Semi simplicity criterion for the Kadar-Yu algebras

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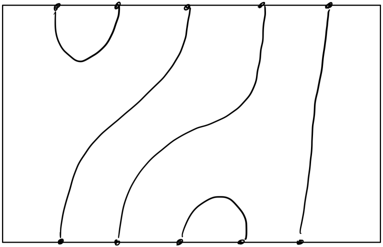
For $\alpha \in \mathbb{C}$, the (complex) Temperley-Lieb algebra on n-strands, $TL_n(\alpha)$, has basis given by "non-crossing diagrams", *e.g.* $n = 5$



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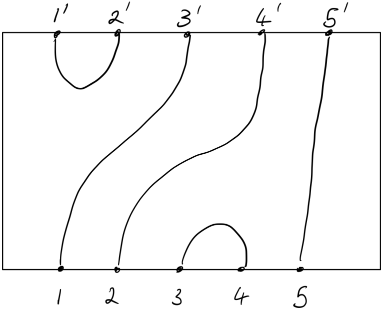

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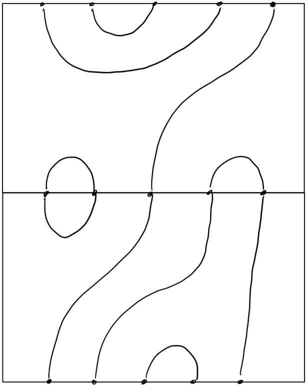

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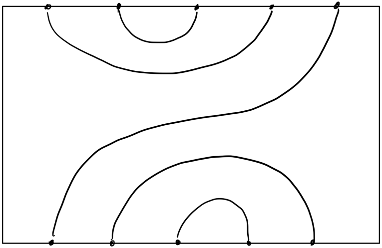

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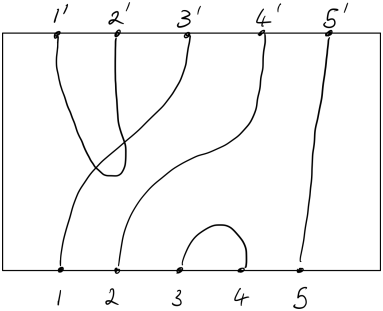

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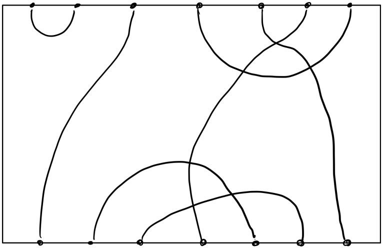

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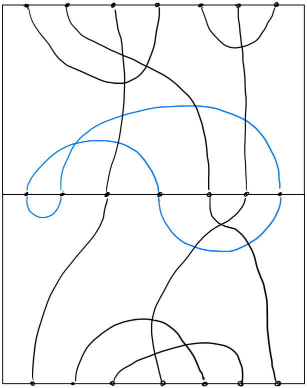

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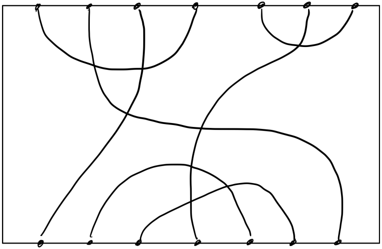

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$$\frac{\dim(TL_n)}{\dim(Br_n)} = \frac{C_n}{(2n-1)!!} = \frac{2^n}{(n+1)!} \to 0, \qquad \text{as} \qquad n \to \infty.$$

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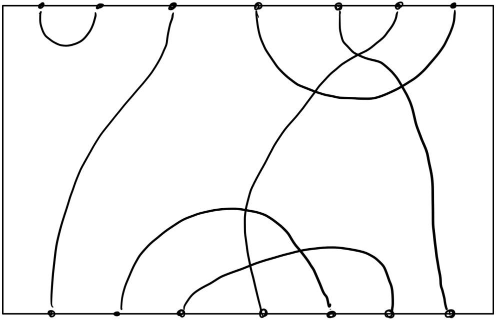

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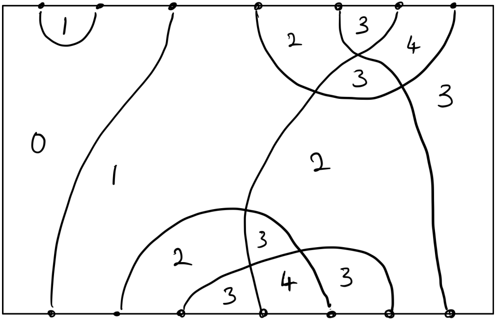

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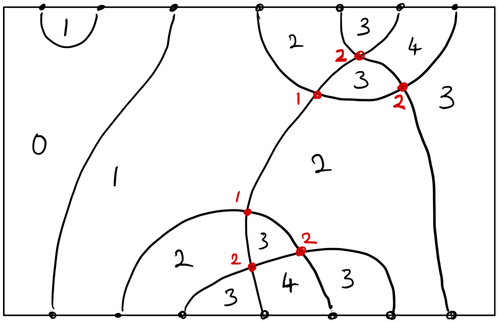

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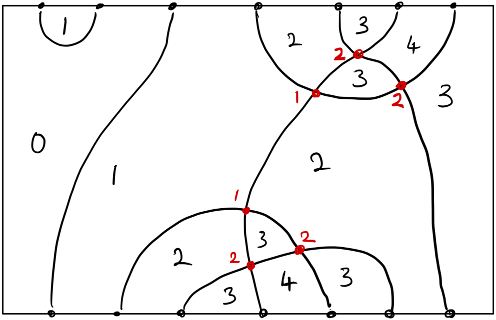

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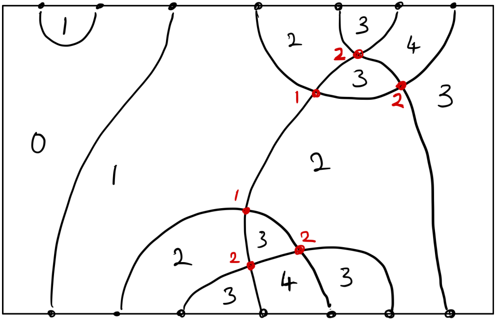

$ht(diag.) =$ max. height over all crossings (-1 if no crossings).

$ht(Br.\,ptn) =$ min. height over all diagrams.

The Kadar-Yu Algebras

Proposition [Kadar, Martin, and Yu, 2014]

Let $P_1, P_2 \in Br_n(\alpha)$ be pair partitions, and let $P_2 \# P_1$ denote their vertical juxtaposition. Then $ht(P_2 \# P_1) \leq \max(ht(P_1), ht(P_2))$.

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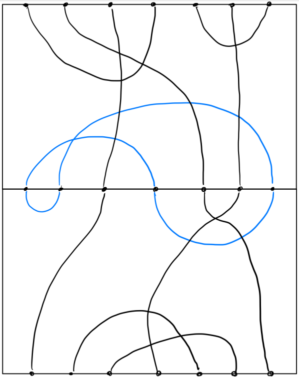

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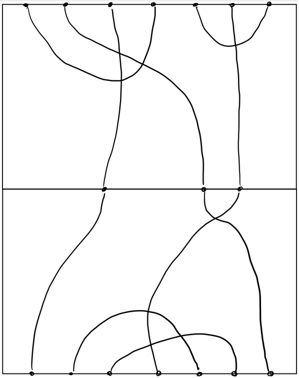

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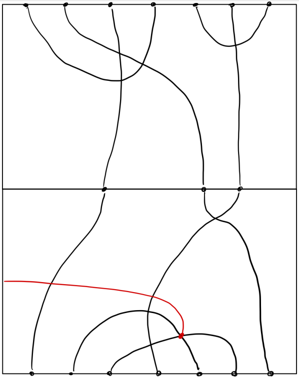

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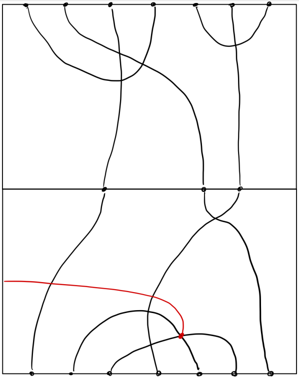

Define the Kadar-Yu (KY) algebra, $J_{l,n}(\alpha)$, to be the subalgebra of $Br_n(\alpha)$ consisting of all pair partitions of height $\leq l$ ($l = -1, 0, 1, \ldots$).

The Kadar-Yu Algebras

Observe

$$TL_n = J_{-1,n} \subset J_{0,n} \subset J_{1,n} \subset \ldots \subset J_{n-2,n} = J_{\infty,n} = Br_n$$

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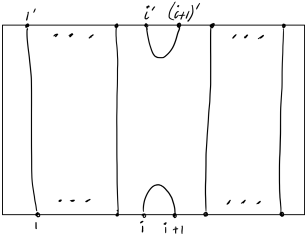
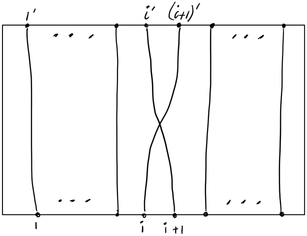
Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}$ are generated by the elements e_i for $1 \leq i \leq n-1$ and s_k for $1 \leq k \leq l+1$.

$e_i =$ 

,

$s_i =$ 

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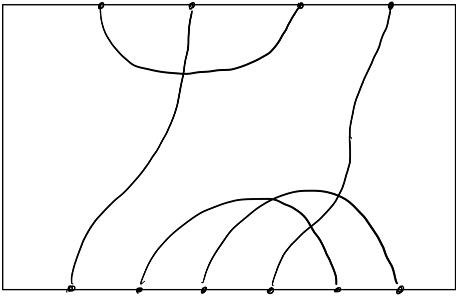
Label Set for Simples

Our aim is to determine a labelling set for the (complex) simple, $J_{l,n}$-modules, say, $\Lambda_{l,n}$. Consider

$$J_l(n,m) := \mathbb{C}\{\text{pair partitions of type (n,m), with height } \leq l\},$$

e.g.



$\in J_2(6,4)$

Label Set for Simples

Note that $J_l(n-2,n)$ is a left $J_{l,n-2}$, right $J_{l,n}$ bimodule. Thus, we may define:

$$\mathrm{mod}(J_{l,n}) \xrightarrow{F} \mathrm{mod}(J_{l,n-2})$$

$$M \xmapsto{F} J_l(n-2,n) \otimes_{J_{l,n}} M$$

Similarly, using $J_l(n,n-2)$, we can define $G : \mathrm{mod}(J_{l,n-2}) \to \mathrm{mod}(J_{l,n})$. Note that $FG = \mathrm{id}$, and $GF(M) = J_{l,n}^{(n-2)} \cdot M \subset M$. Therefore,

$$\Lambda_{l,n} = \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_{l,n}^{(n-2)}\right)$$

i.e. we can inductively build $\Lambda_{l,n}$, by considering the quotients $J_{l,n}/J_{l,n}^{(n-2)}$.

What is $J_{l,n}/J_{l,n}^{(n-2)}$?

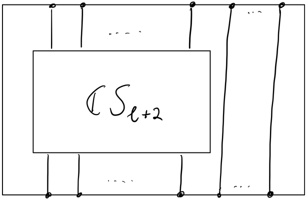
- $l = -1$ (TL): One fully propagating height -1 diagram:

$$J_{-1,n}/J_{-1,n}^{(n-2)} \simeq \mathbb{C} \quad \Rightarrow \quad \Lambda_{-1,n} = \{n, n-2, n-4, \ldots, 0/1\},$$

- $l = \infty$ (Brauer): Can realise any permutation:

$$J_{\infty,n}/J_{\infty,n}^{(n-2)} \simeq \mathbb{C}\, S_n \quad \Rightarrow \quad \{\lambda \vdash p \mid p \in \Lambda_{-1,n}\},$$

- $l \in \mathbb{N}$ (KMY): Any word in the $s_1, \ldots, s_{l+1}$:



,

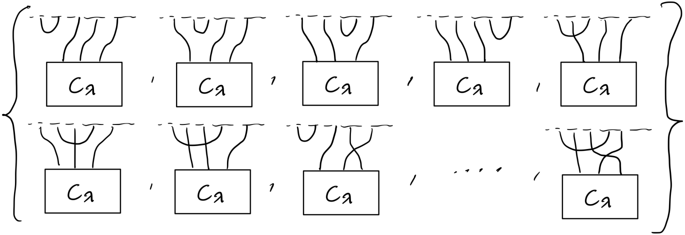
$$J_{\infty,n}/J_{\infty,n}^{(n-2)} \simeq \mathbb{C}\, S_{\min(n,l+2)}$$

$$\Rightarrow \Delta_{l,n} \left\{ (p, \lambda) \mid \begin{array}{c} p \in \Lambda_{-1,n}, \\ \lambda \vdash \min(p, l+2) \end{array} \right\}$$

Standard Modules

For each $(p, \lambda) \in \Lambda_{l,n}$, there is a **standard module** (for $J_{l,n}$) $\Delta_{(p,\lambda)}$. It has a basis given by pairs; a half-diagram with $k = (n-p)/2$ cups, with height $\leq l$ paired with a basis element for the Specht-module $\mathcal{S}^{\lambda}$:

e.g. $n = 5$, $l = 1$, $p = 3$, $\lambda = (2, 1)$:



Young-symmetriser $c_\lambda \in \mathbb{C}S_n$: $c_\lambda^* = c_\lambda$, $c_\lambda^2 = c_\lambda$, $\mathbb{C}S_n c_\lambda = \mathcal{S}^{\lambda}$, $c_\lambda \mathbb{C}S_n c_\lambda \simeq \mathbb{C}c_\lambda$.

Standard Modules - Bratelli/Rollet Diagrams

Induction/restriction rules $\text{mod}(J_{l,n}) \underset{Ind}{\overset{Res}{\rightleftarrows}} \text{mod}(J_{l,n-1})$ for std modules, encoded in Bratelli/Rollet Diagrams: *e.g.*

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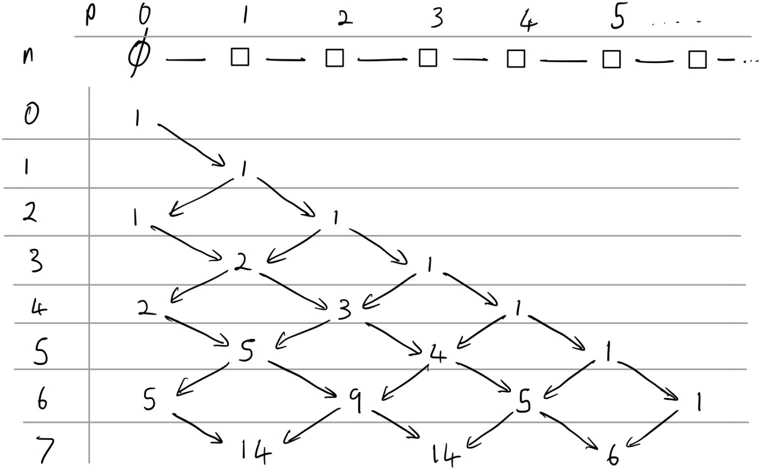

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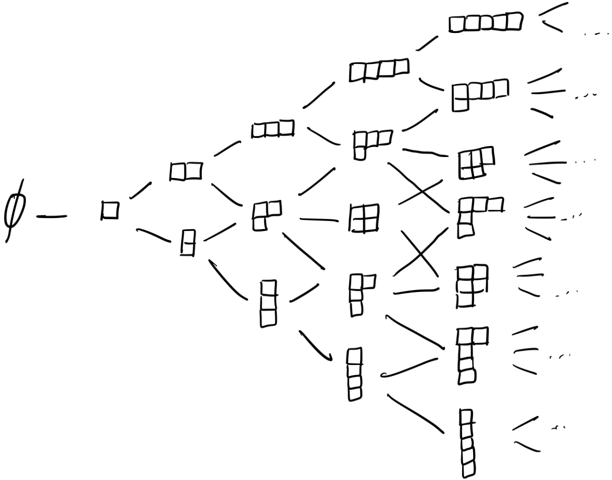

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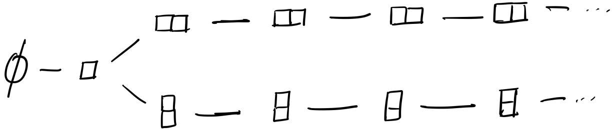
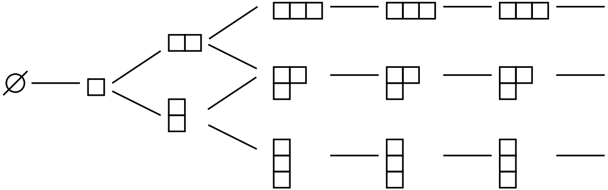
$l = 0$:



$l = 1$:



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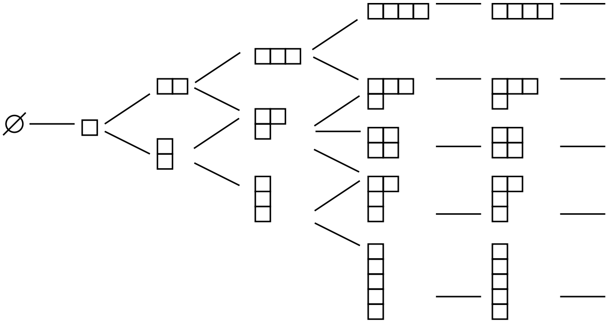
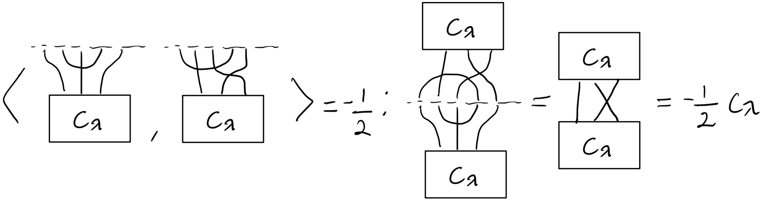
$l = 2$:



Standard Modules

Can determine semi-simplicity criteria by studying certain bilinear forms on the $\Delta_{(p,\lambda)}$:



Can find the simple heads $L_{(p,\lambda)} = \Delta_{(p,\lambda)}/\mathrm{rad}(\langle \_, \_ \rangle)$. Therefore, when $\langle \_, \_ \rangle$ is non-degenerate, simples and standards coincide: Semi-simplicity!

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1 & \alpha & 1 & 1 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & 1 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha & 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{\alpha}{2} \\
-\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \alpha & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 & \alpha & 1 & 1 & 0 & -\frac{1}{2} & 0 \\
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-\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\
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0 & 0 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{\alpha}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha
\end{pmatrix}$$

Standard Modules - Gram Matrices

Can study these forms by their matrices: *e.g.* $n = 5$, $l = 1$, $p = 3$, $\lambda = (2, 1)$:

$$
\begin{pmatrix}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 0 & 1 & 1 & \alpha & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 \\
0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \alpha
\end{pmatrix}
$$

$$\mathrm{Det}(\alpha) = (\alpha - 2)^3 \alpha (\alpha + 2)(\alpha + 4) \left(\alpha^4 - 7\alpha^2 + 3\right)^2.$$

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1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 1 & \alpha & 0 & 1 & -1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 0 & 1 & 1 & \alpha & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 \\
0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \alpha
\end{pmatrix}
$$

$\mathrm{Det}(\alpha) = (\alpha - 2)^3 \alpha (\alpha + 2)(\alpha + 4)\left(\alpha^4 - 7\alpha^2 + 3\right)^2$. Passing to an o/n basis for $\mathcal{S}^\lambda$, we see this has real roots [Alraddadi and Parker, 2024].

Standard Modules - Gram Matrices

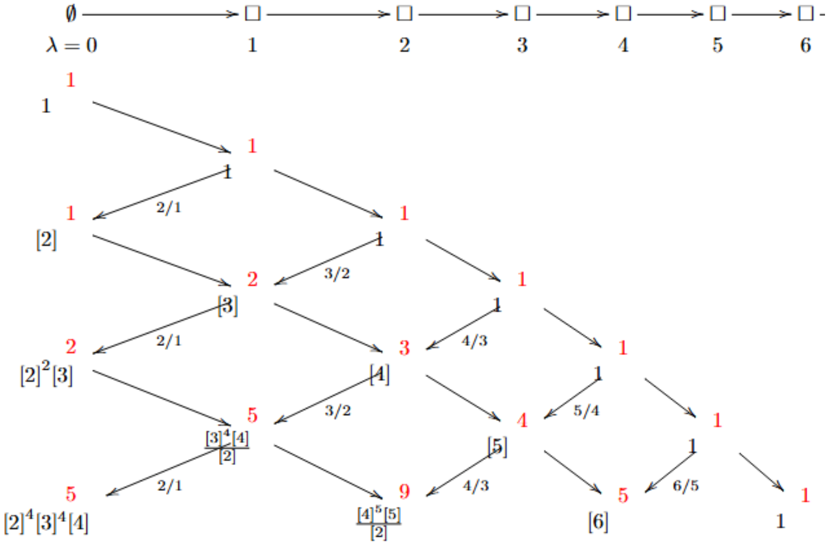
Systematic approach to computing determinants? An "easy" case:
$n \mapsto n+2$, $l = -1$, $p = n$

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & \dots & 0 \\ 1 & \alpha & 1 & 0 & \dots & 0 \\ 0 & 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow |\Delta_n^{(n+2)}| = U_n(\alpha/2)$$

Parametrise as $\alpha = [2]_q = q + q^{-1} \Rightarrow |\Delta_n^{(n+2)}| = [n+1]_q$. Vanishes when $q^{2(n+1)} = 1$ (or at $\alpha = 2\cos(k\pi/(n+1))$ for $k = 1, \dots, n$).

Standard Modules - Gram Matrices

What about modules with $k > 1$ cups?



Key observation:

$$\frac{\left|\Delta^{(n)}_{(p)}\right|}{\left|\Delta^{(n-1)}_{(p-1)}\right|\left|\Delta^{(n-1)}_{(p+1)}\right|} = \left(\frac{[p+2]_q}{[p+1]_q}\right)^d, \qquad d = \dim\left(\Delta^{(n-1)}_{(p+1)}\right)$$

For generic $l \in \mathbb{N}$, consider the ratio:

$$\mathcal{V}^{(n)}_{(p,\lambda)} := \frac{\left|\Delta^{(n)}_{(p,\lambda)}\right|}{\prod_{v\in\mathrm{Res}(p,\lambda)}\left|\Delta^{(n-1)}_{v}\right|}$$

For $l = 0$, the Rollet graph labelled with $\mathcal{V}^{(p+4)}_{(p,\lambda)}$:

$$\alpha^3 - \frac{(\alpha-1)^8(\alpha+2)^4}{\alpha^4}$$

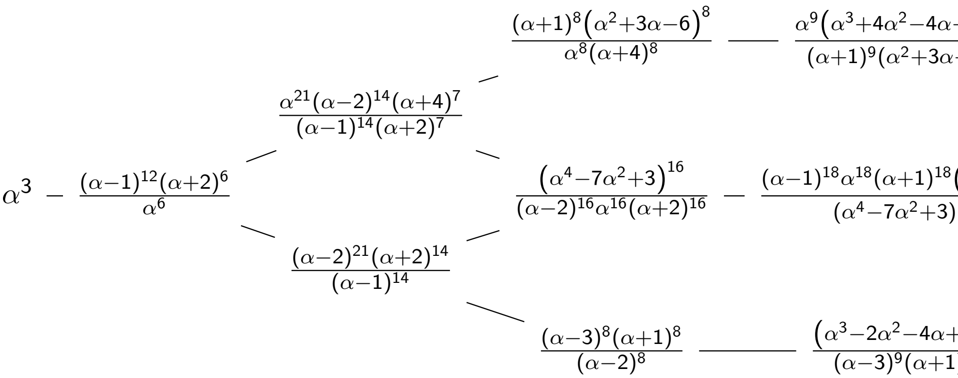
Upper branch:

$$\frac{\alpha^5\left(\alpha^2+\alpha-4\right)^5}{(\alpha-1)^5(\alpha+2)^5} - \frac{\left(\alpha^4+\alpha^3-5\alpha^2-\alpha+2\right)^6}{\alpha^6(\alpha^2+\alpha-4)^6} - \frac{(\alpha-1)^7\alpha^7\left(\alpha^3+2\alpha^2-4\alpha-\right.}{(\alpha^4+\alpha^3-5\alpha^2-\alpha+2)^7}$$

Lower branch:

$$\frac{(\alpha-2)^5(\alpha+1)^5}{(\alpha-1)^5} - \frac{\left(\alpha^3-\alpha^2-3\alpha+1\right)^6}{(\alpha-2)^6(\alpha+1)^6} - \frac{(\alpha-1)^7\left(\alpha^3-4\alpha-2\right)^7}{(\alpha^3-\alpha^2-3\alpha+1)^7}$$

For $l = 1$, the Rollet graph labelled with $\mathcal{V}^{(p+4)}_{(p,\lambda)}$:



Standard Modules - Gram Matrices

What is going on? Conjecture:

$$\mathcal{V}^{(n)}_{(p,\lambda)} = \frac{\left|\Delta^{(n)}_{(p,\lambda)}\right|}{\prod_{v\in\text{Res}(p,\lambda)}\left|\Delta^{(n-1)}_{v}\right|} = \prod_{(p+1,\mu)\in\text{Res}^+(p,\lambda)} \left(\frac{F^{(p+1)}_{\mu}(\alpha)}{F^{(p)}_{\mu}(\alpha)}\right)^{d_{(p+1,\mu)}},$$

where for each partition μ we have introduced a series of (monic) polynomials, $F^{(p)}_{\mu}(\alpha)$, for $p \overset{?}{\geq} |\mu|-1$:

$$\begin{aligned}\deg(F^{(p+1)}_{\mu}(\alpha)) &= \deg(F^{(p)}_{\mu}(\alpha)) + 1\\ F^{(p+1)}_{\mu}(\alpha) &= \alpha F^{(p)}_{\mu}(\alpha) - F^{(p-1)}_{\mu}(\alpha)\end{aligned}$$

Standard Modules - Gram Matrices

Examples of the $F_\mu^{(n)}$

- $\mu = (1) \vdash 1$, we have

$$F_\mu^{(0)}(\alpha) = 1, \quad F_\mu^{(1)}(\alpha) = \alpha, \ldots, F_\mu^{(p)}(\alpha) = U_p(\alpha/2)$$

- $\mu = (2) \vdash 2$, we have

$$(\alpha+2)(\alpha+1), \quad \alpha(\alpha^2+\alpha-4), \quad \alpha^4+\alpha^3-5\alpha^2+2$$
$$\alpha(\alpha-1)(\alpha^3+2\alpha^2-4\alpha-6), \quad \alpha^6+\alpha^5-7\alpha^4-3\alpha^3+11\alpha^2-2$$

- $\mu = (21) \vdash 3$, we have

$$(\alpha-2)\alpha(\alpha+2), \quad \alpha^4-7\alpha^2+3, \quad (\alpha-1)\alpha(\alpha+1)\left(\alpha^2-7\right)$$
$$\alpha^6-9\alpha^4+14\alpha^2-3, \quad \alpha\left(\alpha^6-10\alpha^4+22\alpha^2-10\right)$$

Standard Modules - Gram Matrices

Examples of the $F_\mu^{(n)}$

- $\mu = (1) \vdash 1$, we have

$$F_\mu^{(0)}(\alpha) = 1, \quad F_\mu^{(1)}(\alpha) = \alpha, \ldots, F_\mu^{(p)}(\alpha) = U_p(\alpha/2)$$

- $\mu = (2) \vdash 2$, we have

$$(\alpha+2)(\alpha+1), \quad \alpha(\alpha^2+\alpha-4), \quad \alpha^4+\alpha^3-5\alpha^2+2$$
$$\alpha(\alpha-1)(\alpha^3+2\alpha^2-4\alpha-6), \quad \alpha^6+\alpha^5-7\alpha^4-3\alpha^3+11\alpha^2-2$$

- $\mu = (21) \vdash 3$, we have

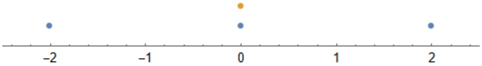
$$(\alpha-2)\alpha(\alpha+2), \quad \alpha^4-7\alpha^2+3, \quad (\alpha-1)\alpha(\alpha+1)\left(\alpha^2-7\right)$$
$$\alpha^6-9\alpha^4+14\alpha^2-3, \quad \alpha\left(\alpha^6-10\alpha^4+22\alpha^2-10\right)$$

Standard Modules - Gram Matrices

Roots of the $F_\mu^{(p)}$: $\lambda = (2,1)$

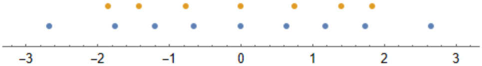
```
In[66]:= NumberLinePlot[{α /. NSolve[F21[4, α] == 0, α], α /. NSolve[ChebyshevU[1, α / 2] == 0, α]}]
```

Out[66]=



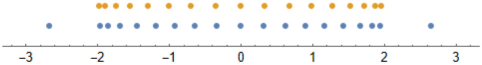

```
In[67]:= NumberLinePlot[{α /. NSolve[F21[10, α] == 0, α], α /. NSolve[ChebyshevU[7, α / 2] == 0, α]}]
```

Out[67]=



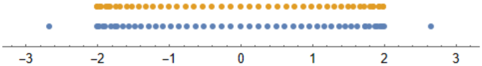

```
In[68]:= NumberLinePlot[{α /. NSolve[F21[20, α] == 0, α], α /. NSolve[ChebyshevU[17, α / 2] == 0, α]}]
```

Out[68]=




```
In[69]:= NumberLinePlot[{α /. NSolve[F21[50, α] == 0, α], α /. NSolve[ChebyshevU[47, α / 2] == 0, α]}]
```

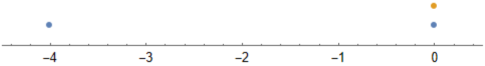
Out[69]=



Roots of the $F_\mu^{(p)}$: $\lambda = (3)$

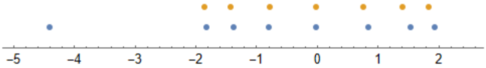
```
In[70]:= NumberLinePlot[{α /. NSolve[F3[4, α] == 0, α], α /. NSolve[ChebyshevU[1, α / 2] == 0, α]}]
```

Out[70]=



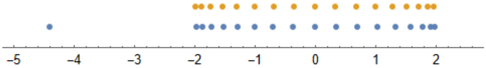

```
In[71]:= NumberLinePlot[{α /. NSolve[F3[10, α] == 0, α], α /. NSolve[ChebyshevU[7, α / 2] == 0, α]}]
```

Out[71]=



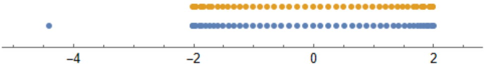

```
In[72]:= NumberLinePlot[{α /. NSolve[F3[20, α] == 0, α], α /. NSolve[ChebyshevU[17, α / 2] == 0, α]}]
```

Out[72]=




```
In[73]:= NumberLinePlot[{α /. NSolve[F3[50, α] == 0, α], α /. NSolve[ChebyshevU[47, α / 2] == 0, α]}]
```

Out[73]=



QUESTION: Suppose we have two monic polynomials f_0, f_1 of consecutive degree. Then consider the a family f_n generated from them by Chebyshev recursion. Do we expect this behaviour????

Table of Contents

THANK YOU!

Questions?

References

Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv: 2407.07028 [math.RT]. URL: https://arxiv.org/abs/2407.07028.

Kadar, Zoltan, Paul P. Martin, and Shona Yu (2014). *On geometrically defined extensions of the Temperley-Lieb category in the Brauer category*. DOI: 10.48550/ARXIV.1401.1774. URL: https://arxiv.org/abs/1401.1774.