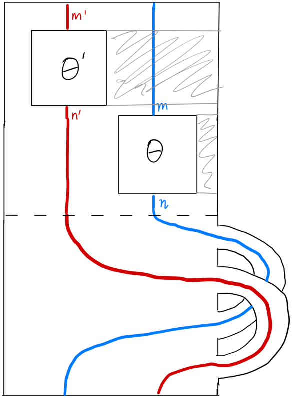

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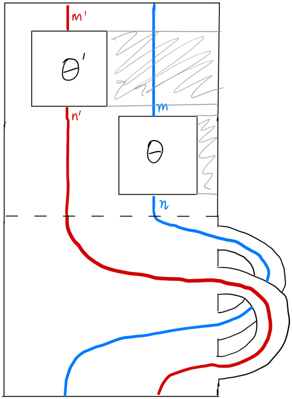


$$= (\Theta' \otimes \Theta) \circ \mathbb{T}_{n',n}$$

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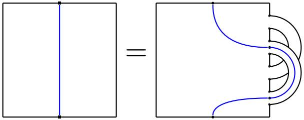
**However**, $\mathcal{SQ}$ is NOT a braided mon.cat. The smallest such quotient is obtained by imposing the relation $\mathcal{R}_2$ (as well as $\mathcal{R}_1^+$ *):



NOTE: This implies $\alpha = \gamma$. * not necessary if $\alpha$ invertible.

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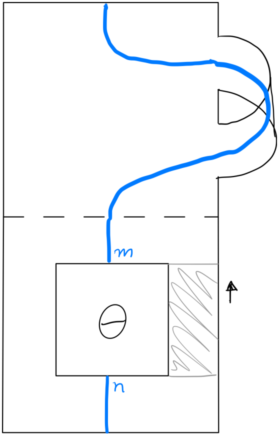

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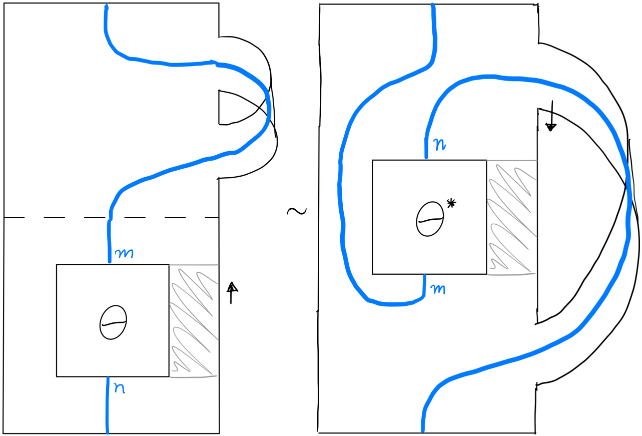

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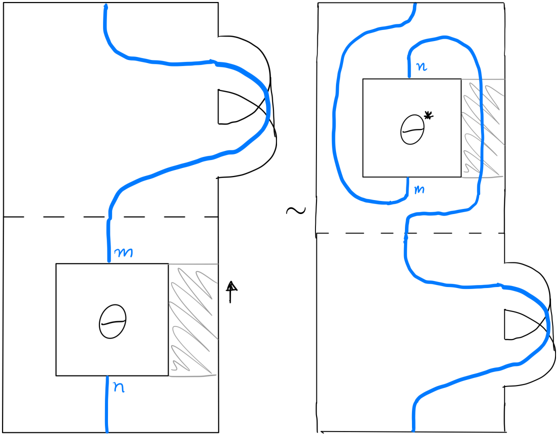

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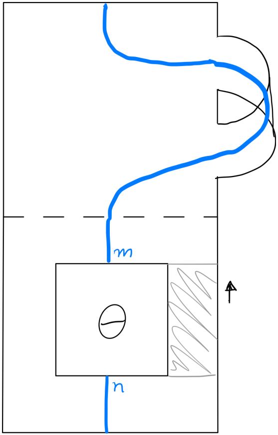
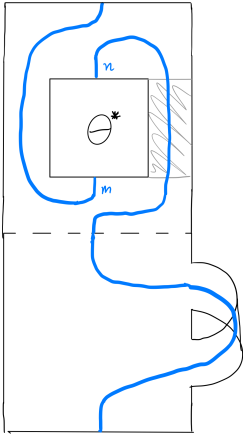



$\sim$

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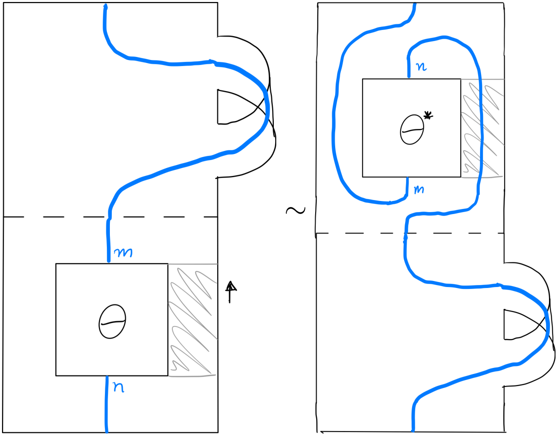
$\mathbb{M}_m \circ \Theta =$  $:= F(\Theta) \circ \mathbb{M}_n$



where $F^2 = \mathrm{id}$, $F \circ (\Theta \otimes \Theta') = F(\Theta') \otimes F(\Theta)$.

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- What does the (monoidal) representation theory look like?
- What about non-functorial quotients? e.g. terminate at a finite genus

# Thank You!

Questions?