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Towards a Factorised Solution of the Yang-Baxter Equation with $U_q(\mathfrak{sl}_n)$ Symmetry

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École de Physique des Houches, 2023

Introduction

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  - Symmetric Group Relations

Yang-Baxter Equation

The (parameter dependent) YBE on $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ is

$$R_{V_1,V_2}(u_1,u_2)R_{V_1,V_3}(u_1,u_3)R_{V_2,V_3}(u_2,u_3) = R_{V_2,V_3}(u_2,u_3)R_{V_1,V_3}(u_1,u_3)R_{V_1,V_2}(u_1,u_2),$$

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Additive dependence $\Rightarrow R_{V_i,V_j}(u_i,u_j) = R_{V_i,V_j}(u_i - u_j)$

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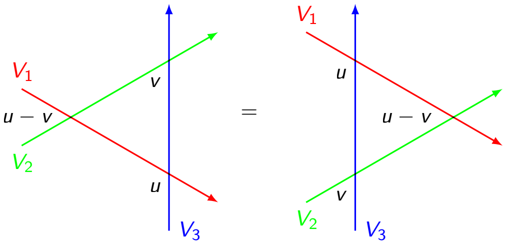
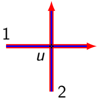


RLL-Method

Our Goal: construct an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$,

$\mathcal{R}_{12}(u) =$



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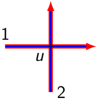
$\mathcal{R}_{12}(u) =$ 

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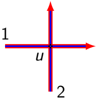
$\mathcal{R}_{12}(u) =$ 

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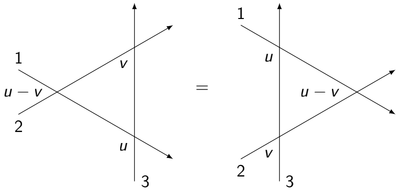

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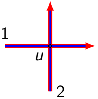
$\mathcal{R}_{12}(u) =$ 

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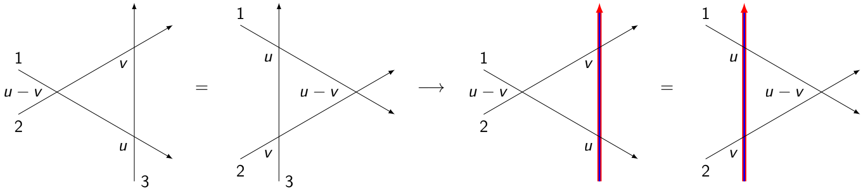

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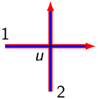
$\mathcal{R}_{12}(u) =$ 

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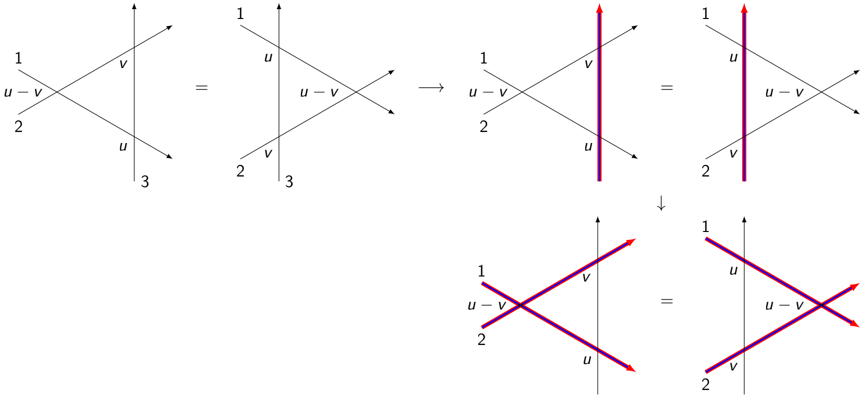

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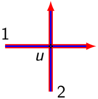
$\mathcal{R}_{12}(u) =$ 

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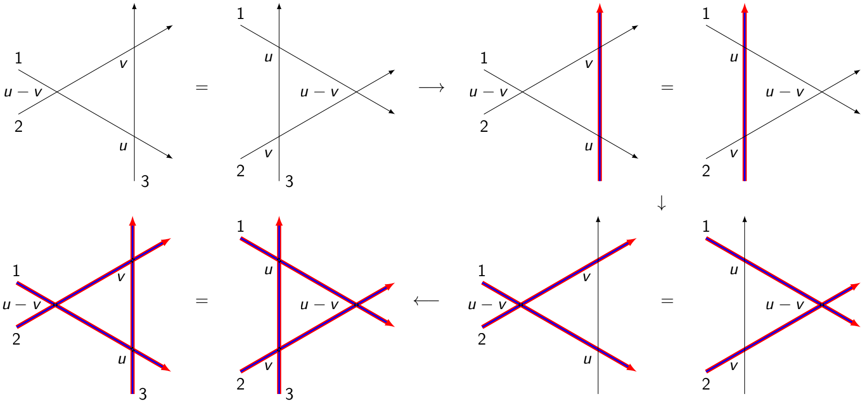

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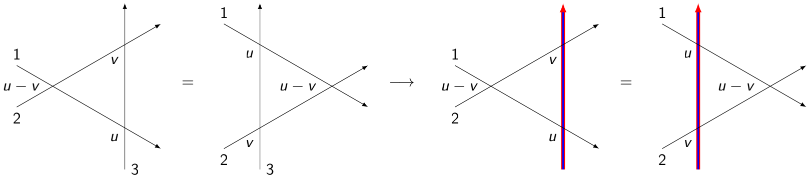
Fundamental R-Matrix and L-operators



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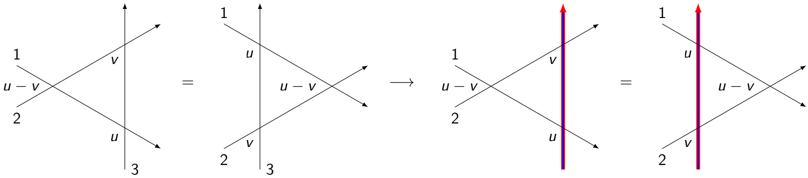

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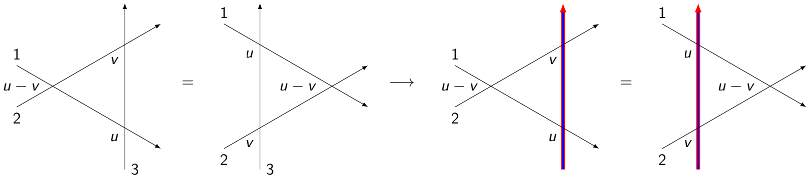

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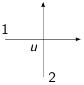
- $R_{12}(u) =$ 

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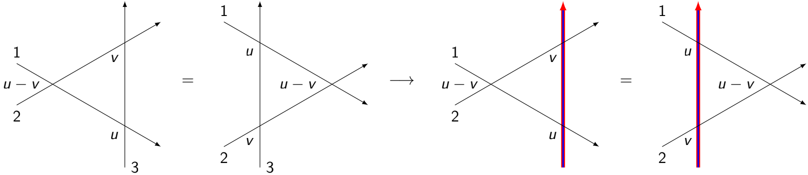

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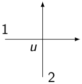
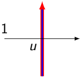
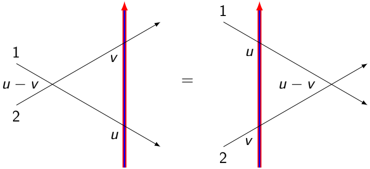
- $R_{12}(u) =$  $\in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (an $n^2 \times n^2$ matrix).
- $L_1(u) =$  $\in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$, where $\mathcal{A} \subset \mathrm{End}(\mathcal{V})$. An $n \times n$ matrix with values in $\mathcal{A}$.

Fundamental R-Matrix and Universal L-operators



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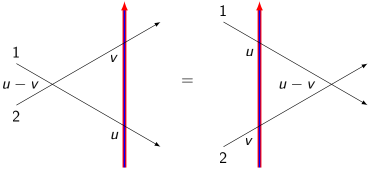

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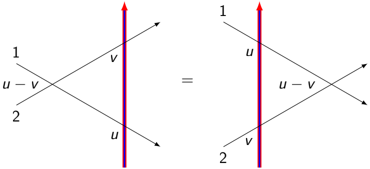

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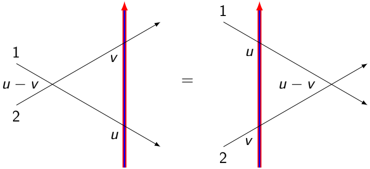

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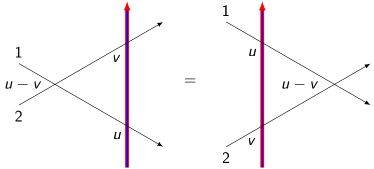

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Why YBE for R? This is a consistency condition for associativity of $\mathcal{A}$.

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The universal enveloping algebra (UEA) $\mathcal{A} = U(\mathfrak{sl}_n)$ has a fundamental R-matrix

$$R_{12}(u) = u \cdot \mathrm{id}_{n^2} + P_{12} : \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n,$$

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$$L(u) = u \cdot \mathrm{id}_n \otimes 1_{\mathcal{A}} + \sum_{i,j=1}^{n} e_{ij} \otimes E_{ji},$$

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where e_{ij} is the matrix unit. Here $\{E_{ij}\}$ is the Cartan-Weyl basis for $\mathfrak{sl}_n$:

$$h_i = E_{ii} - E_{i+1,i+1}, \quad \sum\nolimits_i E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i,$$
$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{ik} E_{lj}.$$

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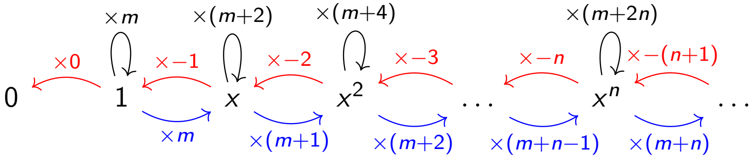

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$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(-\boldsymbol{\rho}) = \begin{pmatrix} -\rho_n & P_{21} & P_{31} & \dots & P_{n1} \\ & -\rho_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & -\rho_2 & P_{n,n-1} \\ & & & & -\rho_1 \end{pmatrix},$$

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where the P_{ij} are first order linear differential operators:

$$P_{ij} = -\partial_{ij} - \sum_{k=j+1}^{n} x_{kj} \cdot \partial_{ki}.$$

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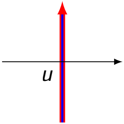
- It is reducible if some $m_i \in \mathbb{Z}_{\leq 0}$. It contains a finite dimensional irreducible subrep iff true for all m_i.
- It has a factorised L-operator!

$$L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1} =$$ ,

$\boldsymbol{u} = (u_i)$, where $u_i = u - \rho_i$.

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- Relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm 1} - (q + q^{-1}) g_i g_{i\pm 1} g_i + g_{i\pm 1} g_i^2 = 0,$$

$g_i = e_i, f_i$.

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The q-deformed UEA $U_q(\mathfrak{sl}_n)$: For some $q = e^h \in \mathbb{C} \setminus \{0, \pm 1\}$

- Generators: e_i, f_i, and invertible $k_i = q^{h_i}$ for $i = 1, 2, \ldots, n-1$
- Relations:

$$[k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{a_{ij}} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} = \delta_{ij} [h_i]_q,$$

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i - j| > 1,$$

$$g_i^2 g_{i\pm1} - (q + q^{-1}) g_i g_{i\pm1} g_i + g_{i\pm1} g_i^2 = 0,$$

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- Notation: $[x]_q = (q^x - q^{-x})/(q - q^{-1})$

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[1] M. Jimbo. "A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation". In: *Lett. Math. Phys.* 11 (1986), pp. 247–252.

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$$R(u) = q^u R + q^{-u} R^{-1} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n),$$

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Is there an analogous class of representations for $U_q(\mathfrak{sl}_n)$? How about a factorised L-operator?

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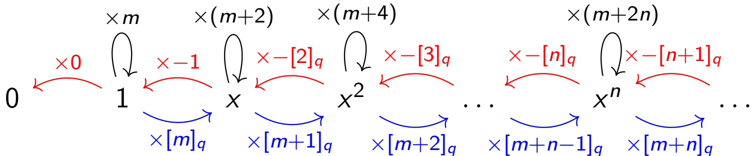

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- An Explicit formula: $m_i = \rho_{n-i} - \rho_{n+1-i} + 1$

$$E_{ii}^{(n)} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji}+1)\,,$$

$$f_i^{(n)} = -D_{i+1,i}\, q^{\sum_{j=1}^{i-1}(N_{ij}-N_{i+1,j})} - \sum_{j=1}^{i-1} x_{ij} D_{i+1,j}\, q^{\sum_{k=1}^{j-1}(N_{ik}-N_{i+1,k})}\,,$$

$$e_i^{(n)} = \begin{array}{c} x_{i+1,i}\Big[\, m_i + N_{i+1,i} + \sum_{j=i+2}^{n}(N_{ji}-N_{j,i+1})\,\Big]_q + q^{-m_i} \sum_{j=i+2}^{n} x_{ji} D_{j,i+1}\, q^{\sum_{k=j}^{n}(N_{k,i+1}-N_{k,i})} \\ - q^{m_i+2N_{i+1,i}} \sum_{j=1}^{i-1} x_{i+1,j} D_{ij}\, q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) + \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})} \end{array}\,,$$

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$\underline{\mathfrak{sl}_n}$: $L(\boldsymbol{u}) = ZD(\boldsymbol{u})Z^{-1}$

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \dots & x_{n,n-1} & 1 \end{pmatrix}, \quad D(\boldsymbol{u}) = \begin{pmatrix} u_n & P_{21} & P_{31} & \dots & P_{n1} \\ & u_{n-1} & P_{32} & \dots & P_{n2} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & P_{n,n-1} \\ & & & & u_1 \end{pmatrix},$$

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$\underline{U_q(\mathfrak{sl}_n)}$: Postulate $L(\boldsymbol{u}) = Z_1(\boldsymbol{u})D(\boldsymbol{u})Z_2(\boldsymbol{u})^{-1_R}$

$$D(\boldsymbol{u}) = \begin{pmatrix} [u_n]_q q^{b_1} & P_{21} & \dots & P_{n1} \\ & \ddots & \ddots & \vdots \\ & & [u_2]_q q^{b_{n-1}} & P_{n,n-1} \\ & & & [u_1]_q q^{b_n} \end{pmatrix},$$

$$P_{ij} = -D_{ij} q^{b_{ij}} - \sum_{k=i+1}^{n} x_{ki} D_{kj} q^{b_{ijk}}, \quad Z_i(\boldsymbol{u}) = \begin{pmatrix} 1 & & & \\ x_{21} q^{a^{(i)}_{21}} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{n1} q^{a^{(i)}_{n1}} & \dots & x_{n,n-1} q^{a^{(i)}_{n,n-1}} & 1 \end{pmatrix},$$

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[3] S. Derkachov, D. Karakhanyan, and R. Kirschner. "Yang-Baxter-operators & parameter permutations". In: *Nucl. Phys. B* 785.3 (2007), pp. 263–285.

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n=2: Yes[3]

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ q^{u_1 - N_x} x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x - 1} & -D_x q^{N_x} \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{u_2 - N_x} x & 1 \end{pmatrix}.$$

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n=3: Yes[4], $L(u_1,u_2,u_3)=Z_1 D Z_2^{-1_R}$ with

$$D=\begin{pmatrix}[u_3]_q q^{-N_{21}+N_{31}} & (D_{21}+x_{32}D_{31}q^{N_{31}-N_{32}-1})q^{N_{21}+N_{31}} & D_{31}q^{N_{31}}\\ 0 & [u_2]_q q^{N_{21}-N_{32}} & D_{32}q^{u_2-N_{31}+N_{32}}\\ 0 & 0 & [u_1]_q q^{N_{32}+N_{31}}\end{pmatrix},$$

$$Z_1=\begin{pmatrix}1 & 0 & 0\\ q^{u_2-N_{31}+N_{32}-N_{21}}x_{21} & 1 & 0\\ q^{-u_1-N_{31}+N_{32}}x_{31} & q^{u_1-u_2-N_{32}}x_{32} & 1\end{pmatrix},\quad Z_2=\begin{pmatrix}1 & 0 & 0\\ q^{c_{21}}x_{21} & 1 & 0\\ q^{c_{31}}x_{31} & q^{c_{32}}x_{32} & 1\end{pmatrix},$$

$$c_{21}=u_3-N_{21},\ c_{31}=-u_3-N_{31}-N_{21}-1,\ c_{32}=N_{21}+N_{31}-N_{32}.$$

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"Controlled deformation" breaks: For an example we compute

$$E_{42} = [f_3, f_2]_q^{-1} = -D_{42} q^{N_{21}-N_{32}-N_{41}-1} - x_{21} D_{41} q^{-(1+N_{31})} + (q - q^{-1}) x_{31} D_{41} D_{32} q^{N_{21}-N_{31}-1}.$$

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Such terms cannot arise from our ansatz!

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Parameter Permutations and YBE

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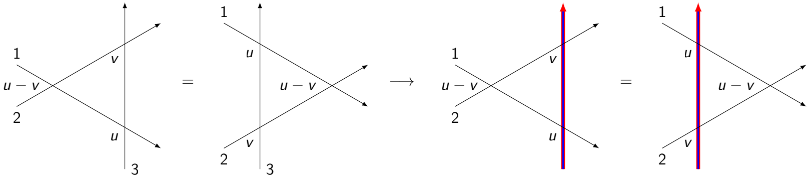

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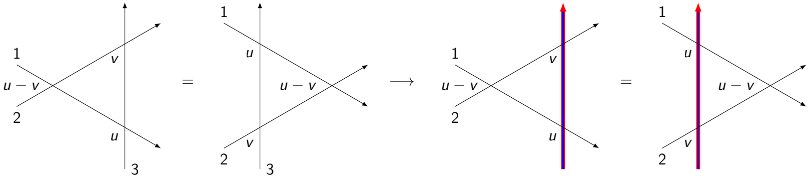
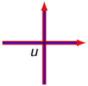
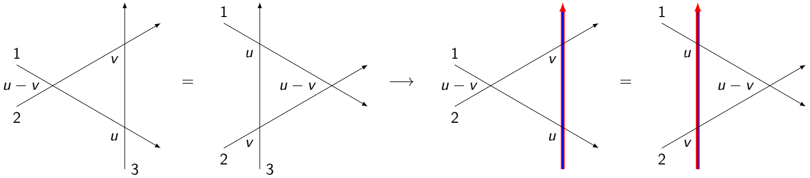
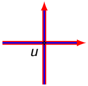


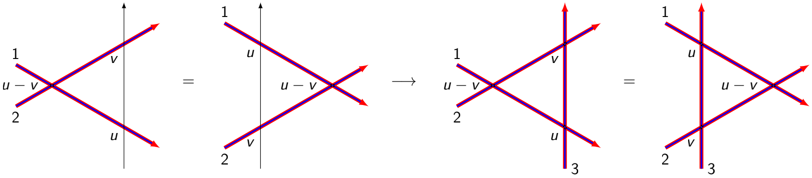
Now we're seeking $\mathcal{R}(u) =$  $\in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$

Parameter Permutations and YBE

So far



Now we're seeking $\mathcal{R}(u) =$  $\in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$



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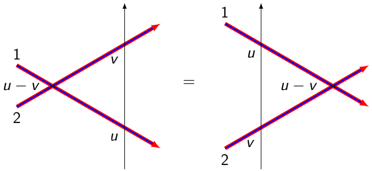

$$\sim \quad \begin{aligned} &\check{\mathcal{R}}(u-v) L_1(\boldsymbol{u}) L_2(\boldsymbol{v}) \\ &= L_1(\boldsymbol{v}) L_2(\boldsymbol{u}) \check{\mathcal{R}}(u-v). \end{aligned}$$

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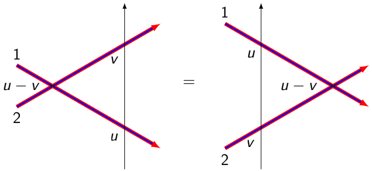

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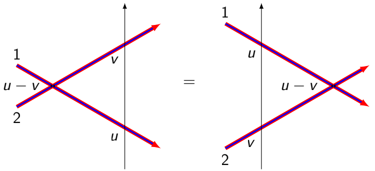

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<u>IDEA:</u> Factorise $\check{\mathcal{R}}(u-v)$ in terms of elementary transposition operators $\mathcal{S}_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$

$$\mathcal{S}_i L_{12}(\boldsymbol{u}, \boldsymbol{v}) = L_{12}(s_i(\boldsymbol{u}, \boldsymbol{v})) \mathcal{S}_i, \quad (L_{12}(\boldsymbol{u}, \boldsymbol{v}) = L_1(\boldsymbol{u}) L_2(\boldsymbol{v}))$$

$(s_i(\alpha_1, \ldots \alpha_{2n}) = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots \alpha_{2n}))$ for $i = 1, \ldots, 2n-1$.

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Simplification: Can just find $n-1$-"intertwining" operators $\mathcal{T}_i \in \mathrm{End}(\mathcal{V}_\rho)$:

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and a single "exchange" operator:

$$\mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(\boldsymbol{u},\boldsymbol{v}) = \mathcal{S}_n(\boldsymbol{u},\boldsymbol{v})L_{12}(u_1,\dots,u_{n-1},v_1,u_n,v_2,\dots,v_n).$$

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These will both be answered provided we can prove these operators define an action of S_{2n}, that is,

$$s_{i_j} \ldots s_{i_2} s_{i_1} \mapsto \mathcal{S}_{i_j}(s_{i_{j-1}} \ldots s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \ldots \mathcal{S}_{i_2}(s_{i_1}(\boldsymbol{u}, \boldsymbol{v}))\, \mathcal{S}_{i_1}(\boldsymbol{u}, \boldsymbol{v}),$$

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YBE then follows from equivalence of the decompositions

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \xmapsto{\check{\mathcal{R}}_{12}} (\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}) \xmapsto{\check{\mathcal{R}}_{23}} (\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) \xmapsto{\check{\mathcal{R}}_{12}} (\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}),$$

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Undeformed Case

[5] S. Derkachov and A. N. Manashov. "$\mathcal{R}$-Matrix and Baxter $\mathcal{Q}$-Operators for the Noncompact $SL(N, C)$ Invariant Spin Chain". In: *SIGMA* (Dec. 2006).

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q-Deformed Case

Proposition

The intertwiners for the $U_q(\mathfrak{sl}_n)$ L-operator are given by

$$\mathcal{T}^{(n)}_{n-i}(\alpha) = \left(\Lambda^{(n)}_{n-i}\right)^{\alpha} \frac{e_{q^2}(q^{2(N_{i+1,i}+1)}\boldsymbol{X}^{(n)}_{n-i})}{e_{q^2}(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}^{(n)}_{n-i})},$$

$$e_{q^2}(\boldsymbol{Z}) = ((\boldsymbol{Z};q^2)_\infty)^{-1} = \left[(1-\boldsymbol{Z})(1-q^2\boldsymbol{Z})(1-q^{2\cdot 2}\boldsymbol{Z})\ldots\right]^{-1},$$

$$\frac{e_{q^2}(\boldsymbol{Z})}{e_{q^2}(q^{-\alpha}\boldsymbol{Z})} = \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j}\boldsymbol{Z}^j,$$

where $\alpha = u_{n-i} - u_{n+1-i}$, $\Lambda^{(n)}_{n-i} = (x_{i+1,i})q^{\beta_i}$, *and*
$\boldsymbol{X}^{(n)}_{n-i} = 1 + x_{i+1,i}\sum_{j=i+2}^{n} \frac{x_{j,i+1}}{x_{ji}}(q^{N_{ij}} - q^{-N_{ij}})q^{\gamma_i}$. $(|q| < 1)$

[6] P. A. Valinevich et al. "Factorization of the $\mathcal{R}$-matrix for the quantum algebra $U_q(\mathfrak{sl}_3)$". In: *J. Math. Sci.* 151 (2008), pp. 2848–2858.

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There formulae are obtained using an approach from[6].

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Proof.

The only non-trivial relation is the braid relation

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After a series expansion it is reduced to a family of (terminating) q-series identity relating rank $i+1$ and rank $2i-1$ q-Lauricella series. □

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q-Series Identity

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For $n \geq 1$ and non-negative integer tuples

$$\boldsymbol{k} = (k_0, \dots, k_n) = (k_0, \tilde{\boldsymbol{k}}), \quad \boldsymbol{l} = (l_1, \dots, l_n), \quad \boldsymbol{m} = (m_1, \dots, m_{n-1}),$$

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with $K = \sum_{j=0}^{n} k_j$ and L, M. Define n-tuples $\boldsymbol{r} = (r_i)$ and $\boldsymbol{p} = (p_i)$

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The identity we need is the equality $\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}}$

$$\Theta_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \frac{(\xi;q)_{L+M}}{(\xi\,\zeta;q)_{L+M}} \Phi_D^{(2n-1)}\big[\,\zeta; q^{-\boldsymbol{l}}, q^{-\boldsymbol{m}}\,; q^{1-L-M}/\xi;\, q^{\boldsymbol{r}+\boldsymbol{l}+(\boldsymbol{m},0)}, q^{(r_i,\hat{r}_n)+\boldsymbol{m}}\,\big],$$

$$\Omega_{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}} = \zeta^{k_0} \frac{(\xi;q)_K}{(\xi\,\zeta;q)_K} \Phi_D^{(n+1)}\big[\,\zeta; q^{-\boldsymbol{k}}\,; q^{1-K}/\xi;\, q^{1+k_0-K}/(\xi\,\zeta), q^{\boldsymbol{p}+\tilde{\boldsymbol{k}}}\,\big],$$

for arbitrary complex parameters ξ, ζ.

Exchange Operator

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The defining relation for the exchange operator $\mathcal{S}_n$ is

$$\mathcal{S}_n L_1(\boldsymbol{u}) L_2(\boldsymbol{v}) = L_1(u_1, \tilde{\boldsymbol{u}}, v_1) L_2(u_n, \tilde{\boldsymbol{v}}, v_n) \mathcal{S}_n.$$

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Recall the (postulated) factorisation for $L(\boldsymbol{u})$. This can be put into the form:

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Now we can reduce the defining relation to

$$\left[D^{(x)}(\tilde{\boldsymbol{u}})^{-1_L}\, \mathcal{S}_n\, D^{(x)}(\tilde{\boldsymbol{u}})\right] Z_2^{(x)}(\tilde{\boldsymbol{u}}, u_n)^{-1_R} Z_1^{(y)}(v_1, \tilde{\boldsymbol{v}}) = Z_2^{(x)}(\tilde{\boldsymbol{u}}, v_1)^{-1_R} Z_1^{(y)}(u_n, \tilde{\boldsymbol{v}}) \left[D^{(y)}(\tilde{\boldsymbol{v}})\, \mathcal{S}_n\, D^{(y)}(\tilde{\boldsymbol{v}})^{-1_R}\right],$$

if $\mathcal{S}_n^{(x,y)}$ commutes (element wise) with $Z_1^{(x)}$ and $Z_2^{(y)}$.

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This seems to represent a serious obstruction to constructing the exchange operator... So far hasn't been obtained in the $n = 4$ case (or general).

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- We described explicitly all but one of the transposition operators in the $U_q(\mathfrak{sl}_n)$ case, and prove they obey the necessary symmetric group relations.
- We explain how the failure of the factorisation property for the $U_q(\mathfrak{sl}_4)$ L-operator represents an obstruction to constructing the missing "exchange" operator.

Thank You!

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Questions?