Semi simplicity criteria for the Kadar-Yu algebras

(based on work in progress arxiv2512.XXXXX)

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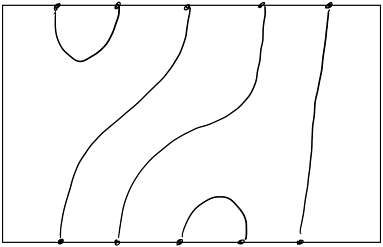

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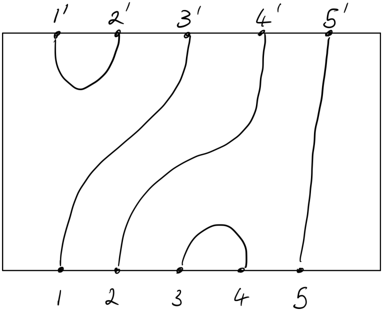

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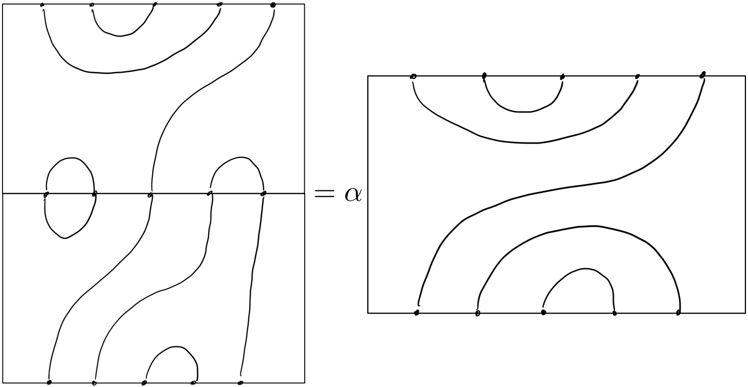

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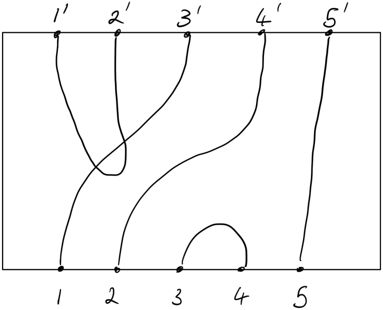

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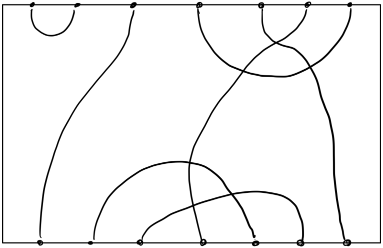

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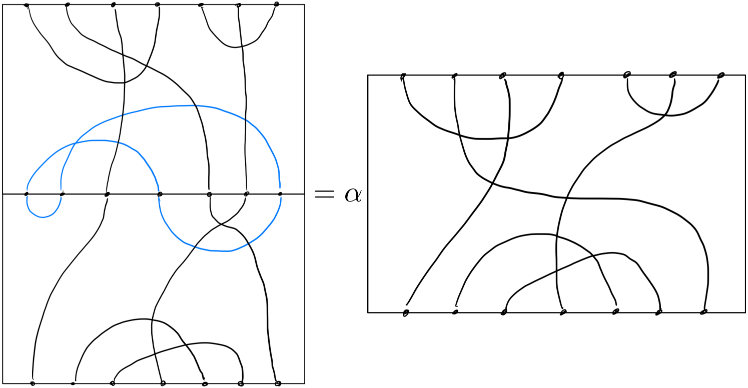

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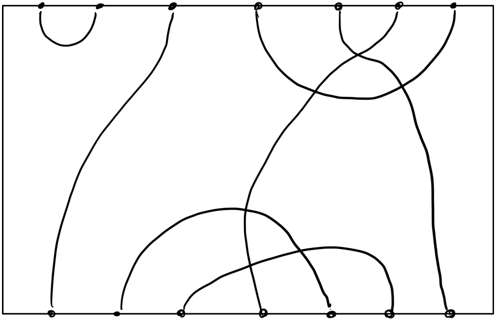

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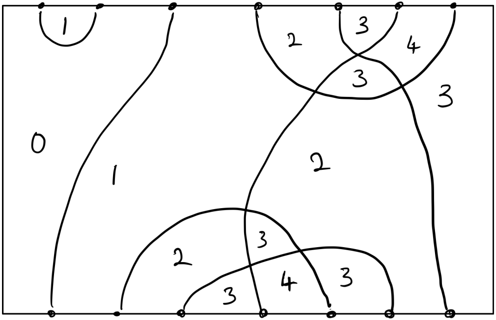

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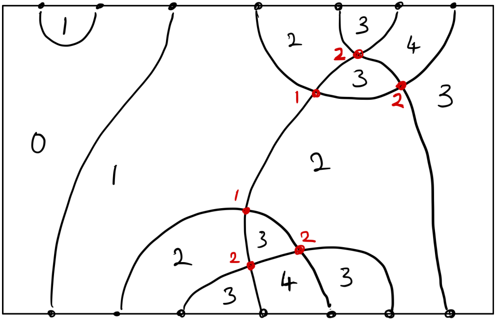

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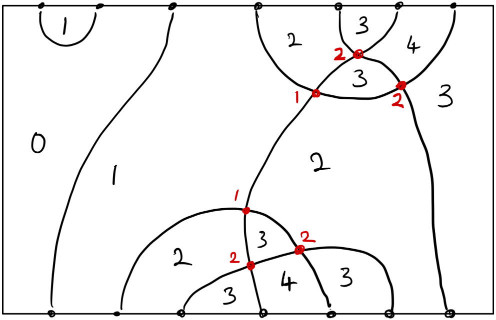

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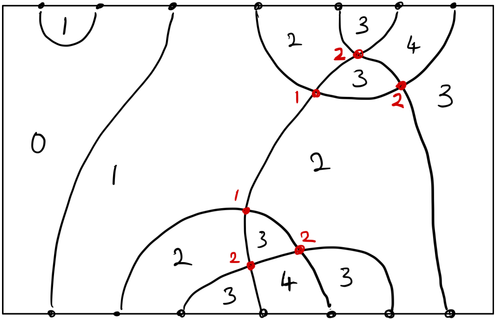

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$ht(Br.\,ptn)$ = min. height over all diagrams.

The Kadar-Yu Algebras

Proposition [Kadar, Martin, and Yu, 2014]

Let $P_1, P_2 \in Br_n(\alpha)$ be pair partitions, and let $P_2 \# P_1$ denote their vertical juxtaposition. Then $ht(P_2 \# P_1) \leq \max\left(ht(P_1), ht(P_2)\right)$.

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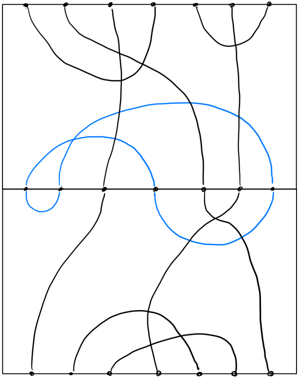

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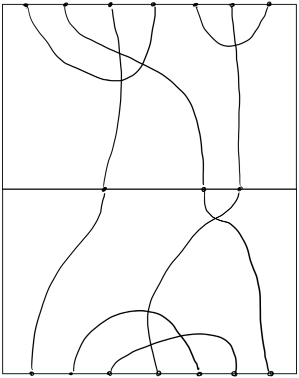

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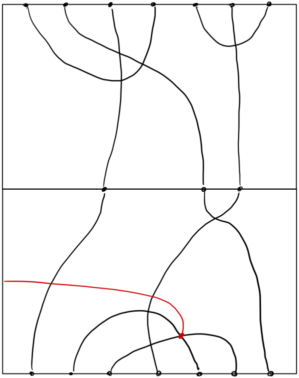

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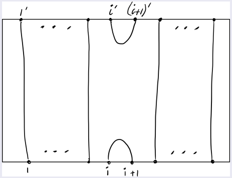
Proposition [Alraddadi and Parker, 2024]

The KY-algebra $J_{l,n}$ are generated by the elements e_i for $1 \leq i \leq n-1$ and s_k for $1 \leq k \leq l+1$.

$e_i =$



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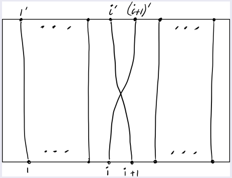

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The Kadar-Yu Algebras

Some dimensions:

	$n = 0$	1	2	3	4	5	6
$l = -1$	1	1	2	5	14	42	132
0	1	1	3	11	43	173	707
1	1	1	3	15	87	525	2947
2	1	1	3	15	105	849	5007
3	1	1	3	15	105	945	9795
∞	1	1	3	15	105	945	10395

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Labelling Set for Simples

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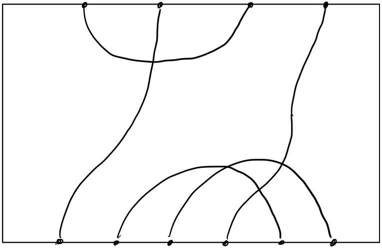
$$J_l(n,m) := \mathbb{C}\{\text{pair partitions of type } (n,m), \text{ with height } \leq l\}, \quad \textit{e.g.}$$



$\in J_2(6,4)$

In fact we can regard $J_l(n,m)$, as the hom-spaces in a $\mathbb{C}$-linear category J_l which has the algebras $J_{l,n}$ as its endomorphism algebras: $J_{l,n} = J_l(n,n)$.

Labelling Set for Simples

Globalisation/Localisation:

Note that $J_l(n, n-2)$ is a left $J_{l,n-2}$, right $J_{l,n}$ bimodule. Thus, we may define:

$$\mathrm{mod}(J_{l,n}) \xrightarrow{F} \mathrm{mod}(J_{l,n-2})$$

$$M \xmapsto{F} J_l(n-2, n) \otimes_{J_{l,n}} M$$

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Similarly, using $J_l(n-2, n)$, we can define $G : \mathrm{mod}(J_{l,n-2}) \to \mathrm{mod}(J_{l,n})$. Note that $FG = \mathrm{id}$, and $GF(M) = J_{l,n}^{(n-2)}M \subset M$. Therefore,

$$\Lambda_{l,n} = \Lambda_{l,n-2} \sqcup \Lambda\left(J_{l,n}/J_{l,n}^{(n-2)}\right)$$

i.e. we can inductively build $\Lambda_{l,n}$, by considering the quotients $J_{l,n}/J_{l,n}^{(n-2)}$.

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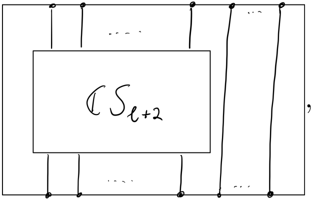
- $l \in \mathbb{N}$ (KMY): Any word in the $s_1, \dots, s_{l+1}$:



$$J_{l,n}/J^{(n-2)}_{l,n} \simeq \mathbb{C}\, S_{\min(n,l+2)}$$

$$\Rightarrow \Delta_{l,n} = \left\{ (p, \lambda) \mid \begin{array}{c} p \in \Lambda_{-1,n}, \\ \lambda \vdash \min(p, l+2) \end{array} \right\}$$

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These questions can all be addressed by studying a nice class of modules: standard/cell modules arising from the quasi-heredity chain:

$$J_{l,n}^{(0/1)} \subset \ldots \subset J_{l,n}^{(n-4)} \subset J_{l,n}^{(n-2)} \subset J_{l,n}$$

We can build these by successively globalising Specht-modules.

Standard Modules

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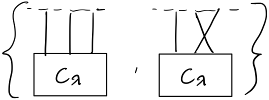

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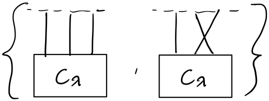
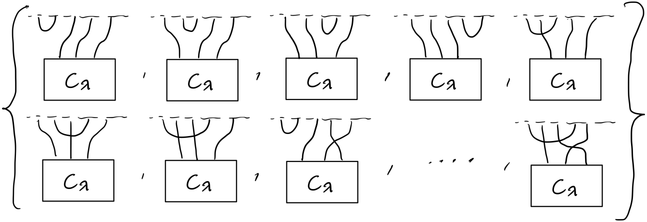


To get $\Delta^{(1,5)}_{(3,\lambda)}$ (*i.e.* a one cup module) we globalise:



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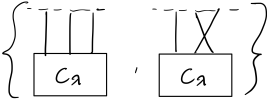

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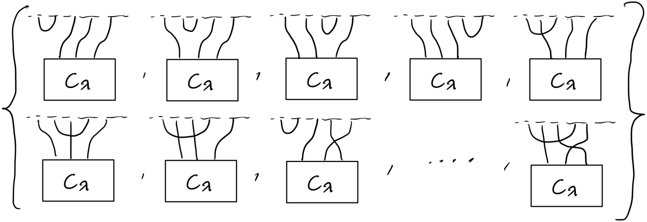

(Young-symmetriser $c_\lambda \in \mathbb{C}S_n$: $c_\lambda^* = c_\lambda$, $c_\lambda^2 = c_\lambda$, $\mathbb{C}S_n c_\lambda = \mathcal{S}^\lambda$, $c_\lambda \mathbb{C}S_n c_\lambda \simeq \mathbb{C}c_\lambda$. Really using $\mathbb{C}$!!.)

Standard Modules - Bratelli/Rollet Diagrams

Induction/restriction rules $\mathrm{mod}(J_{l,n}) \underset{Ind}{\overset{Res}{\rightleftarrows}} \mathrm{mod}(J_{l,n-1})$ for std modules, encoded in Bratelli/Rollet Diagrams: *e.g.*

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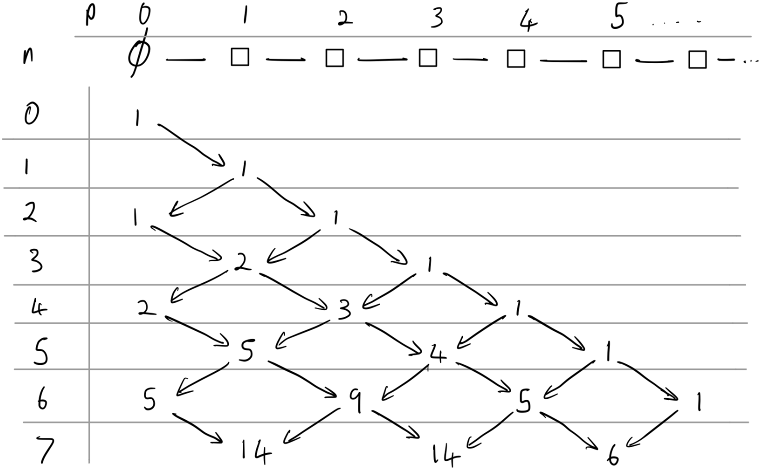

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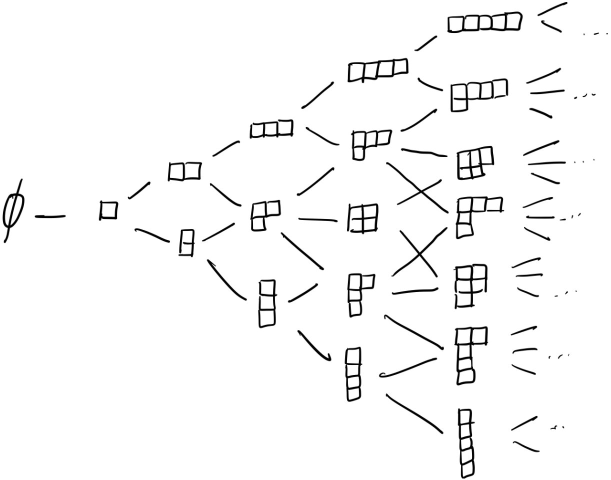

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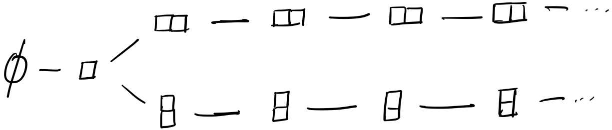

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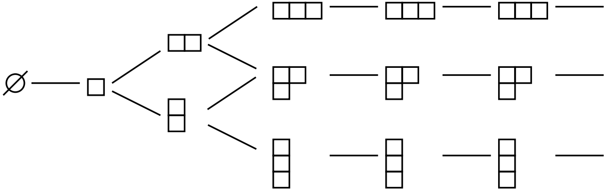

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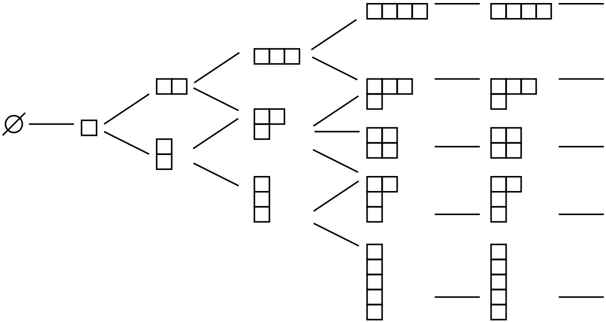

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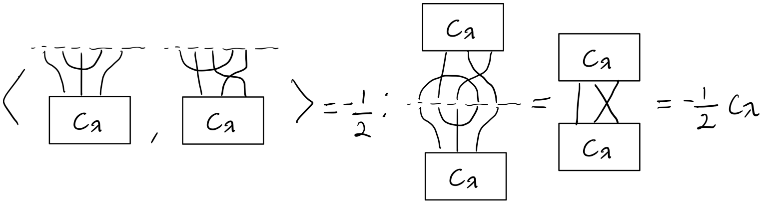

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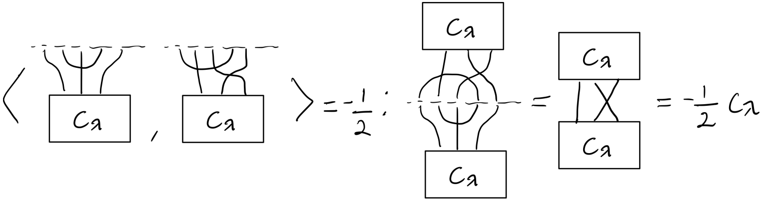

Can find the simple heads $L_{(p,\lambda)} = \Delta_{(p,\lambda)}/\mathrm{rad}(\langle \_, \_ \rangle)$. Therefore, when $\langle \_, \_ \rangle$ is non-degenerate, simples and standards coincide... Semi-simplicity!

Standard Modules - Gram Matrices

Can study these forms by their matrices: *e.g.* $n = 5$, $l = 1$, $p = 3$, $\lambda = (2, 1)$:

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$$\begin{pmatrix}
\alpha & 1 & 1 & 1 & 1 & 0 & 0 & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & 1 \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha & 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{\alpha}{2} \\
-\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \alpha & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 & \alpha & 1 & 1 & 0 & -\frac{1}{2} & 0 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 1 & \alpha & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\alpha}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\
-\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} \\
0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\alpha}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 1 & \alpha & 1 \\
0 & 0 & -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{\alpha}{2} & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha
\end{pmatrix}$$

Standard Modules - Gram Matrices

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\alpha & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 1 & \alpha & 0 & 1 & -1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 0 & 1 & 1 & \alpha & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 \\
0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \alpha
\end{pmatrix}$$

$$\mathrm{Det}(\alpha) = (\alpha - 2)^3 \alpha (\alpha + 2)(\alpha + 4) \left(\alpha^4 - 7\alpha^2 + 3\right)^2.$$

Standard Modules - Gram Matrices

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1 & \alpha & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \alpha & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
1 & 1 & 0 & \alpha & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & \alpha & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\
0 & 1 & 1 & 1 & 1 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & \alpha & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 1 & \alpha & 0 & 1 & -1 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \alpha & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -1 & 0 & 1 & 1 & \alpha & 1 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & \alpha & 1 \\
0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & \alpha
\end{pmatrix}$$

$\mathrm{Det}(\alpha) = (\alpha - 2)^3\alpha(\alpha + 2)(\alpha + 4)\left(\alpha^4 - 7\alpha^2 + 3\right)^2$. Passing to an o/n basis for $\mathcal{S}^\lambda$, we see this has real roots [Alraddadi and Parker, 2024].

Standard Modules - Gram Matrices

Systematic approach to computing determinants?

Standard Modules - Gram Matrices

Systematic approach to computing determinants? An "easy" case: $n \mapsto n+2$, $l = -1$, $p = n$

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & \dots & 0 \\ 1 & \alpha & 1 & 0 & \dots & 0 \\ 0 & 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow |\Delta_n^{(n+2)}| = U_n(\alpha/2)$$

Standard Modules - Gram Matrices

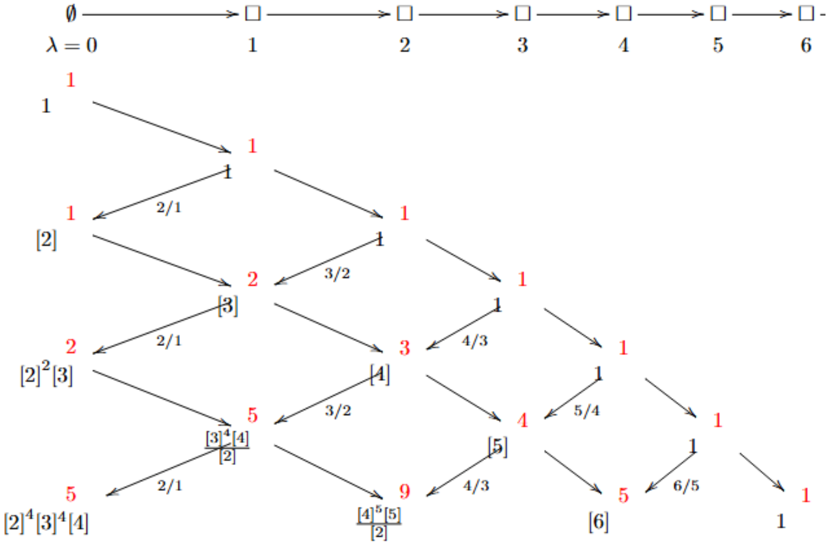
Systematic approach to computing determinants? An “easy” case: $n \mapsto n+2$, $l=-1$, $p=n$

$$\begin{pmatrix} \alpha & 1 & 0 & 0 & \dots & 0 \\ 1 & \alpha & 1 & 0 & \dots & 0 \\ 0 & 1 & \alpha & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \alpha & 1 \\ 0 & 0 & \dots & 0 & 1 & \alpha \end{pmatrix} \Rightarrow |\Delta_n^{(n+2)}| = U_n(\alpha/2)$$

Parametrise as $\alpha = [2]_q = q + q^{-1} \Rightarrow |\Delta_n^{(n+2)}| = [n+1]_q$. Vanishes when $q^{2(n+1)} = 1$ (or at $\alpha = 2\cos(k\pi/(n+1))$ for $k = 1, \dots, n$).

Standard Modules - Gram Matrices

What about modules with $k > 1$ cups?



Standard Modules - Gram Matrices

Key observation:

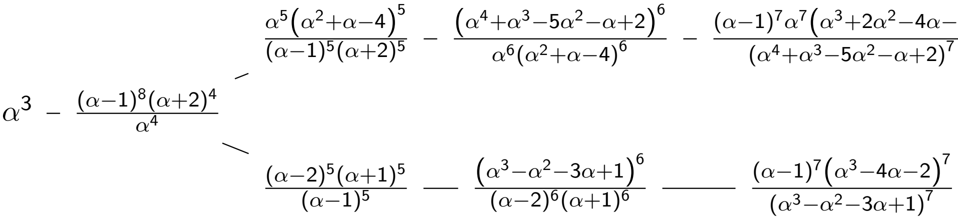
$$\frac{\left|\Delta^{(n)}_{(p)}\right|}{\left|\Delta^{(n-1)}_{(p-1)}\right|\left|\Delta^{(n-1)}_{(p+1)}\right|} = \left(\frac{[p+2]_q}{[p+1]_q}\right)^d, \qquad d = \dim\left(\Delta^{(n-1)}_{(p+1)}\right)$$

For generic $l \in \mathbb{N}$, consider the ratio:

$$\mathcal{V}^{(n)}_{(p,\lambda)} := \frac{\left|\Delta^{(n)}_{(p,\lambda)}\right|}{\prod_{v\in \mathrm{Res}(p,\lambda)}\left|\Delta^{(n-1)}_{v}\right|}$$

Standard Modules - Gram Matrices

For $l = 0$, the Rollet graph labelled with $\mathcal{V}_{(p,\lambda)}^{(p+4)}$:



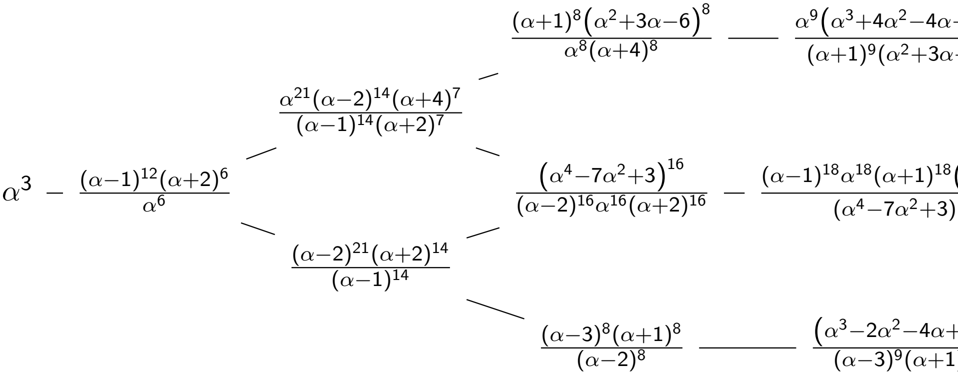
$\alpha^3 - \frac{(\alpha-1)^8(\alpha+2)^4}{\alpha^4}$

$\frac{\alpha^5\left(\alpha^2+\alpha-4\right)^5}{(\alpha-1)^5(\alpha+2)^5}$ — $\frac{\left(\alpha^4+\alpha^3-5\alpha^2-\alpha+2\right)^6}{\alpha^6(\alpha^2+\alpha-4)^6}$ — $\frac{(\alpha-1)^7\alpha^7\left(\alpha^3+2\alpha^2-4\alpha-\right.}{(\alpha^4+\alpha^3-5\alpha^2-\alpha+2)^7}$

$\frac{(\alpha-2)^5(\alpha+1)^5}{(\alpha-1)^5}$ — $\frac{\left(\alpha^3-\alpha^2-3\alpha+1\right)^6}{(\alpha-2)^6(\alpha+1)^6}$ — $\frac{(\alpha-1)^7\left(\alpha^3-4\alpha-2\right)^7}{(\alpha^3-\alpha^2-3\alpha+1)^7}$

Standard Modules - Gram Matrices

For $l = 1$, the Rollet graph labelled with $\mathcal{V}^{(p+4)}_{(p,\lambda)}$:



α^3

$\frac{(\alpha-1)^{12}(\alpha+2)^6}{\alpha^6}$

$\frac{\alpha^{21}(\alpha-2)^{14}(\alpha+4)^7}{(\alpha-1)^{14}(\alpha+2)^7}$

$\frac{(\alpha-2)^{21}(\alpha+2)^{14}}{(\alpha-1)^{14}}$

$\frac{(\alpha+1)^8\left(\alpha^2+3\alpha-6\right)^8}{\alpha^8(\alpha+4)^8}$

$\frac{\left(\alpha^4-7\alpha^2+3\right)^{16}}{(\alpha-2)^{16}\alpha^{16}(\alpha+2)^{16}}$

$\frac{(\alpha-3)^8(\alpha+1)^8}{(\alpha-2)^8}$

$\alpha^9\left(\alpha^3+4\alpha^2-4\alpha\right.$ / $(\alpha+1)^9(\alpha^2+3\alpha$

$(\alpha-1)^{18}\alpha^{18}(\alpha+1)^{18}($ / $(\alpha^4-7\alpha^2+3)$

$\left(\alpha^3-2\alpha^2-4\alpha+\right.$ / $(\alpha-3)^9(\alpha+1$

Standard Modules - Gram Matrices

What is going on? Conjecture:

$$\mathcal{V}^{(n)}_{(p,\lambda)} = \frac{\left|\Delta^{(n)}_{(p,\lambda)}\right|}{\prod_{v\in \mathrm{Res}(p,\lambda)}\left|\Delta^{(n-1)}_{v}\right|} = \prod_{(p+1,\mu)\in \mathrm{Res}^{+}(p,\lambda)} \left(\frac{F^{(p+1)}_{\mu}(\alpha)}{F^{(p)}_{\mu}(\alpha)}\right)^{d_{(p+1,\mu)}},$$

where for each partition μ we have introduced a series of (monic) polynomials, $F^{(p)}_{\mu}(\alpha)$, for $p \overset{?}{\geq} |\mu|-1$:

$$\begin{aligned}\deg(F^{(p+1)}_{\mu}(\alpha)) &= \deg(F^{(p)}_{\mu}(\alpha))+1\\ F^{(p+1)}_{\mu}(\alpha) &= \alpha F^{(p)}_{\mu}(\alpha) - F^{(p-1)}_{\mu}(\alpha)\end{aligned}$$

Standard Modules - Gram Matrices

Examples of the $F_\mu^{(n)}$

- $\mu = (1) \vdash 1$, we have

$$F_\mu^{(0)}(\alpha) = 1, \quad F_\mu^{(1)}(\alpha) = \alpha, \ldots, F_\mu^{(p)}(\alpha) = U_p(\alpha/2)$$

- $\mu = (2) \vdash 2$, we have

$$(\alpha + 2)(\alpha + 1), \quad \alpha(\alpha^2 + \alpha - 4), \quad \alpha^4 + \alpha^3 - 5\alpha^2 + 2$$
$$\alpha(\alpha - 1)(\alpha^3 + 2\alpha^2 - 4\alpha - 6), \quad \alpha^6 + \alpha^5 - 7\alpha^4 - 3\alpha^3 + 11\alpha^2 - 2$$

- $\mu = (21) \vdash 3$, we have

$$(\alpha - 2)\alpha(\alpha + 2), \quad \alpha^4 - 7\alpha^2 + 3, \quad (\alpha - 1)\alpha(\alpha + 1)\left(\alpha^2 - 7\right)$$
$$\alpha^6 - 9\alpha^4 + 14\alpha^2 - 3, \quad \alpha\left(\alpha^6 - 10\alpha^4 + 22\alpha^2 - 10\right)$$

Standard Modules - Gram Matrices

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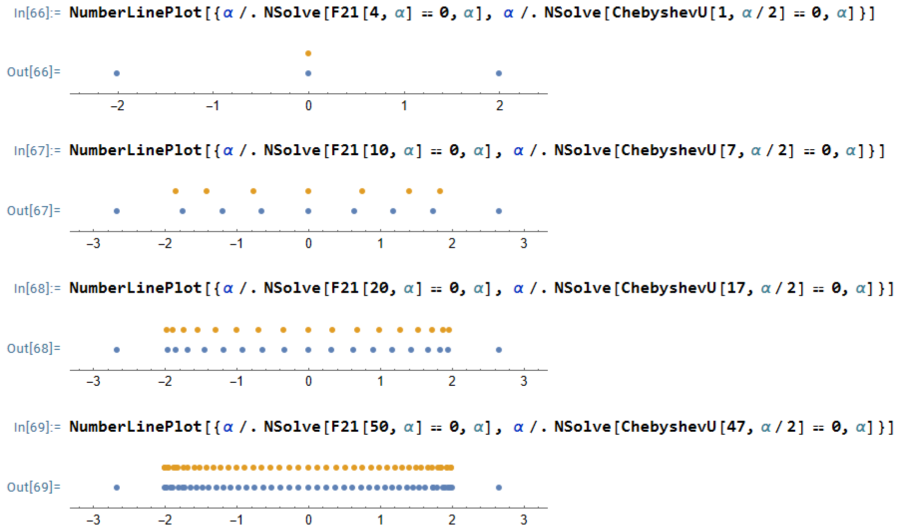
Standard Modules - Gram Matrices

Roots of the $F_{\mu}^{(p)}$: $\lambda = (2, 1)$



```
In[66]:= NumberLinePlot[{α /. NSolve[F21[4, α] == 0, α], α /. NSolve[ChebyshevU[1, α / 2] == 0, α]}]
```

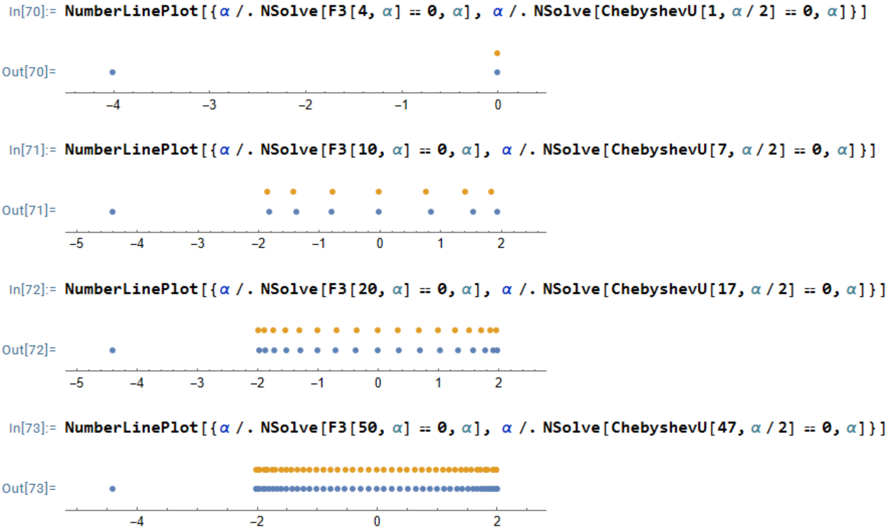
```
In[67]:= NumberLinePlot[{α /. NSolve[F21[10, α] == 0, α], α /. NSolve[ChebyshevU[7, α / 2] == 0, α]}]
```

```
In[68]:= NumberLinePlot[{α /. NSolve[F21[20, α] == 0, α], α /. NSolve[ChebyshevU[17, α / 2] == 0, α]}]
```

```
In[69]:= NumberLinePlot[{α /. NSolve[F21[50, α] == 0, α], α /. NSolve[ChebyshevU[47, α / 2] == 0, α]}]
```

Standard Modules - Gram Matrices

Roots of the $F_{\mu}^{(p)}$: $\lambda = (3)$



```
In[70]:= NumberLinePlot[{α /. NSolve[F3[4, α] == 0, α], α /. NSolve[ChebyshevU[1, α / 2] == 0, α]}]

Out[70]=

In[71]:= NumberLinePlot[{α /. NSolve[F3[10, α] == 0, α], α /. NSolve[ChebyshevU[7, α / 2] == 0, α]}]

Out[71]=

In[72]:= NumberLinePlot[{α /. NSolve[F3[20, α] == 0, α], α /. NSolve[ChebyshevU[17, α / 2] == 0, α]}]

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In[73]:= NumberLinePlot[{α /. NSolve[F3[50, α] == 0, α], α /. NSolve[ChebyshevU[47, α / 2] == 0, α]}]

Out[73]=
```
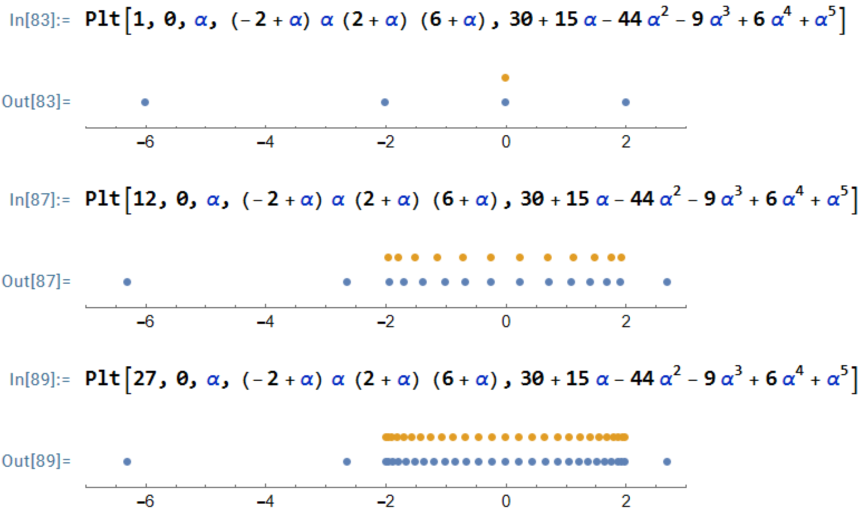

Standard Modules - Gram Matrices

Roots of the $F_\mu^{(p)}$: $\lambda = (4,1)$



```
In[83]:= Plt[1, 0, α, (-2 + α) α (2 + α) (6 + α), 30 + 15 α - 44 α^2 - 9 α^3 + 6 α^4 + α^5]
```

```
In[87]:= Plt[12, 0, α, (-2 + α) α (2 + α) (6 + α), 30 + 15 α - 44 α^2 - 9 α^3 + 6 α^4 + α^5]
```

```
In[89]:= Plt[27, 0, α, (-2 + α) α (2 + α) (6 + α), 30 + 15 α - 44 α^2 - 9 α^3 + 6 α^4 + α^5]
```

Standard Modules - Gram Matrices

QUESTION: Suppose we have two monic polynomials f_0, f_1 of consecutive degree. Then consider the family f_n generated from them by Chebyshev recursion. Do we expect this behaviour generically?

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NO!

Standard Modules - Gram Matrices

QUESTION: Suppose we have two monic polynomials f_0, f_1 of consecutive degree. Then consider the family f_n generated from them by Chebyshev recursion. Do we expect this behaviour generically?

NO! Consider $f_0 = x - 1$, and $f_1 = x^2 - 2$. Then the next term must be $f_2 = x^3 - x - 1$ which does not even have real roots!

Maps Between Standard Modules

Temperley-Lieb case $l = -1$. Semi-simple unless $\alpha = [2]_q$ and $[n]_q = 0$ for some $n \in \mathbb{N}$.

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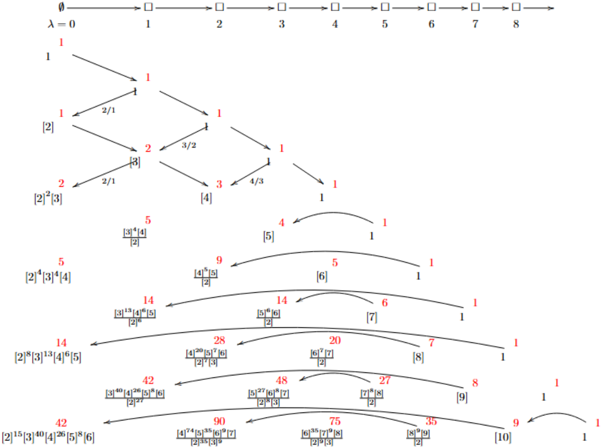

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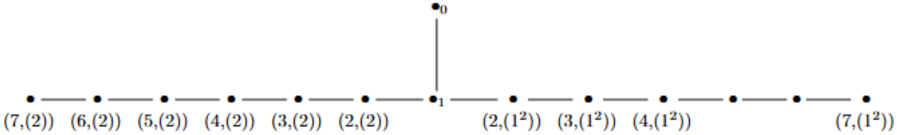

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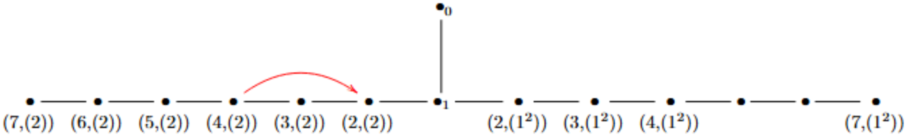

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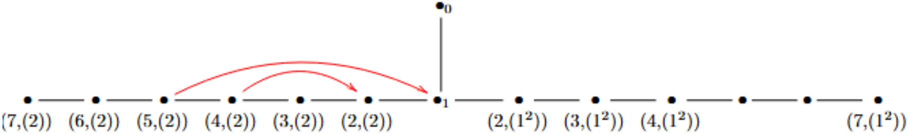

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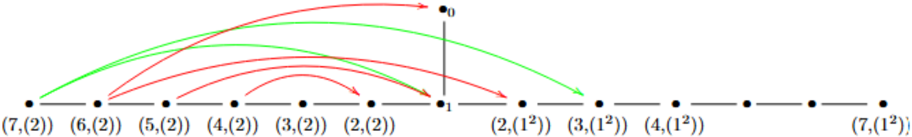

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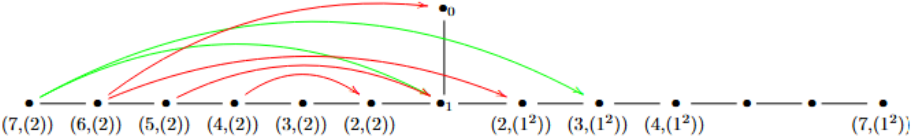

Can this be described by reflections?

THANK YOU!

Questions?

References

Alraddadi, Nouf and Alison Parker (2024). *On semi-simplicity of KMY algebras*. arXiv: `2407.07028 [math.RT]`. URL: `https://arxiv.org/abs/2407.07028`.

Kadar, Zoltan, Paul P. Martin, and Shona Yu (2014). *On geometrically defined extensions of the Temperley-Lieb category in the Brauer category*. DOI: `10.48550/ARXIV.1401.1774`. URL: `https://arxiv.org/abs/1401.1774`.