



A Diagram Category for Non-Orientable Surfaces

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Joint work with Dionne Ibarra², Gabriel Montoya-Vega³, and
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The Interplay Between Skew Braces and Hopf-Galois Theory,
Leeds 2025



Motivation

Construct interesting low-dim “cobordism categories” amenable to rep. th. study:

- ▶ Linear
- ▶ Combinatorial
- ▶ Finite Dimensional Hom-spaces
- ▶ More structure? (monoidal... etc)

In particular, we consider **nested** $(0, 1, 2)$ - “cobordism categories”.



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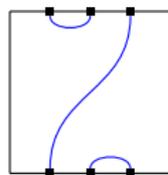
- ▶ **Objects:** $(0, 1)$ part - points in $[0, 1]$ (skeletally \mathbb{N})
- ▶ **Morphisms:** $(1, 2)$ part



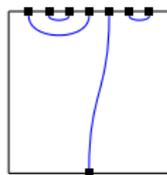
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- **Morphisms:** $(1, 2)$ part - $\text{Hom}(n, m)$ is \mathbb{K} -linear combinations of type n, m “TL-diagrams”, (embedded intervals in $[0, 1]^2$):



$\in \text{Hom}(3, 3)$,



$\in \text{Hom}(1, 7)$.

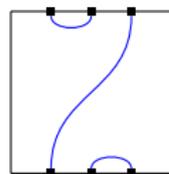
up to homeomorphisms of $[0, 1]^2$ (ambient isotopy).



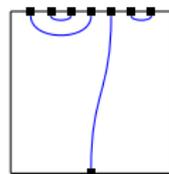
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$\{ \text{classes of diagrams} \} \leftrightarrow \{ \text{xless pair ptns of } V(n, m) \}$



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$((\phi, \psi) \mapsto \psi \circ \phi)$:

$$D_2 \circ D_1 = \text{[Diagram 1]} \circ \text{[Diagram 2]} = \text{[Diagram 3]} = \alpha \text{[Diagram 4]} = \alpha D_2 \# D_1$$

The diagram equation illustrates the composition of two diagrams, D_2 and D_1 , in the Temperley-Lieb category. The composition is defined by vertically stacking the diagrams. The first diagram, D_2 , is a square with a blue curve that starts at the bottom left, goes up, and then down to the bottom right. The second diagram, D_1 , is a square with a blue curve that starts at the top left, goes down, and then up to the top right. The composition $D_2 \circ D_1$ is shown as the two diagrams stacked vertically. The resulting diagram is a square with a blue curve that starts at the bottom left, goes up, and then down to the bottom right. A red circle is drawn around the intersection of the two curves in the middle of the square. The diagram is then shown to be equal to $\alpha D_2 \# D_1$, where α is a scalar factor and $\#$ is a specific operation on the diagrams.



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Generically $D_2 \circ D_1 = \alpha^{L(D_1, D_2)} D_2 \# D_1$.



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The diagram shows the composition of two diagrams, D_2 and D_1 , in the Temperley-Lieb category. The first diagram, D_2 , is a square with a blue strand that starts at the bottom left, goes up, loops back down, and ends at the top right. The second diagram, D_1 , is a square with a blue strand that starts at the bottom left, goes up, loops back down, and ends at the top right. The composition $D_2 \circ D_1$ is shown as the two diagrams stacked vertically. The resulting diagram is a square with a blue strand that starts at the bottom left, goes up, loops back down, and ends at the top right. A red circle highlights the intersection of the two strands in the middle of the diagram. The final result is $\alpha D_2 \# D_1$, where α is a scalar factor and $\#$ denotes the tensor product.

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Tensor Product: “defined” on diagrams by horizontally stacking:

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$(n_1 \otimes n_2 = n_1 + n_2)$.



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we will restrict to surface types Σ with one boundary component.



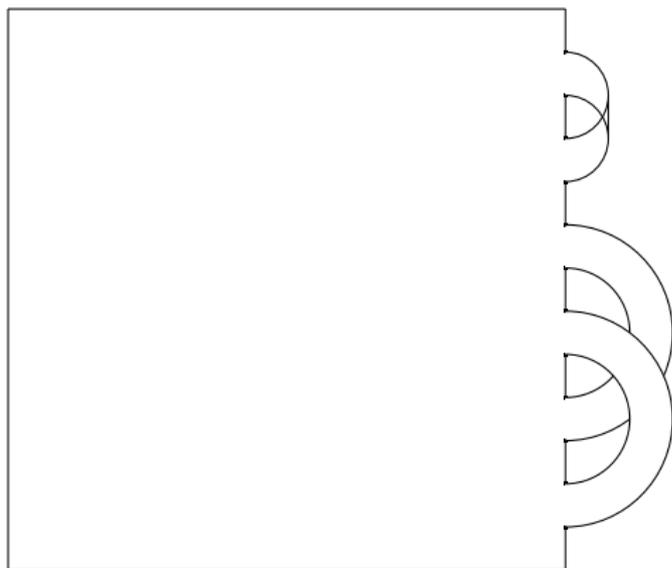
How to draw TL diagrams on other surfaces?

Proceed concretely; attach “handles” to our square frame



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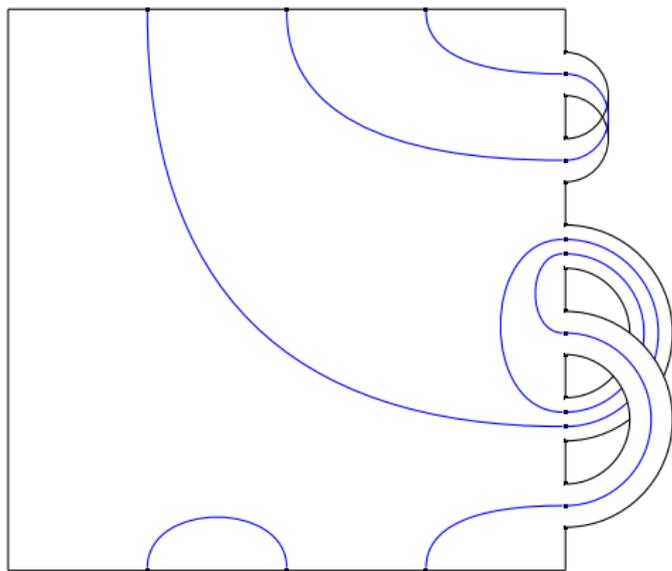
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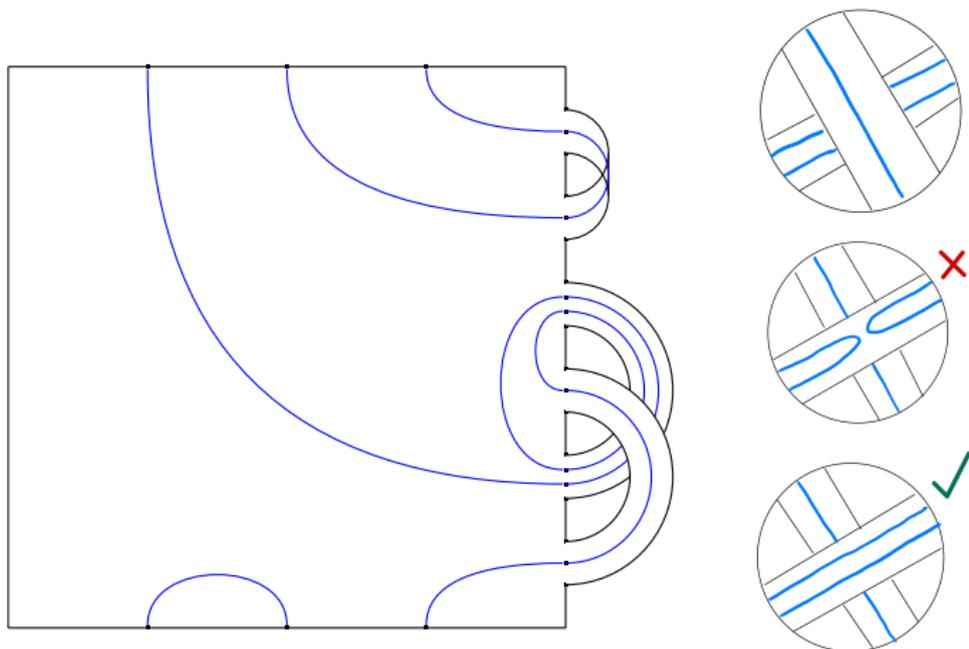
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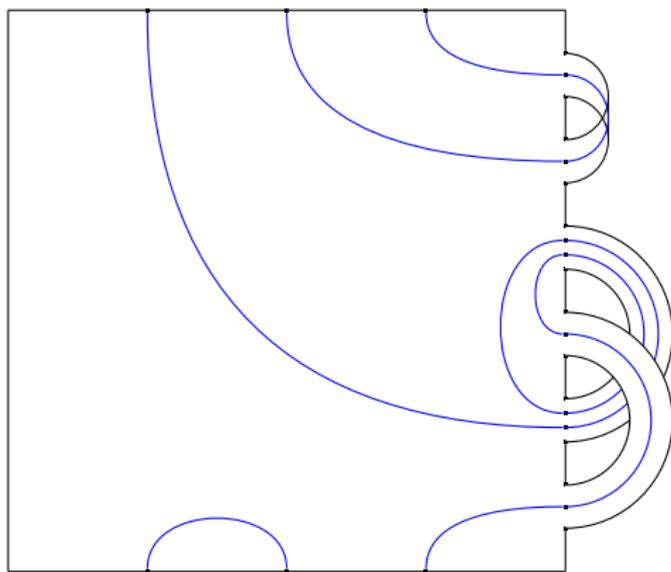
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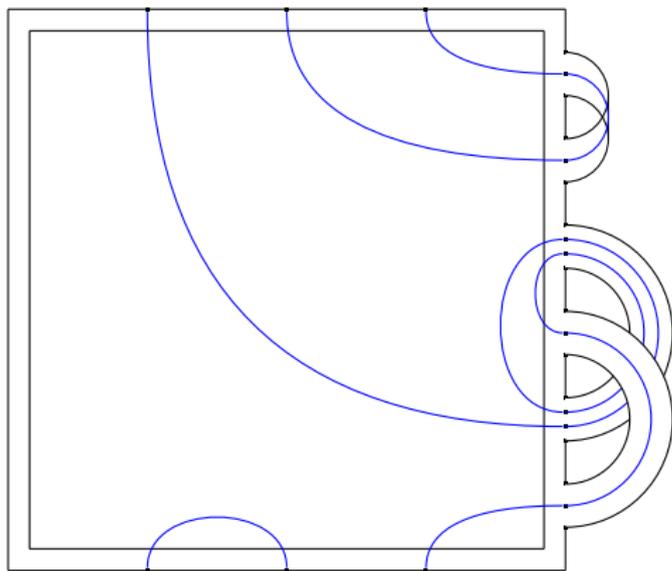
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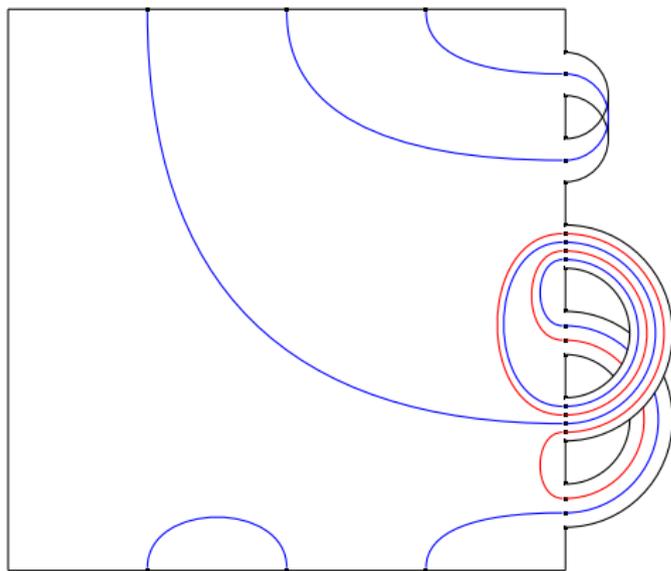


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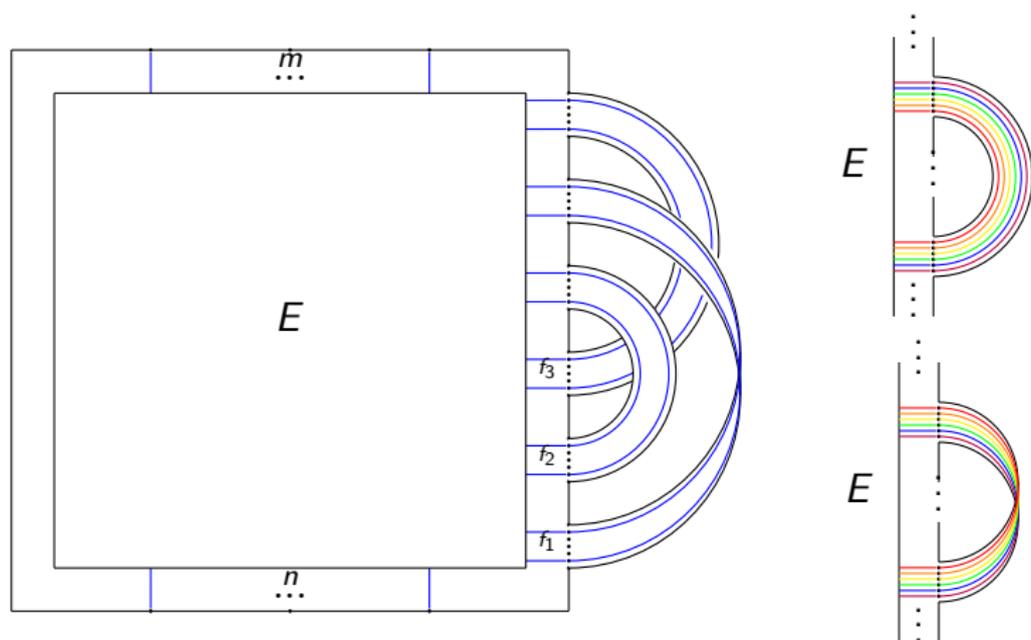
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SWB diagrams

Square with bands (SWB) diagram encoded by $\Theta = (P, s, f, E)$
(type n, m)

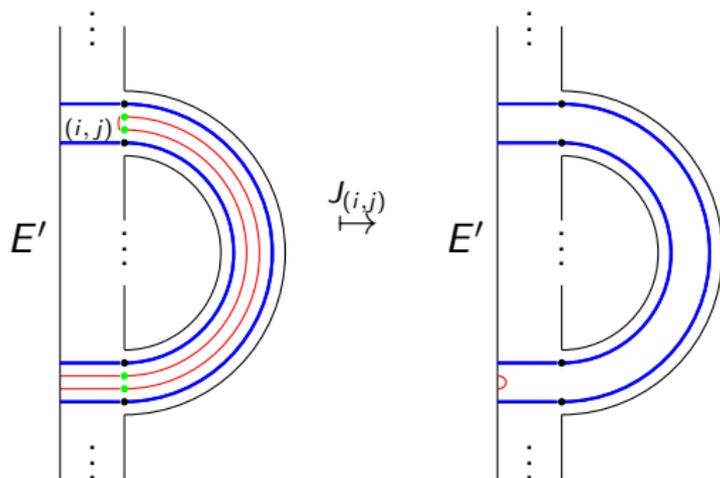




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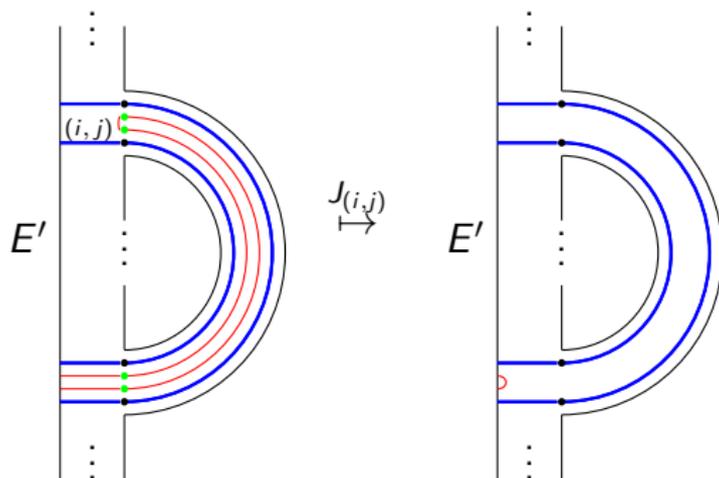
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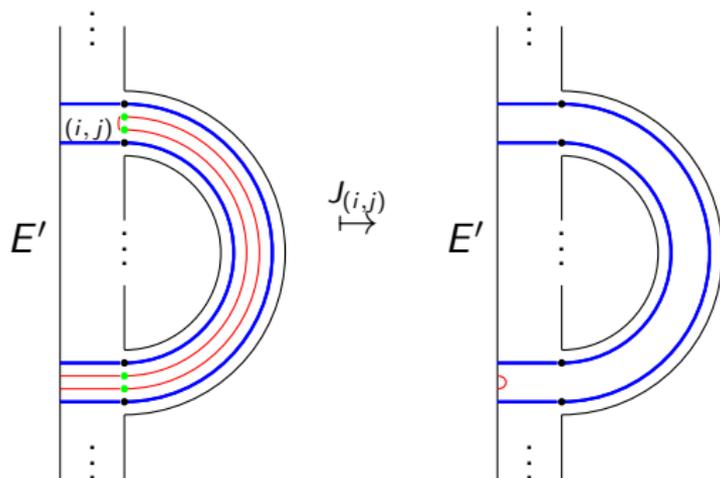


$$(P, s, f, E' \sqcup \{(i, j), (i, j + 1)\}) \mapsto (P, s, f', o(E''))$$



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$$(P, s, f, E' \sqcup \{(i,j), (i,j+1)\}) \mapsto (P, s, f', o(E''))$$

Generate an equivalence relation with this move.

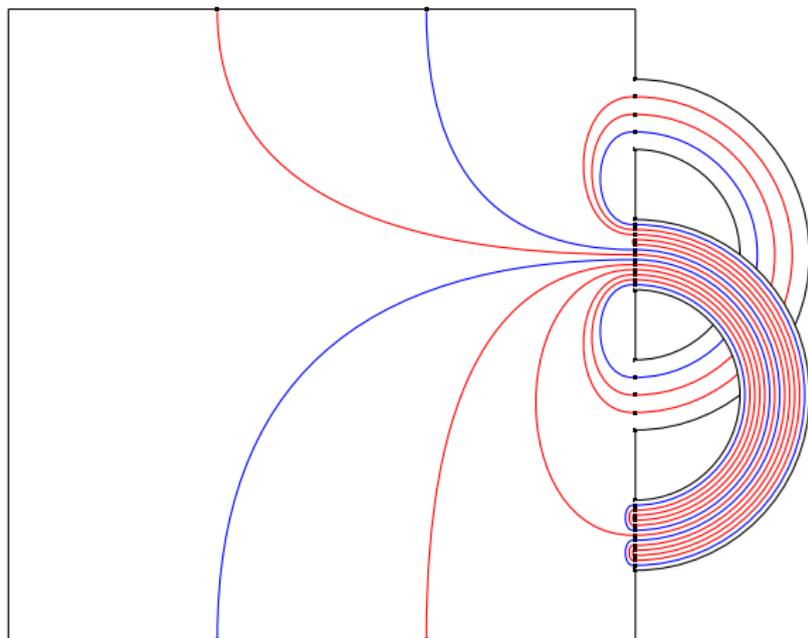


SWB diagrams - Isotopy

Fact: If Θ has no internal components, then its isotopy class has a **unique** representative w/o turnbacks

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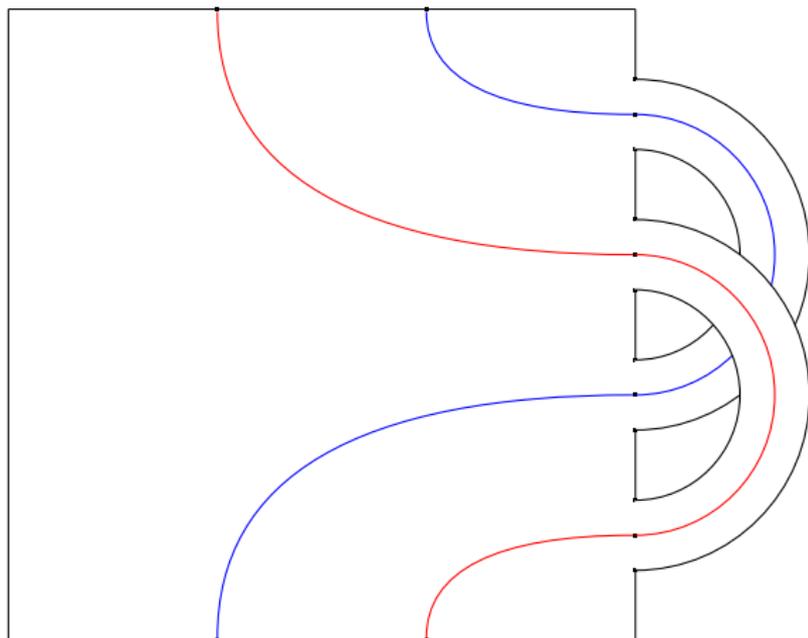
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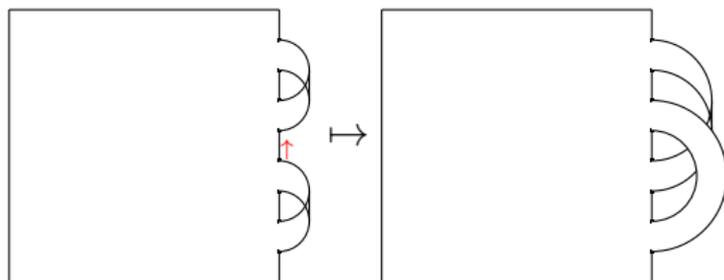


SWB diagrams - Handlesliding

Different realisations of a surface are related by **handleslides**:

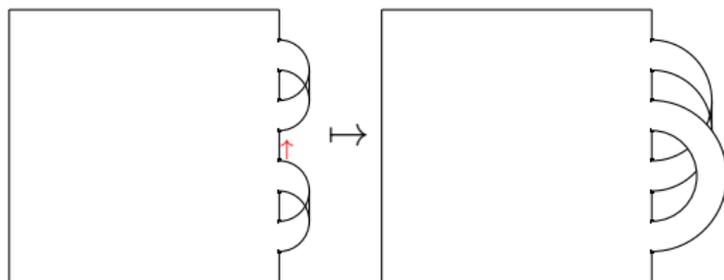
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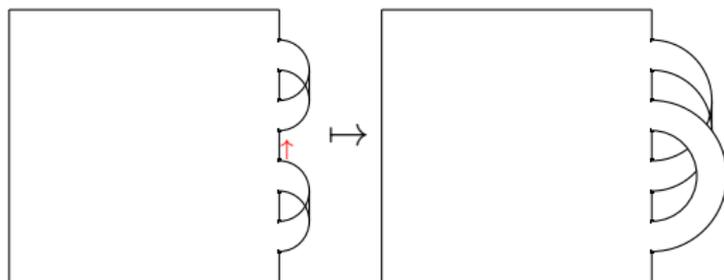
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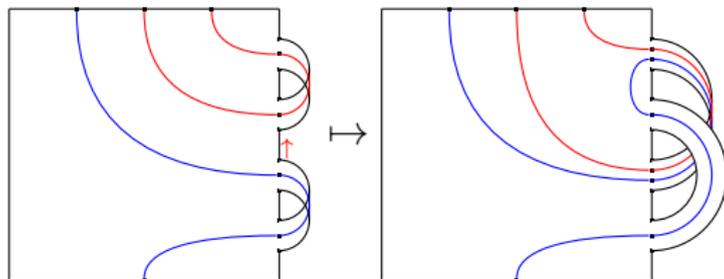
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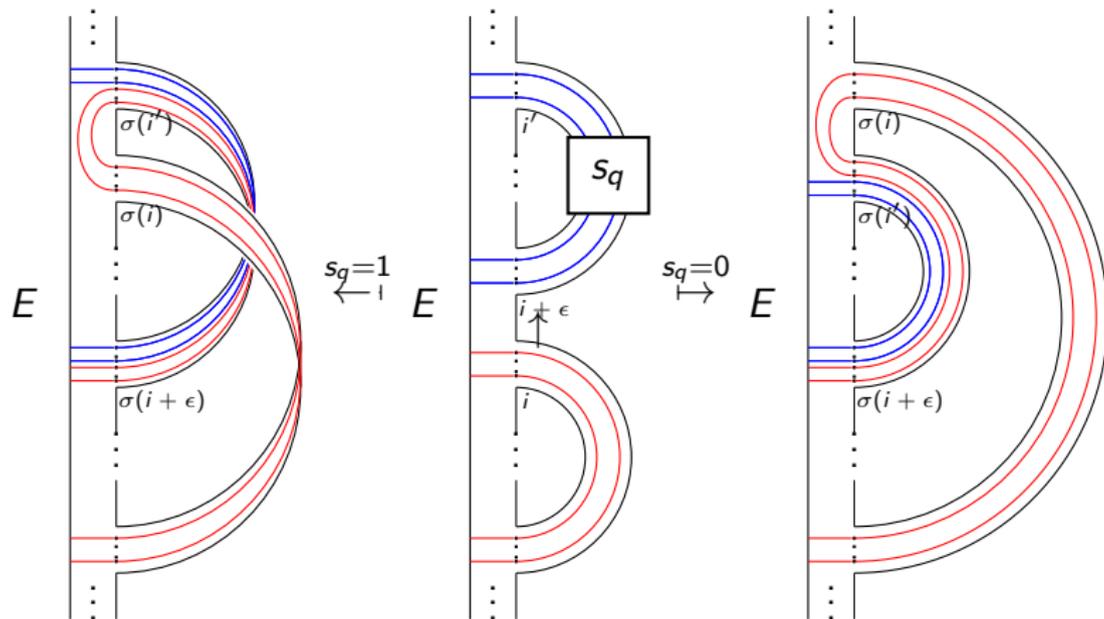
SWB diagrams - Handlesliding

Generically: “Two bands involved”



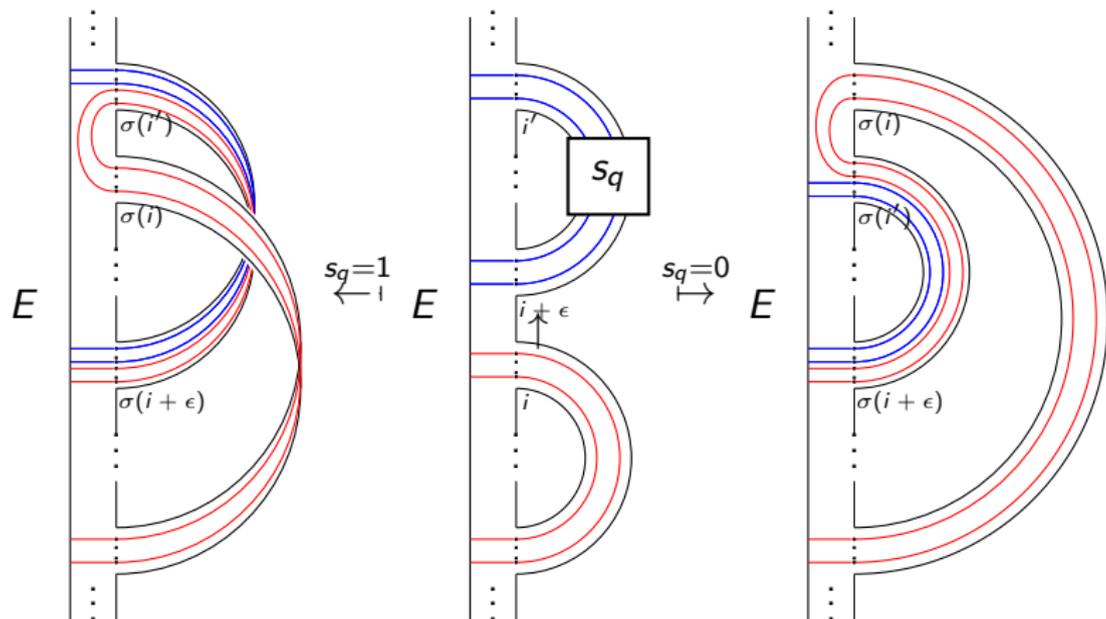
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$$(P, s, f, E) \mapsto (\sigma(P), s' \circ \sigma^{-1}, f' \circ \sigma^{-1}, o(E) \cup \{\text{"new red arcs"}\})$$



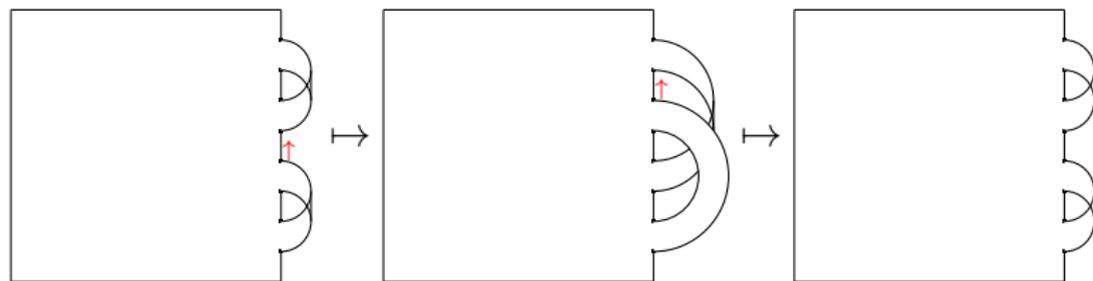
SWB diagrams - Handleslide Equivalence

On the level of the surface, we can define an equivalence relation by $(P, s) \sim (P', s')$ if (P', s') can be obtained from (P, s) by a finite sequence of handleslides, e.g.



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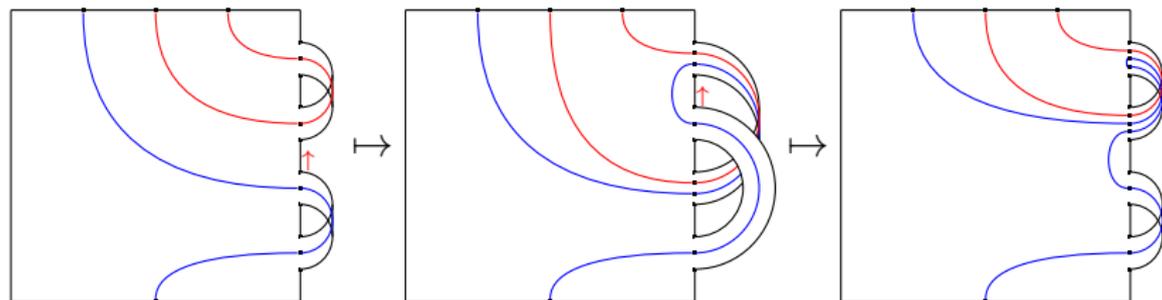
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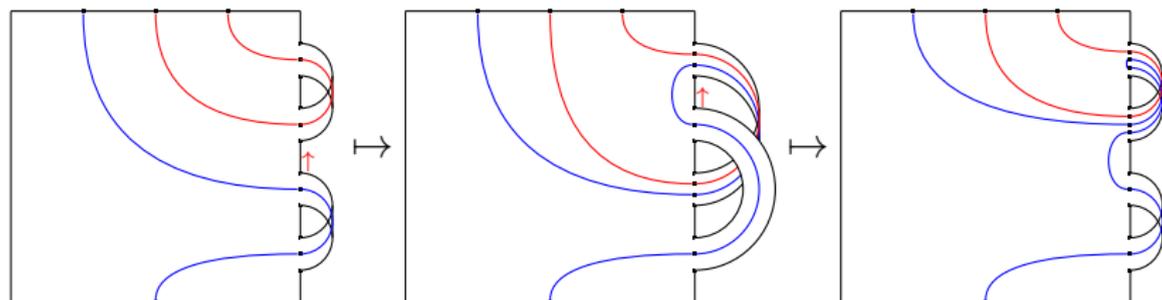
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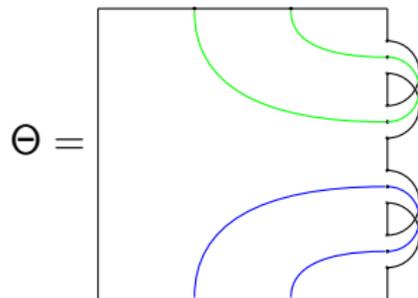
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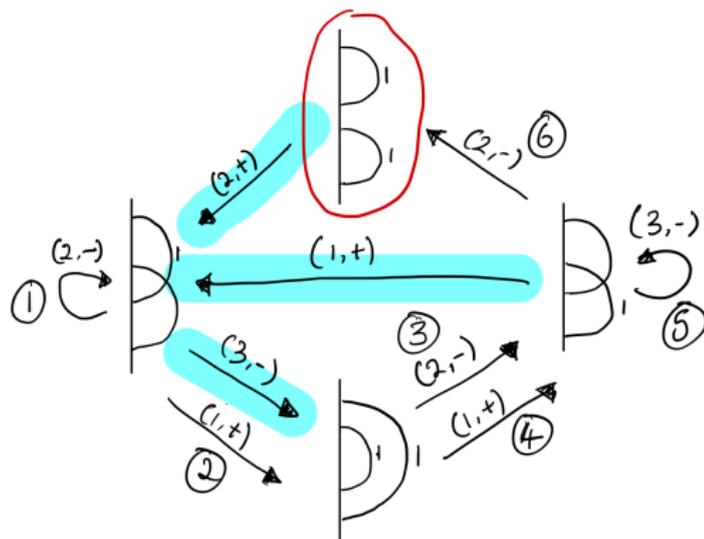
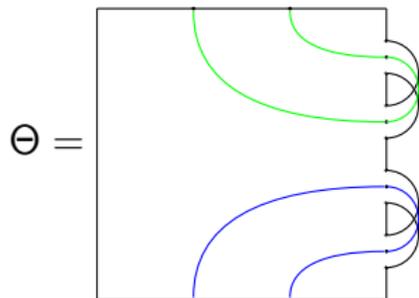
Defines an equivalence relation on **isotopy classes** of SWB diagrams - call this **Handleslide (HS) Equivalence**.

SWB diagrams - Handleslide Equivalence

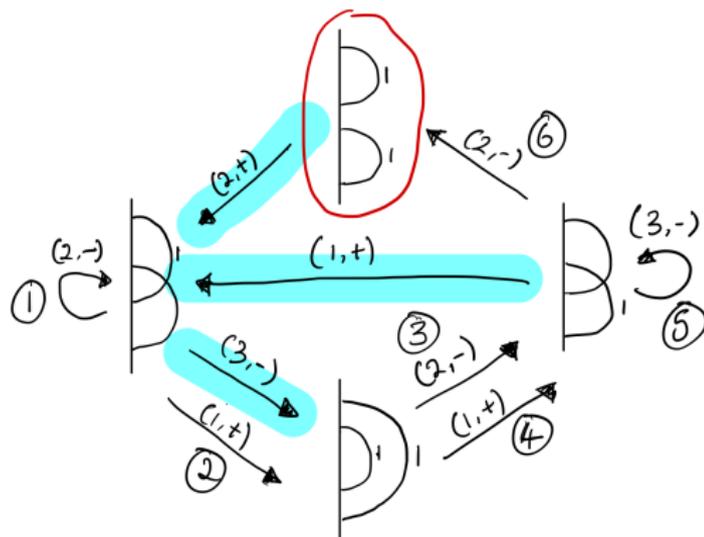
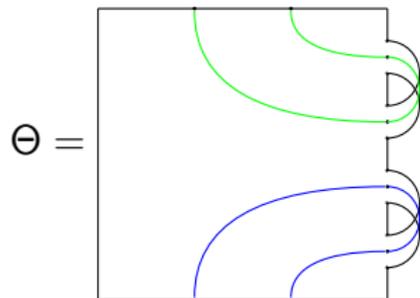




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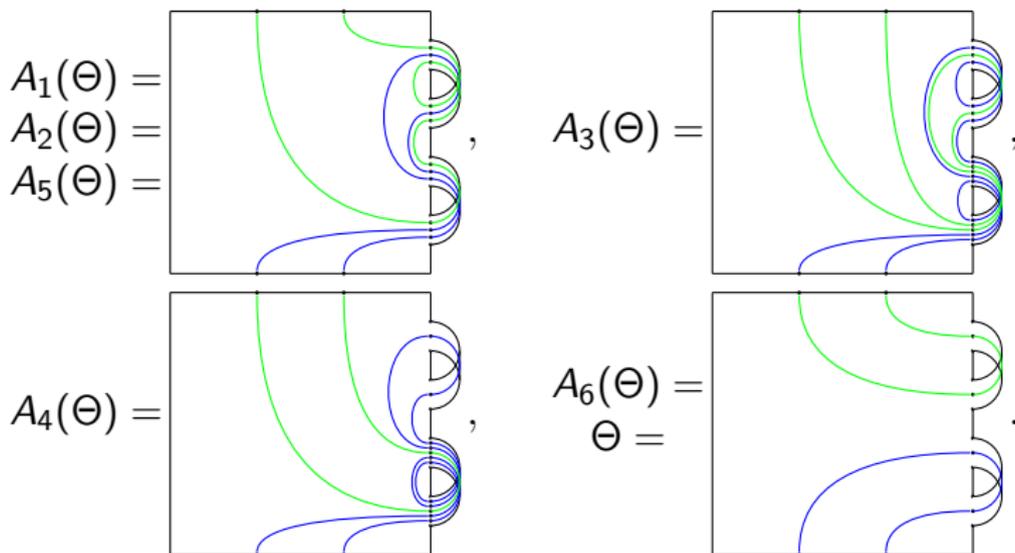
SWB diagrams - Handleslide Equivalence



Associate the “reduced” sequence A_i for each edge outside the tree, e.g.

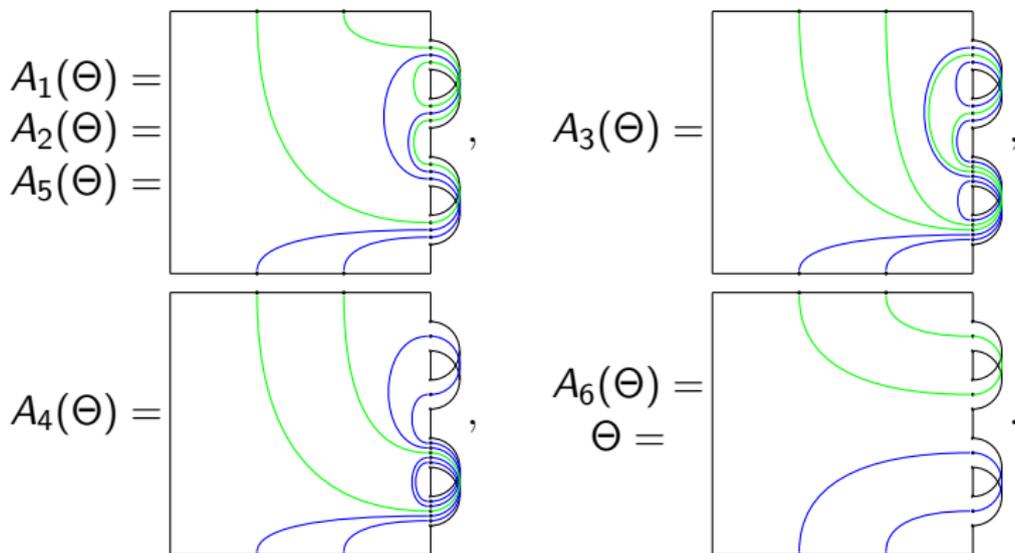
$$A_2 = (3, +) \circ (4, -) \circ (1, +) \circ (2, +)$$

SWB diagrams - Handleslide Equivalence



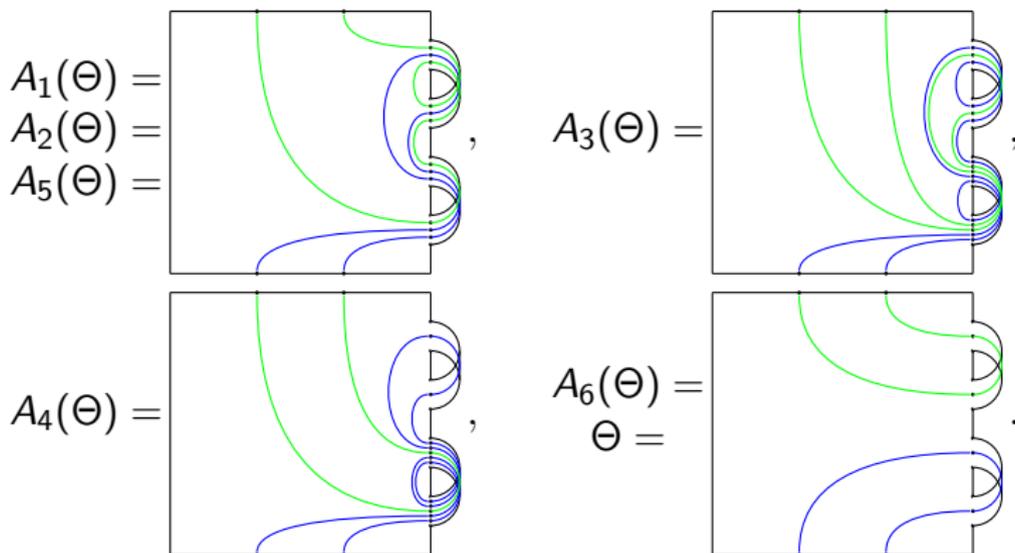


SWB diagrams - Handleslide Equivalence



$$\langle A_2, A_3, A_4 \mid A_3 A_2 = A_4, A_2 A_4 = A_4 A_2^{-1} \rangle \simeq \mathbb{Z} \times \mathbb{Z}.$$

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(Chord Diag. Pres. of Mapping Class Group - Bene 2009)



Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:

$$(P, s) \sim \begin{array}{c} \left. \begin{array}{c} D_1 \\ \vdots \\ D_1 \end{array} \right\} t \\ \left. \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \end{array} \right\} g \\ \left. \begin{array}{c} D_0 \\ \vdots \\ D_0 \end{array} \right\} b \end{array}$$

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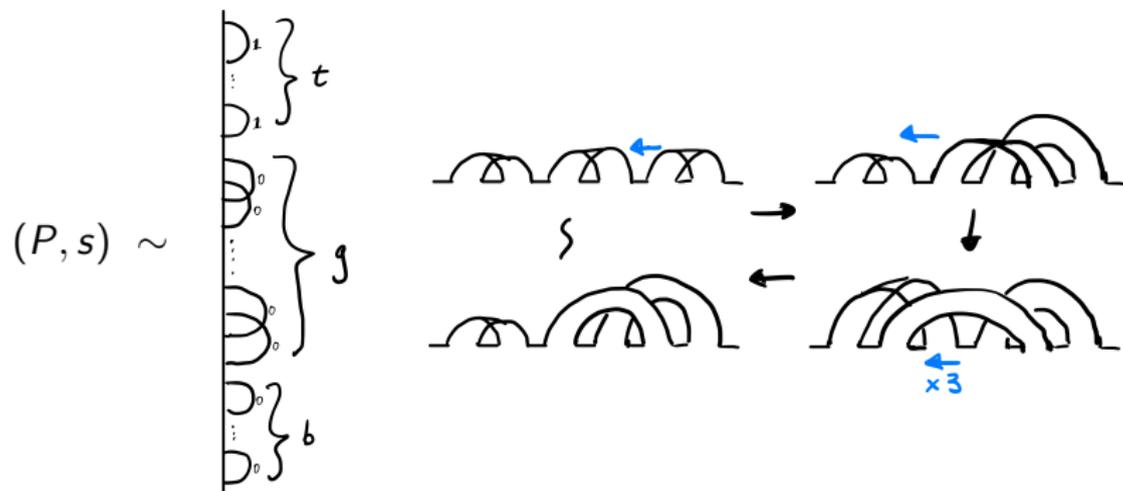
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where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.



Handleslide Equivalence - Caravan form

FACT: Any surface (P, s) has a unique representative in the following **caravan form**:



where $g, b \in \mathbb{Z}_{\geq 0}$, AND $t \in \{0, 1, 2\}$.



The Category \mathcal{SQ}

Fix \mathbb{K} a comm. ring with $\alpha, \beta, \gamma \in \mathbb{K}$.



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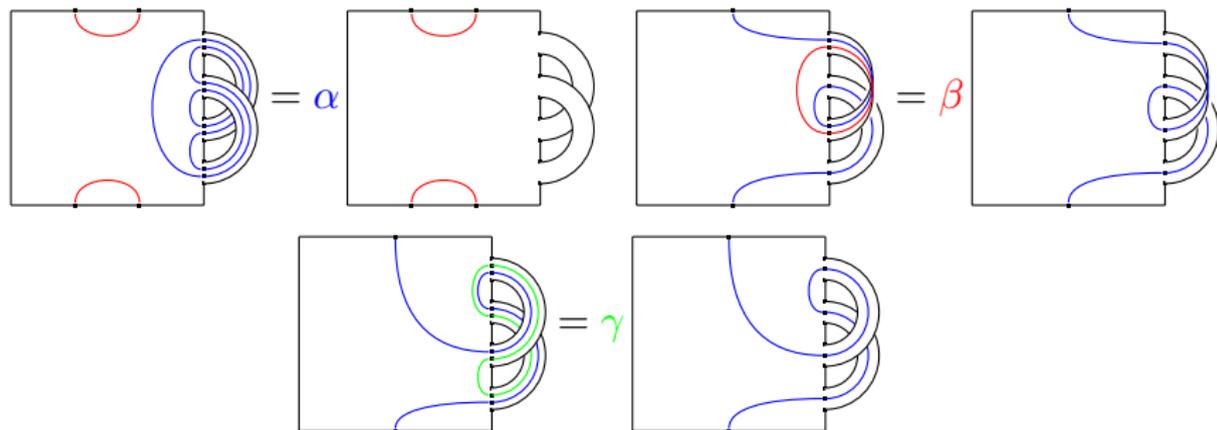
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Composition: $\text{Hom}(n, m) \times \text{Hom}(m, l) \rightarrow \text{Hom}(n, l)$ is given by

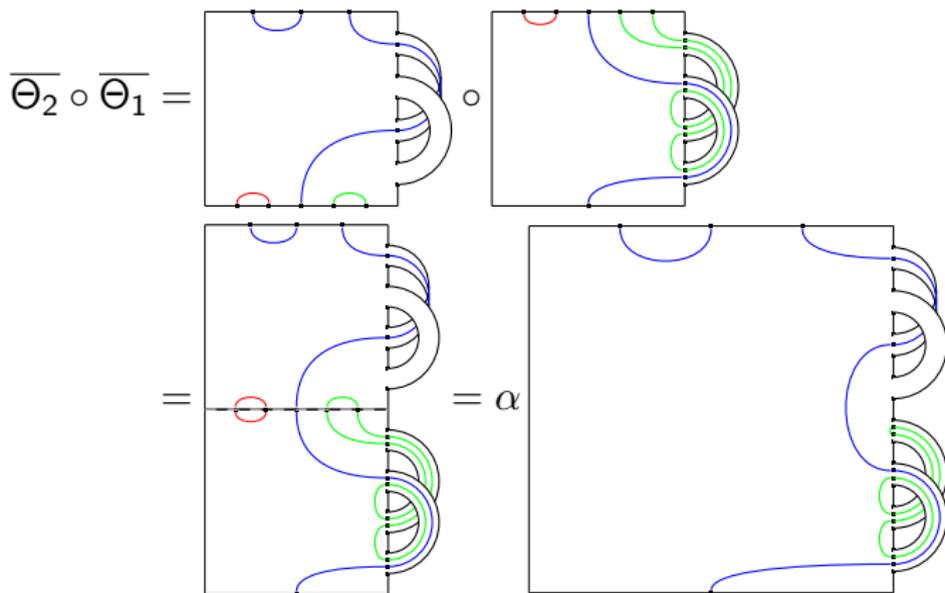
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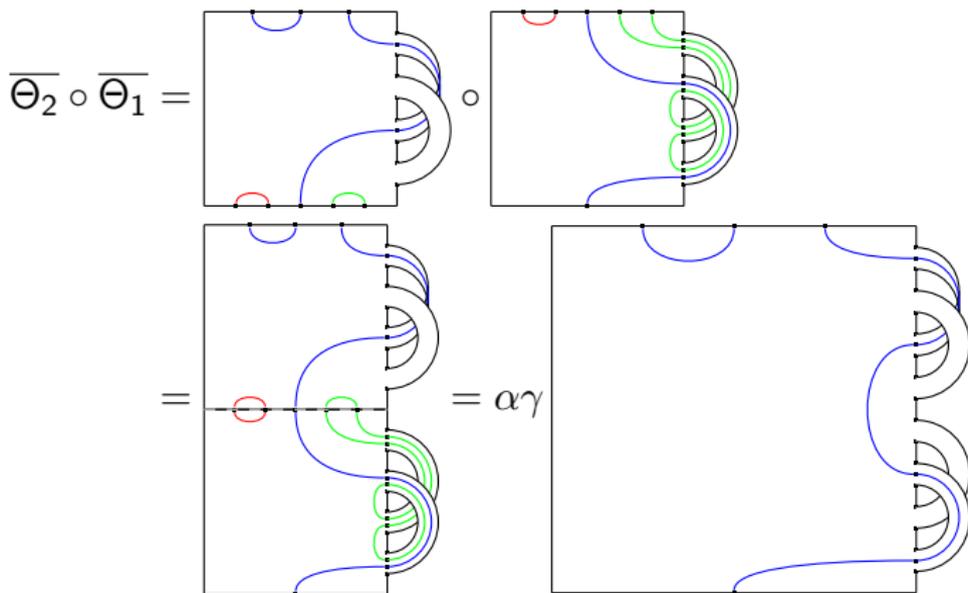




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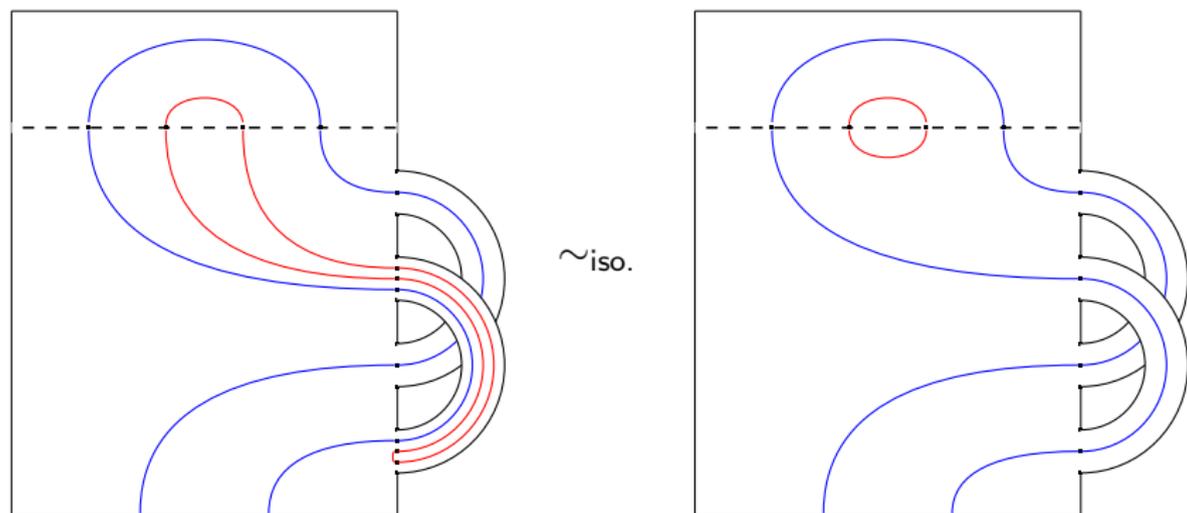


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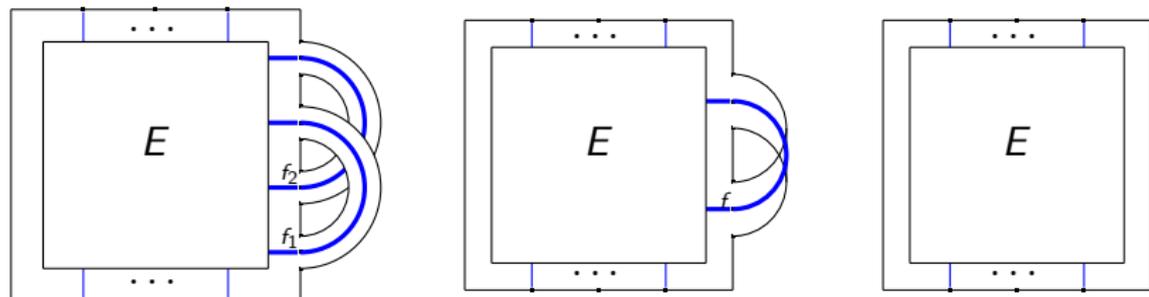
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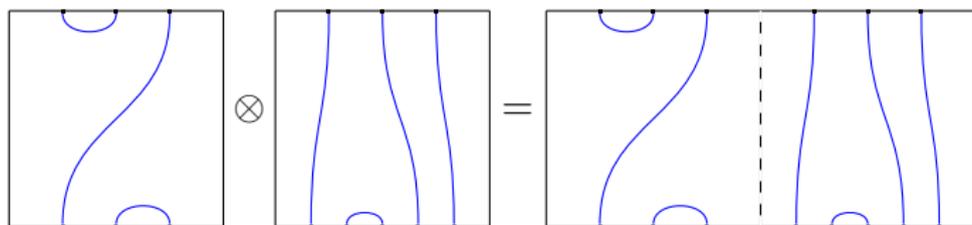
The Category \mathcal{SQ} - Tensor Product

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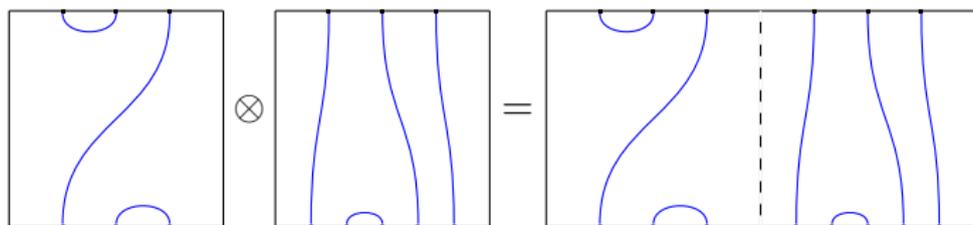
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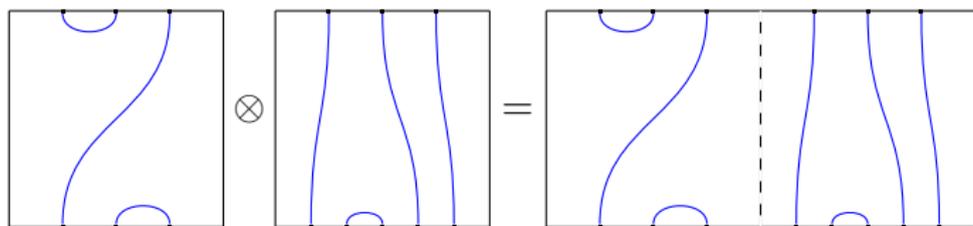
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Can we extend this to a tensor product on \mathcal{SQ} which has $n_1 \otimes n_2 = n_1 + n_2$ on objects. What should $\overline{\Theta} \otimes \overline{\Theta}'$ be for SWB diagrams??



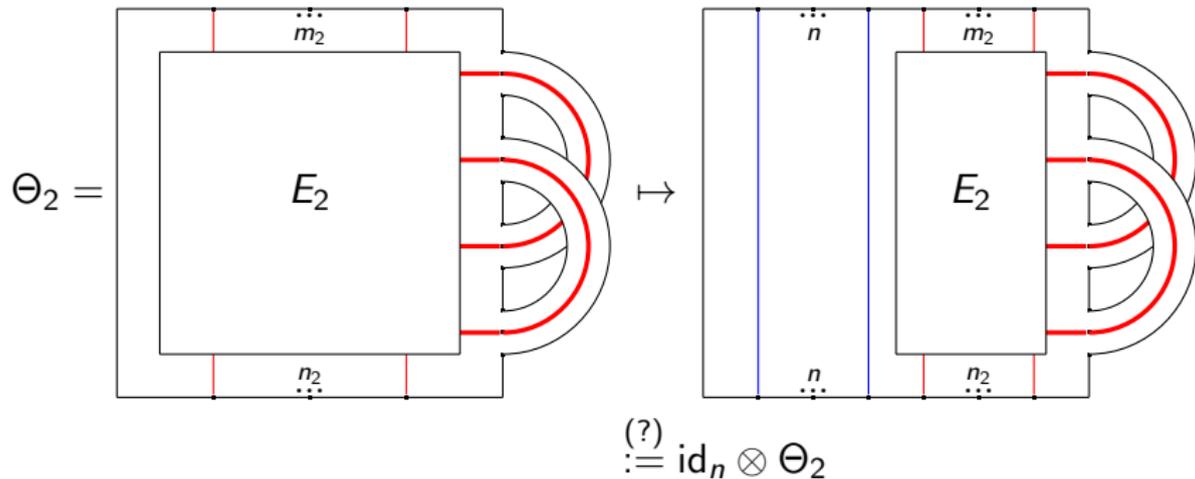
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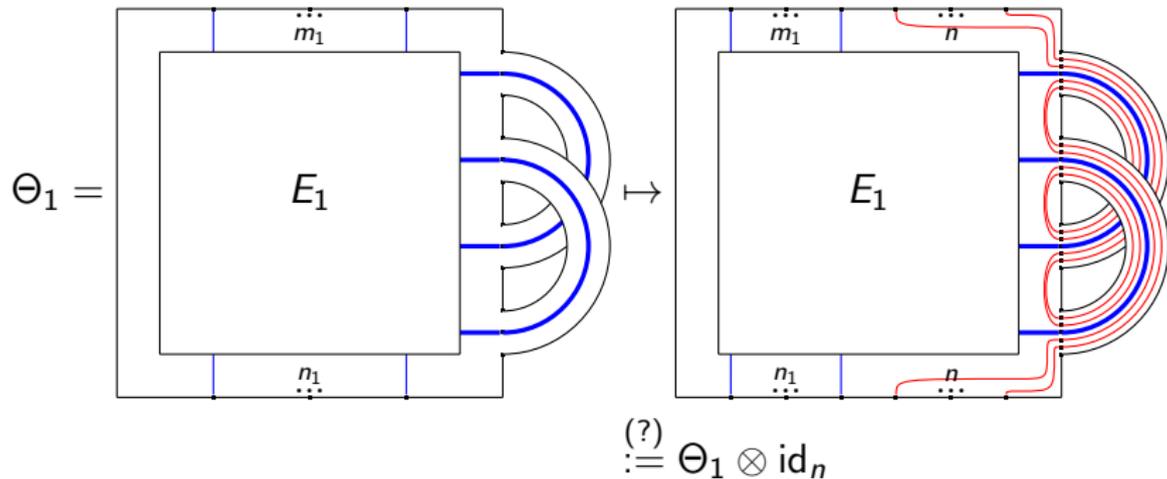
Indirect answer: Step 1 - Put the identity diagram on the left:





The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 2 - Put the identity diagram on the right:





The Category \mathcal{SQ} - Tensor Product

Indirect answer: Step 3 - Insist upon functoriality:



The Category \mathcal{SQ} - Tensor Product

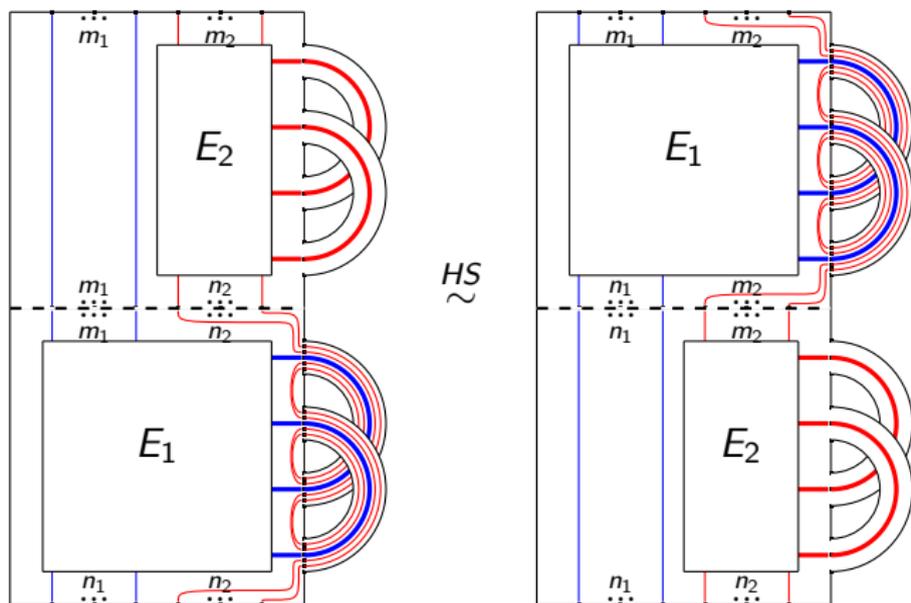
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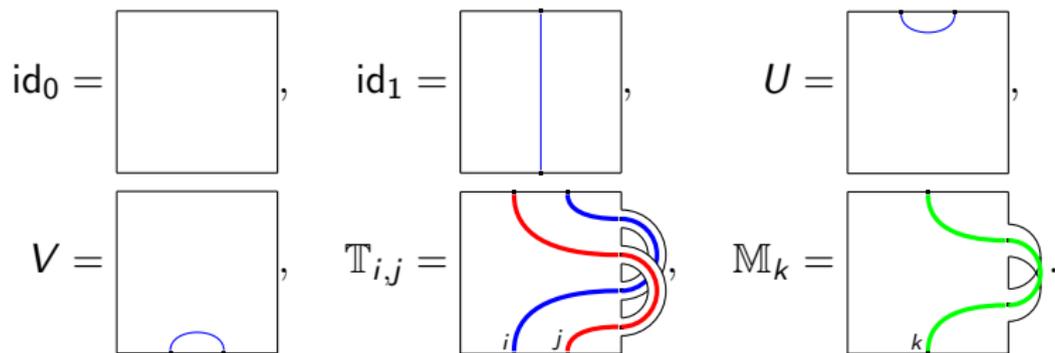
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Monoidal Generating Set?

Conjecture: The following is a monoidal generating set for $\mathcal{SQ}(\alpha, \beta, \gamma)$:





The Infinity problem

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A potential “scheme” for finitising:

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F a full and essentially surjective, monoidal functor, and T a target monoidal \mathbb{K} -linear category with f.d. hom spaces. Call F a **finitising functor**.



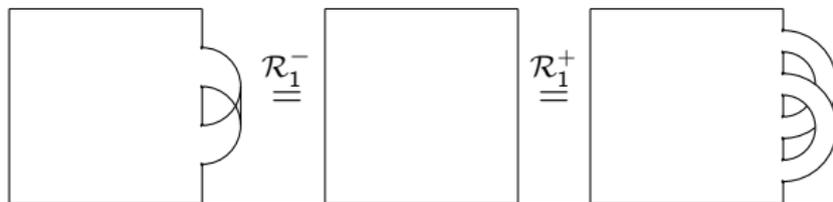
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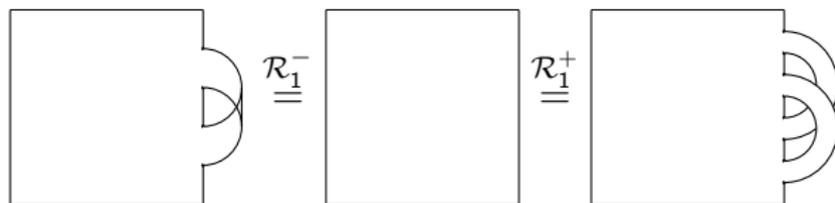
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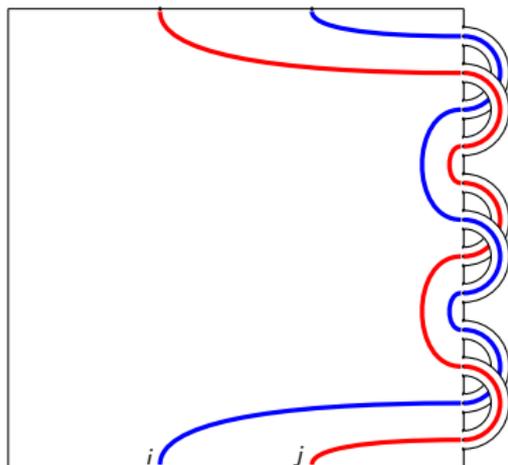
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- ▶ $\dim(\text{Hom}(2, 2)) \geq 23$ *



Example in $\mathcal{SQ}/\mathcal{R}_1$

Sample calculation in $\mathcal{SQ}/\mathcal{R}_1$:

$$\mathbb{T}_{i,j} \circ \mathbb{T}_{j,i} \circ \mathbb{T}_{i,j} =$$

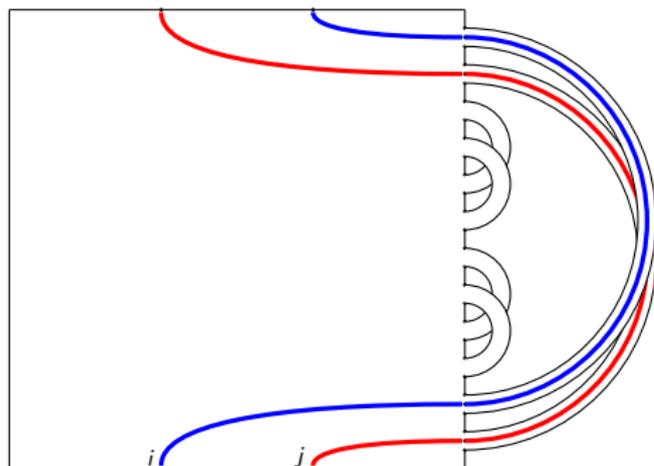




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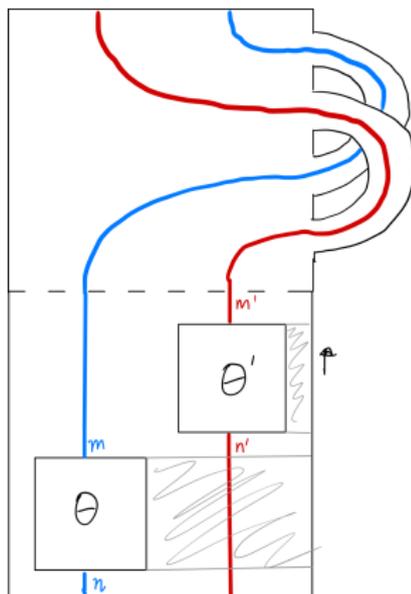
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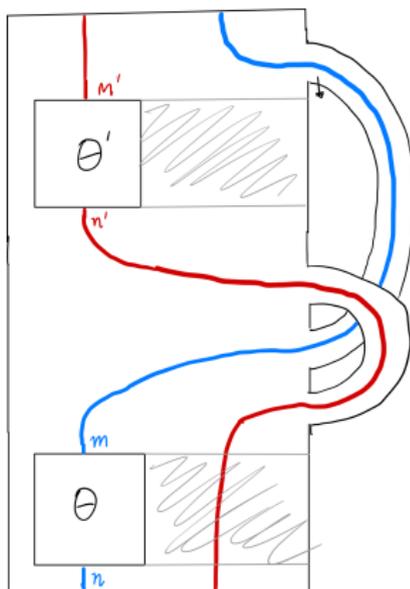




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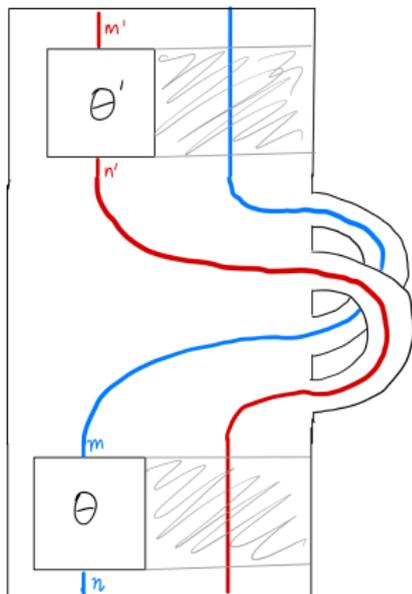




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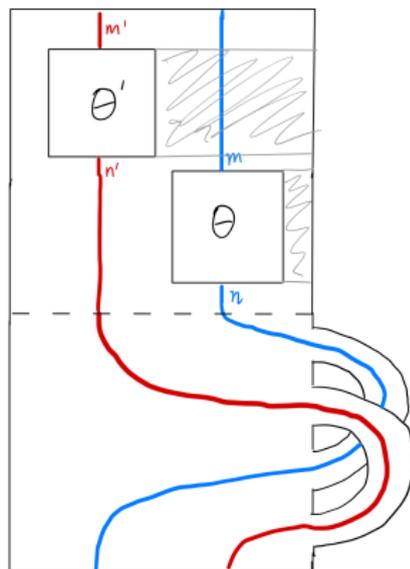




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The diagram illustrates the equality of two compositions of braiding and twisting operations. On the left, a braiding $\mathbb{T}_{m,m'}$ is composed with a tensor product of two twists $\Theta \otimes \Theta'$. On the right, a tensor product of two twists $\Theta' \otimes \Theta$ is composed with a braiding $\mathbb{T}_{n',n}$. The central diagram shows a rectangular box divided into two horizontal sections by a dashed line. The top section contains a box labeled Θ' on the left and a shaded region on the right. A red vertical line labeled m' enters the top of the Θ' box, and a blue vertical line labeled m enters the top of the shaded region. The bottom of the Θ' box is connected to the top of a box labeled Θ in the bottom section by a red line labeled n' . The bottom of the Θ box is connected to the bottom of the shaded region by a blue line labeled n . From the bottom of the box, two thick, wavy lines emerge: a red one and a blue one, which cross each other in a complex, intertwined pattern.



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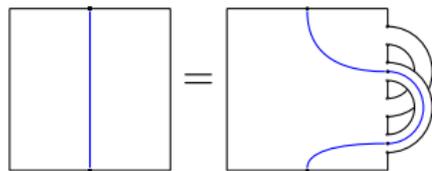
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However, \mathcal{SQ} is NOT a braided mon.cat. The smallest such quotient is obtained by imposing the relation \mathcal{R}_2 (as well as \mathcal{R}_1^+ *):



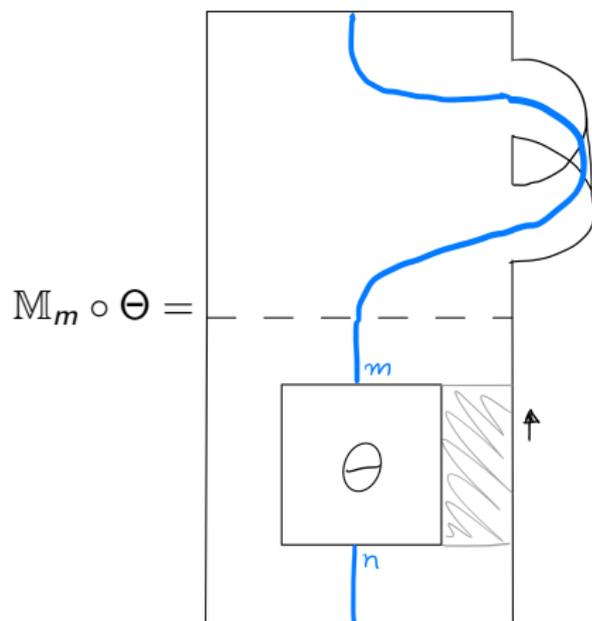
NOTE: This implies $\alpha = \gamma$. * not necessary if α invertible.



A Categorical Interpretation of the \mathbb{M}_k

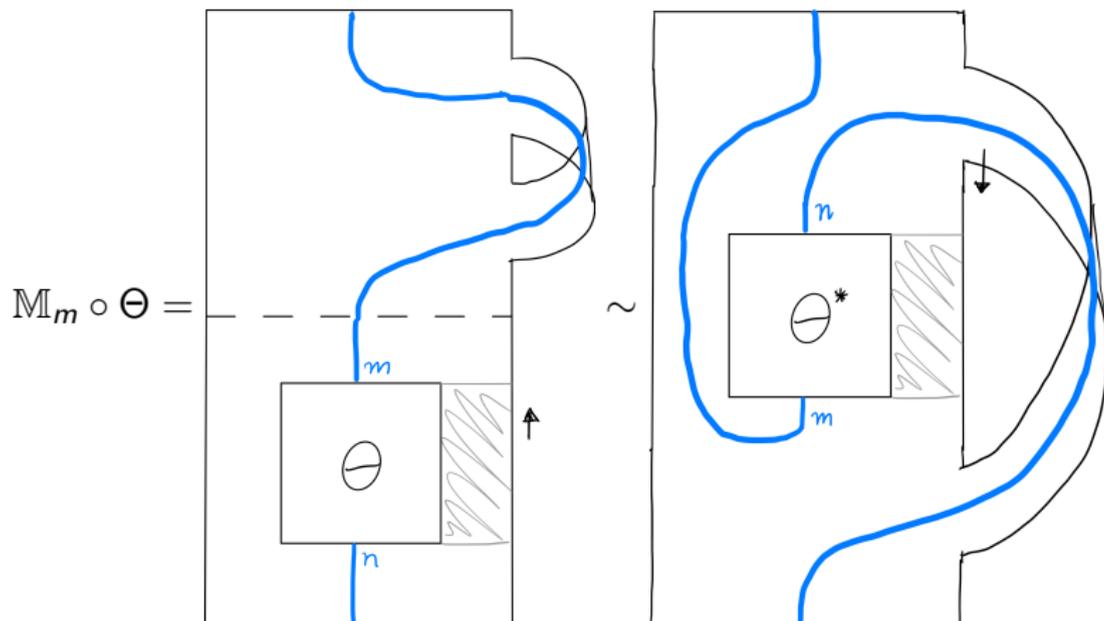


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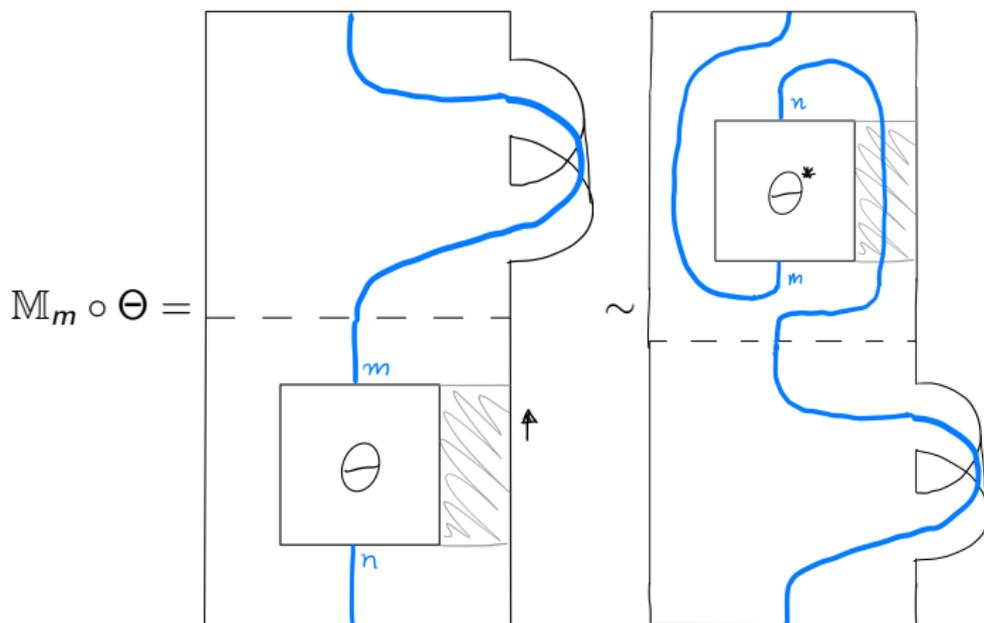


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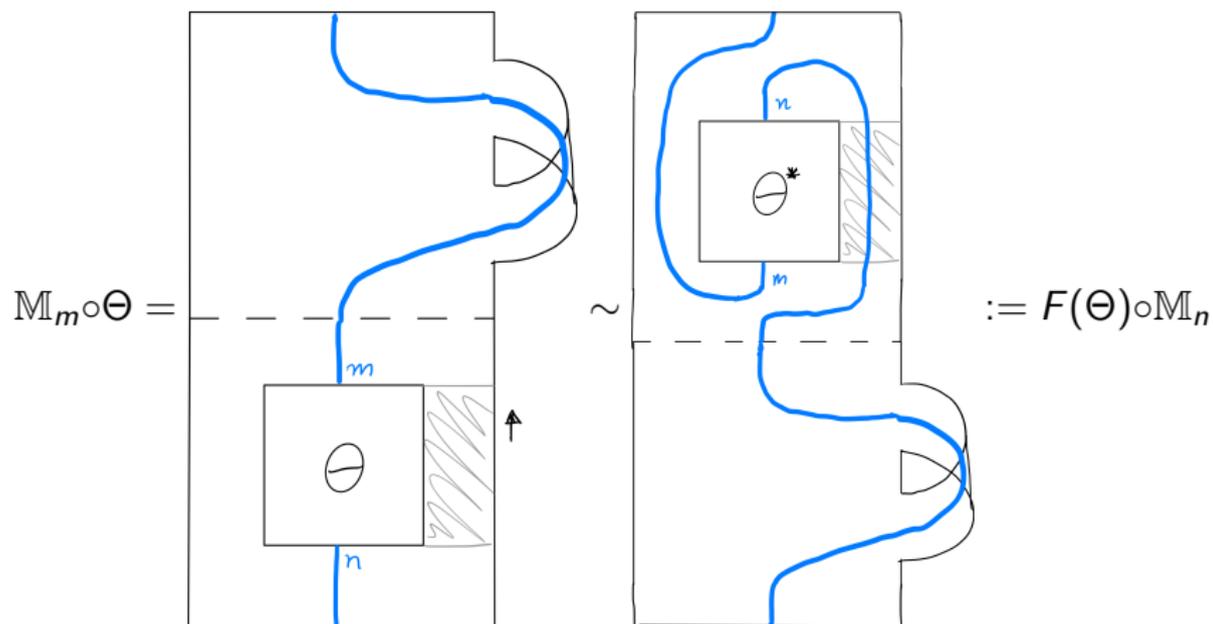


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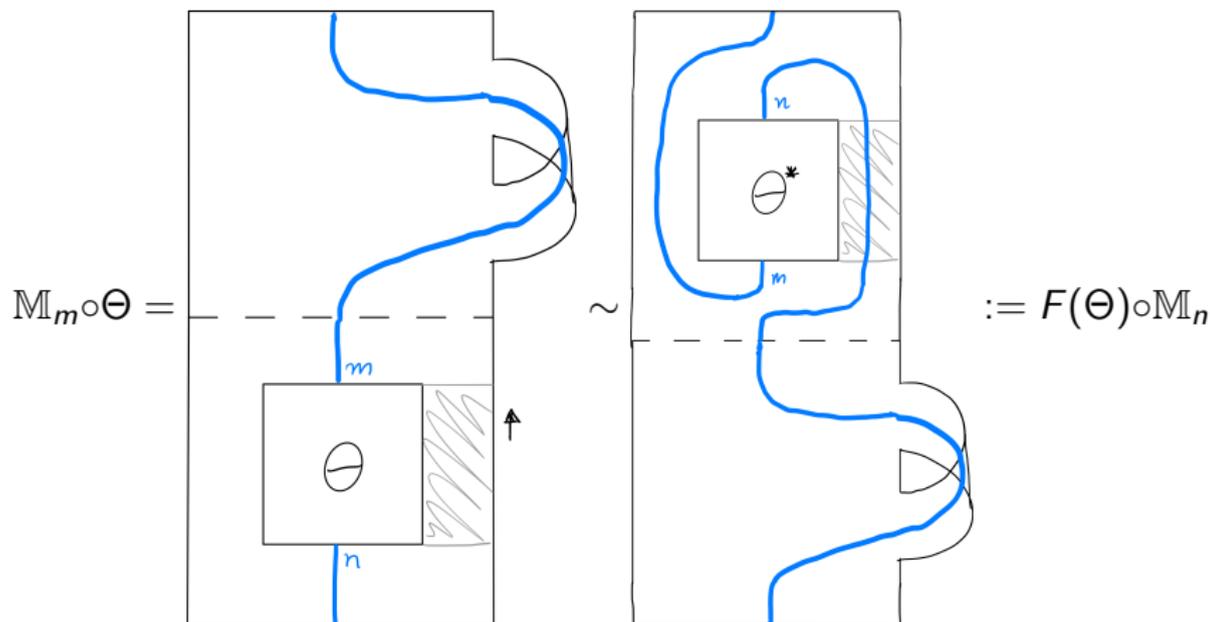


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where $F^2 = \text{id}$, $F \circ (\Theta \otimes \Theta') = F(\Theta') \otimes F(\Theta)$.



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- ▶ What does the (monoidal) representation theory look like?
- ▶ What about non-functorial quotients? e.g. terminate at a finite genus



Thank You!

Questions?