Towards a Factorised Solution of the Yang-Baxter Equation with $U_q(\mathfrak{sl}_4)$ Symmetry

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis presents a study of the parameter permutation method [8, 9] of constructing solutions to the Yang-Baxter equation with $\mathfrak{sl}_n$ or $U_q(\mathfrak{sl}_n)$ symmetry, and which act in the product of spaces $\mathcal{V}^{(n)}$, which are spaces of polynomials in $n(n-1)/2$ variables. The $U_q(\mathfrak{sl}_4)$ case is a particular focus of this thesis in which we obtain a novel factorised L-operator which acts in the tensor product $\mathbb{C}^4 \otimes \mathcal{V}^{(4)}$ and satisfies an RLL-relation with the defining $U_q(\mathfrak{sl}_4)$ R-matrix. This factorisation, which generalises a known expression in the $\mathfrak{sl}_4$ case, is a source of pecularity since it exhibits behaviour unseen in the $U_q(\mathfrak{sl}_n)$ cases for $n < 4$. We are also able to obtain and prove the necessary Coxeter relations for most elementary transposition operators in this case, which are operators $\mathcal{S}_i \in GL(\mathcal{V}^{(4)} \otimes \mathcal{V}^{(4)})$ that have special commutation relations with the $U_q(\mathfrak{sl}_4)$ L-operator and are the building blocks for a factorised solution of the YBE.

Contents

Notation and terminology

The following is a short list of common notation and initialisms used in this thesis.

Notation

V	A generic complex vector space. Mostly finite dimensional.
$\mathrm{End}(V)$	The vector space of linear maps $T : V \to V$ (endomorphisms).
$GL(V)$	The group of invertible linear maps $T : V \to V$.
$\mathfrak{g}$	A complex finite dimensional Lie algebra, generally simple. Mostly taken to be $\mathfrak{g} = \mathfrak{sl}_n$ from § 2 onwards.
$U(\mathfrak{g})$	The universal enveloping algebra of $\mathfrak{g}$.
$U_q(\mathfrak{g})$	The q-deformed universal enveloping algebra of $\mathfrak{g}$.
$\mathcal{A}$	A complex, unital, associative algebra. Mostly taken to be $\mathcal{A} = U(\mathfrak{g})$ or $U_q(\mathfrak{g})$.
$\mathcal{V}^{(n)}$	The complex vector space of polynomials in the $n(n-1)/2$ variables x_{ij} for $1 \leq j < i \leq n$. Generally the superscript (n) is left implicit to be understood from context.
$\mathcal{V}^{(n)}_{\boldsymbol{\rho}}$	The $\mathcal{A}$ module structure on $\mathcal{V}^{(n)}$ defined uniquely by parameters ρ_i which are components of $\boldsymbol{\rho} \in \mathbb{C}^n$. Generally the superscript (n) is left implicit to be understood from context.
$\hat{\mathcal{V}}^{(n)}$	The space of formal power series in the $n(n-1)/2$ variables x_{ij} (centred at $x_{ij} = 1$) for $1 \leq j < i \leq n$ with $\hat{\mathcal{V}}^{(n)}_{\boldsymbol{\rho}}$ denoting the $\mathcal{A}$ module structure thereon defined by parameters ρ_i which are components of $\boldsymbol{\rho} \in \mathbb{C}^n$.

P	The permutation map $P : V \otimes V \to V \otimes V$ defined on pure tensors by $P(u \otimes v) = v \otimes u$. We will write $\mathcal{P}$ for the permutation map acting on $\mathcal{V} \otimes \mathcal{V}$.
e_{ij}	The matrix unit with a 1 in the i, j-th entry and 0 everywhere else.
q	A complex parameter $q \in \mathbb{C} \setminus \{0, \pm 1\}$. It will not be a root of unity unless specified otherwise and is generally considered an indeterminate.
$[x]_q$	The q-number given by $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. We will allow $[\ ]_q$ to take operator valued arguments where appropriate.

Initialisms

YBE	Yang-Baxter equation.
UEA	Universal enveloping algebra.
PBW	Poincaré-Birkhoff-Witt. Generally refers to the PBW theorem or to a PBW basis.
LHS	Left hand side.
RHS	Right hand side.

Chapter 1

Introduction

1.1 The Yang-Baxter Equation

The Yang-Baxter equation (YBE) is a fundamental equation in the field of integrable models within modern theoretical physics. It has been studied for over five decades [18]. It first appeared in the work of McGuire in 1964 and Yang in 1967 [23, 29] who considered a many-body quantum mechanical system in one dimension with particles interacting via a Dirac-delta function potential. Here the YBE arose as a consistency condition allowing the scattering matrix for the many-body system to be factorised in terms of two-body scattering matrices.

The next appearance of the YBE was in statistical mechanics in the work of Baxter [1, 2]. Here it arose as a sufficient condition for the existence of a continuous family of mutually commuting row transfer matrices $T(u)$ (parameterised entirely), in a 2-dimensional lattice model with toroidal boundary conditions. This allows the YBE to be regarded as a master equation in statistical mechanics, which under mild conditions guarantees the existence of a 1-dimensional quantum spin chain with a hamiltonian H such that $T(u) = I + uH + \mathcal{O}(u^2)$, and with infinitely many "conserved quantities" Q satisfying $[Q, H] = 0$ [24].

Today the Yang-Baxter equation is of just as much mathematical interest as it is physical interest. Its study has given rise to areas such as quantum groups [12], and braided monoidal categories [19], and it has applications in representation theory, knot theory and much more.

In its most general form, the Yang-Baxter equation is an equality in $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ for three complex vector spaces V_1, V_2, V_3 which reads as follows

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2), \qquad (1.1)$$

where each $R_{ij}(u_i, u_j) \in \mathrm{End}(V_i \otimes V_j)$ is a function of the variables $u_i, u_j \in \mathbb{C}$ and is extended to the element of $\mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ which acts as the identity in the remaining tensor factor. In this thesis we limit our consideration to the "additive" YBE in which the dependence is specialised as $R_{ij}(u_i, u_j) = R_{ij}(u_i - u_j)$ allowing for (1.1) to be rewritten as

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \tag{1.2}$$

by taking $u = u_1 - u_3$ and $v = u_2 - u_3$. The variables entering (1.2) are known as spectral parameters.

Note that so far we have not said anything about the vector spaces V_1, V_2 and V_3. As such, a solution to the Yang-Baxter equation (1.2) involves giving three operators $R_{ij}(u)$ for $1 \leq i < j \leq 3$ since the R_{ij} should depend on the vector space on which they are acting. However, in the case $V_1 = V_2 = V_3 = V$, one can make sense of the statement that $R \in \mathrm{End}(V \otimes V)$ is a solution of the YBE by taking $R_{ij} \in \mathrm{End}(V \otimes V \otimes V)$ to mean the operator which acts by R in the i-th and j-th tensor factor and trivially in the remaining one. If some $R(u) \in \mathrm{End}(V \otimes V)$ is refered to as a solution of the YBE this will be what is meant, otherwise we shall strive to make it clear where a Yang-Baxter equation lives.

A solution to the YBE $R(u)$, is known as quasi-classical if it depends on some additional parameter h (often refered to as a Planck constant) so that is has the following expansion for small h

$$R(u) = A(I + hr(u) + \mathcal{O}(h^2)), \tag{1.3}$$

where A is some constant. The operator $r(u)$ is known as the classical limit of $r(u)$ and must obey

$$[r_{12}(u-v), r_{13}(u)] + [r_{12}(u-v), r_{23}(v)] + [r_{13}(u), r_{23}(v)] = 0, \tag{1.4}$$

to be consistent with (1.2). This equation (1.4) is known as the classical Yang-Baxter equation (CYBE).

Since the CYBE is expressed in terms of commutators it only makes use of the Lie algebra structure on $\mathrm{End}(V)$. As such it is natural to study solutions $r(u)$ of (1.4) which are $\mathfrak{g} \otimes \mathfrak{g}$ valued functions for $\mathfrak{g}$ a finite-dimensional complex simple Lie algebra. We will write $r(u) = \sum_{\mu,\nu} r^{\mu\nu}(u) X_\nu \otimes X_\mu$ for $\{X_\nu\}$ a basis of $\mathfrak{g}$ where coefficients $r^{\mu\nu}(u)$ are complex valued functions. The classification of such solutions to (1.4) is known as Belavin-Drinfeld theory [4]. In Belavin-Drinfeld theory, non-degenerate[1] solutions $r(u)$, meromorphic in a neighbourhood of 0, can

[1] Here non-degenerate means $\det(r^{\mu\nu}(u))$ is not identically 0.

be extended meromorphically to the whole complex plane and are classified by their poles. In such solutions, coefficients $r^{\mu\nu}(u)$ can have rational, trigonometric dependence, or elliptic dependence on u and furthermore, it was proved that elliptic solutions only exist for $\mathfrak{g} = \mathfrak{sl}_n$. On the other hand, trigonometric and rational solutions exist for all $\mathfrak{g}$.

In order to construct interesting solutions of the YBE with inherent Lie algebra symmetry, the following question was considered [22]: given a solution $r(u) \in \mathfrak{g} \otimes \mathfrak{g}$ of the CYBE (1.4), does there exist a solution $R(u,h)$ of the YBE (1.2) which contains $r(u)$ as its classical limit? The first order of business is in determining where $R(u,h)$ should live. It cannot be in an endomorphism ring since $r(u)$ is in the tensor square of an abstract Lie algebra. Furthermore, it cannot be in $\mathfrak{g} \otimes \mathfrak{g}$ since (1.2) cannot be formulated solely in terms of commutators. This was first resolved by taking $R(u,h) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ where $U(\mathfrak{g})$ is the universal enveloping algebra (UEA) of $\mathfrak{g}$. However, it was soon realised that each solution $r(u)$ defines a deformation of $U(\mathfrak{g})$, known as a "quantised UEA" which is preferable to work with. This ultimately led Drinfeld to introduce quantum groups [12], by considering the quantised UEA for a class of trigonometric solutions $r(u) \in \mathfrak{g} \otimes \mathfrak{g}$ for any $\mathfrak{g}$.

The result of all this theory was a powerful machinery for constructing solutions to the YBE with Lie algebra symmetry (deformed or otherwise). This was motivated by the quantum inverse scattering method attributed to the Leningrad (Saint Petersburg) group (Faddeev, Reshetikhin, Sklyanin, Takhtadzyan et al.)[13]. The idea of this method can be obtained by considering the equation (1.2) in the case where $V_1 = V_2 = V$ is a finite-dimensional vector space and $\mathrm{End}(V_3) = \mathcal{A}$ is considered as an abstract algebra. Written in terms of $\hat{R}_{12}(u) = PR_{12}(u)$, where $P \in \mathrm{End}(V \otimes V)$ is the permutation map $P(u \otimes v) = v \otimes u$, and $T(u) \in \mathrm{End}(V) \otimes \mathcal{A}$, equation (1.2) takes the form

$$\hat{R}_{12}(u-v)T_1(u)T_2(v) = T_1(v)T_2(u)\hat{R}_{12}(u-v), \tag{1.5}$$

where $T_1(u) = T(u) \otimes I_V \in \mathrm{End}(V \otimes V) \otimes \mathcal{A}$ and $T_2(u) = I_V \otimes T(u) \in \mathrm{End}(V \otimes V) \otimes \mathcal{A}$.

The idea is now to view $T(u)$ as an $\mathcal{A}$-valued matrix, by writing $T(u) = \sum_{ij} e_{ij} t_{ij}(u)$ where e_{ij} are matrix units and $t_{ij}(u) \in \mathcal{A}$. The equation (1.5) is considered an $\mathcal{A}$-valued matrix equation which expresses the defining commutation relations for the algebra $\mathcal{B} \subset \mathcal{A}$ generated by the $t_{ij}(u)$. One can obtain as a sufficient condition for associativity of $\mathcal{B}$ that $R_{12}(u)$ solves the YBE. A key feature of this construction is that the algebra $\mathcal{B}$ comes equipped with a coproduct

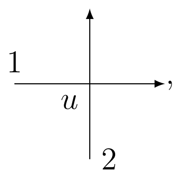
$\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ defined on generators by $\Delta(t_{ij}(u)) = \sum_k t_{ik}(u) \otimes t_{kj}(u)$ such that (1.5) holds in $\mathrm{End}(V \otimes V) \otimes \mathcal{B} \otimes \mathcal{B}$ with $T_1(u)$ $(T_2(v))$ replaced with $\Delta(T_1(u))$ $(\Delta(T_2(v)))$.

In this thesis we will be concerned with constructing solutions to the Yang-Baxter equation in a certain class of representations $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{V})$, of a symmetry algebra $\mathcal{A}$, which will be either the universal enveloping algebra $U(\mathfrak{sl}_n)$ or its trigonometric deformation $U_q(\mathfrak{sl}_n)$. The aforementioned theory provides the following scheme for constructing such solutions, which has proven to be a powerful technique [3].

On the first level we start with a solution of the YBE (1.2) $R(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ where $\mathbb{C}^n$ is the defining representation of $\mathcal{A}$. The $n^2 \times n^2$ matrix $R(u)$ is known as a defining R-matrix for $\mathcal{A}$. Such solutions are well studied and often arise from physical models. For example the R-matrix associated with the 6-vertex model [3]

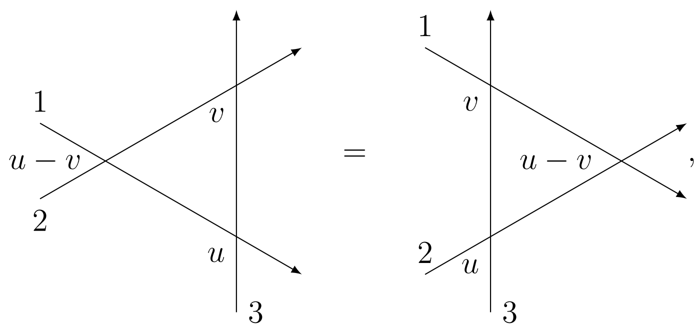
$$R(\rho, \eta, \theta) = \rho \begin{pmatrix} \sin(\theta+\eta) & 0 & 0 & 0 \\ 0 & \sin\theta & e^{i\theta}\sin\eta & 0 \\ 0 & e^{-i\theta}\sin\eta & \sin\theta & 0 \\ 0 & 0 & 0 & \sin(\theta+\eta) \end{pmatrix}, \tag{1.6}$$

is proportional to the defining R-matrix for $U_q(\mathfrak{sl}_2)$ after we identify $e^{i\eta} = q$, and $e^{i\theta} = q^u$. We will use the following graphical representation of $R(u)$ for its mnemonic value

$R_{12}(u) =$ 

, (1.7)

where the single lines denote a copy of the defining representation $\mathbb{C}^n$. The YBE (1.2) for $R(u)$ then takes the graphical form



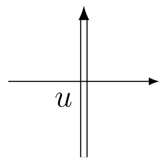
, (1.8)

where the arrows determine the order of multiplication.

On the next level we have a so-called universal L-operator $\tilde{L}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$ which solves the following YBE in $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \otimes \mathcal{A}$

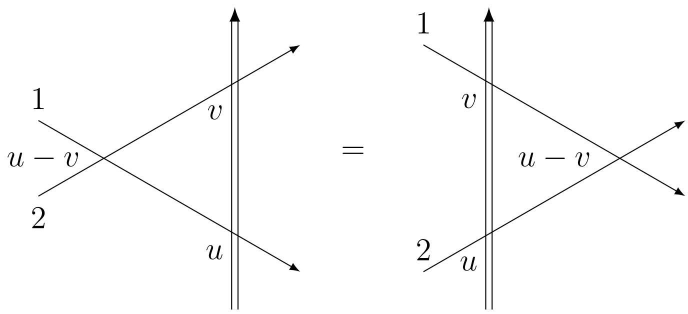
$$R_{12}(u-v)\tilde{L}_1(u)\tilde{L}_2(v) = \tilde{L}_2(v)\tilde{L}_1(u)R_{12}(u-v), \tag{1.9}$$

where the subscript on an L-operator denotes which copy of the defining representation it acts non-trivially on. We will see later that this equation (1.9) (known as the RLL relation) can be considered a compact expression for the defining relations of the symmetry algebra $\mathcal{A}$. We can then evaluate the third tensor factor of (1.9) in the desired representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{V})$ and in doing so specialise our L-operator $L(u) := (\mathrm{id} \otimes \rho)(\tilde{L}(u))$ as an $\mathrm{End}(\mathcal{V})$ valued $n \times n$ matrix. We use the following graphical representation for $L(u)$

$L(u) =$ 

, (1.10)

and the YBE for $L(u)$ and $R(u)$ takes the form



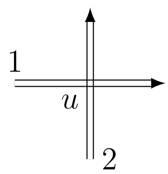
. (1.11)

The double line denotes the vector space $\mathcal{V}$, distinguishing it from $\mathbb{C}^n$.

On the final level we have an R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$ which solves the following YBE in $\mathrm{End}(\mathcal{V} \otimes \mathcal{V} \otimes \mathbb{C}^n)$

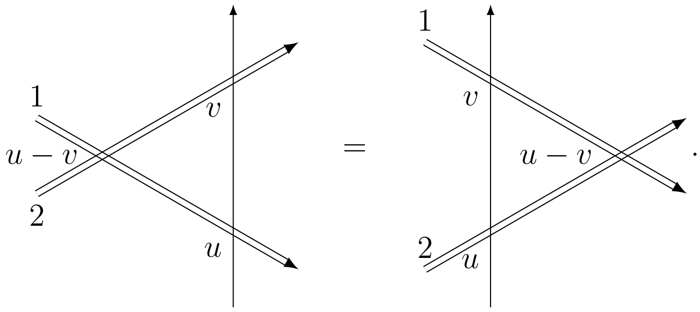
$$\mathcal{R}_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}_{12}(u-v). \tag{1.12}$$

Despite their apparent similarity, equations (1.9) and (1.12) are very different. The equation (1.12) involves two copies of $\mathcal{V}$ and only one of $\mathbb{C}$ as opposed to (1.9) where this is flipped. As such, the subscripts on the L-operators entering (1.12) now denote which copy of $\mathcal{V}$ it acts non-trivially on as opposed to which copy of $\mathbb{C}^n$ in (1.9). This is clarified in the graphical form, where we now have

$\mathcal{R}_{12}(u) =$ 

, (1.13)

and accordingly (1.12) takes the form



(1.14)

That the operator $\mathcal{R}(u)$ solves the YBE in $\mathrm{End}(\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V})$ (represented graphically in Figure 1.1) is not guaranteed by this process, however, it is often obtained as a consequence of the details of its construction.

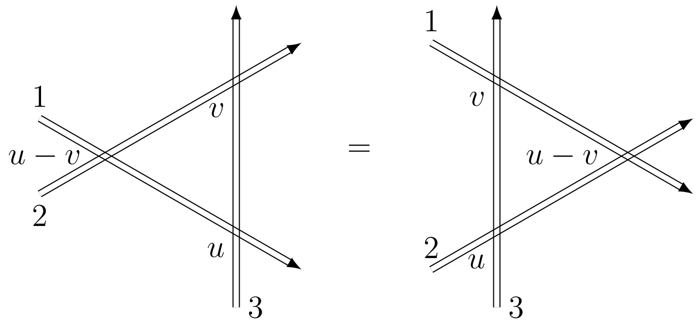


Figure 1.1: Graphical Representation of the YBE (1.2) for $\mathcal{R}(u)$.

1.2 Philosophy of q-deformation

One of the themes of this thesis is the idea of q-deformation or q-analog theory. A q-analog of a result in mathematics is a new result involving a parameter $q \in \mathbb{C}$, such that the original result is recovered in the limit as $q \to 1$. The earliest known examples studied in detail are q-deformed hypergeometric series [14], known as basic hypergeometric series, which date back to the 18th century, and were studied by some of the legends of mathematics including Euler and Gauss. These will play a small role in § 4.

An introduction to q-deformation often begins with the observation that

$$\lim_{q\to 1}\frac{1-q^n}{1-q} = \lim_{q\to 1}(1+q+\cdots+q^{n-1}) = n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.15}$$

This provides the following candidate for a q-deformation of the non-negative integers

$$[\![n]\!]_q := \frac{1-q^n}{1-q}. \tag{1.16}$$

Accordingly, one can define some associated q-deformed combinatorial terms

$$[\![n]\!]_q! := [\![n]\!]_q \cdot [\![n-1]\!]_q \dots [\![2]\!]_q \cdot [\![1]\!]_q, \tag{1.17}$$

$$\left[\!\!\left[\begin{matrix} n \\ k \end{matrix}\right]\!\!\right]_q := \frac{[\![n]\!]_q!}{[\![n-k]\!]_q![\![k]\!]_q!} = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\dots(1-q)}, \tag{1.18}$$

for $n \geq k \geq 0$, where we use the convention $[\![0]\!]_q! = 1$. Using (1.15) it is clear that $[\![n]\!]_q! \to n!$ and $[\![\begin{smallmatrix} n \\ k \end{smallmatrix}]\!]_q \to \binom{n}{k}$ as $q \to 1$.

So far this only shows that formulas (1.16) to (1.18) are q-analogs of the integers and associated combinatorial terms in the sense that they have appropriate limits as $q \to 1$. For these definitions to hold water we would like the q-deformed expressions to have functionality which mirrors that of their undeformed counterparts. For an elementary example of this, recall that the binomial coefficients $\binom{n}{k}$ appear in the binomial theorem

$$(x+y)^n = \sum_k \binom{n}{k} x^{n-k} y^k, \tag{1.19}$$

where $x, y \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$. One of the properties of $\mathbb{C}$ that allows this to hold is commutativity of multiplication. Replacing commutativity with a mild generalisation we obtain the following:

Proposition 1.1. *Let $\mathcal{A}$ be the complex, associative, unital algebra generated by elements x and y subject to the relation $xy = qyx$ for some $q \in \mathbb{C} \setminus \{0, 1\}$. Then for $n \in \mathbb{Z}_{>0}$*

$$(x+y)^n = \sum_{k=0}^{n} \left[\!\!\left[\begin{matrix} n \\ k \end{matrix}\right]\!\!\right]_q y^{n-k} x^k. \tag{1.20}$$

Proof. This insightful proof is from [21]. We begin by writing $(x+y)^n$ as a unique expansion

$$(x+y)^n = \sum_{k=0}^{n} c_{n,k} y^{n-k} x^k, \quad c_{n,k} \in \mathbb{C}$$

since ordered monomials $y^i x^j$ form a basis for $\mathcal{A}$. One can immediately obtain that $c_{n,n} = c_{n,0} = 1$. Otherwise, by making use of the identities $(x+y)^n = (x+y)(x+y)^{n-1} = (x+y)^{n-1}(x+y)$ one obtains the following recurrence relations among the coefficients

$$c_{n,k} = c_{n-1,k} + q^{n-k} c_{n-1,k-1}, \quad c_{n,k} = q^k c_{n-1,k} + c_{n-1,k-1},$$

for $0 < k < n$. By eliminating coefficients $c_{n-1,k}$ we obtain $c_{n,k} = \frac{(1-q^n)}{(1-q^k)} c_{n-1,k-1}$, and it is clear that iteration of this identity gives $c_{n,k} = [\![\begin{smallmatrix} n \\ k \end{smallmatrix}]\!]_q$ by comparing with the right hand side of (1.18). □

This result justifies that the definitions (1.16) to (1.18) are not just contrivances and gives one the flavour of q-analog theory; it considers more general problems which often reveal rich hidden structure. For example, it immediately follows from Proposition 1.1 that $(x+y)^n = x^n + y^n$ provided x and y commute by $xy = \zeta yx$ where ζ is a primitive n-th root of unity.

Before this section concludes we note that in this thesis we will generally use the following convention for q-integers and their associated combinatorial terms

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := [n]_q \dots [2]_q [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}, \tag{1.21}$$

which have the advantage of being symmetric under inversion $q \mapsto q^{-1}$. The definition (1.21) is related to (1.15) by $[n]_q = q^{-n+1} [\![n]\!]_{q^2}$. We can also extend the definition of $[n]_q$ to take any argument $x \in \mathbb{C}$ (or even linear maps provided we are working in a suitable completion) where q^α is always understood to mean the principal value of q^α.

1.3 Structure of Thesis

This thesis is naturally divided into parallel constructions of the L-operator (1.10), and R-matrix (1.13), for the undeformed $U(\mathfrak{sl}_n)$ symmetry and the deformed $U_q(\mathfrak{sl}_n)$ symmetry. The structure of this thesis therefore reflects this. In § 2 we introduce the reader to the algebras $\mathcal{A} = U(\mathfrak{sl}_n), U_q(\mathfrak{sl}_n)$ in §§ 2.1 and 2.2 respectively. We also introduce the representations of interest, which are representations of $\mathcal{A}$ on the space $\mathcal{V}$ of polynomials in $n(n-1)/2$ variables, described by n-parameters $\boldsymbol{\rho} \in \mathbb{C}^n$, and denoted by $\mathcal{V}_{\boldsymbol{\rho}}$.

In § 3 we treat in parallel the defining R-matrix and L-operators $L(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}})$ for both symmetry algebras and in § 4 we treat the R-matrix $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$ by means of the parameter permutation method introduced in [8, 9]. Since the $\mathfrak{sl}_n$ case is well studied, our focus will be on the deformed $U_q(\mathfrak{sl}_n)$ case, particularly the $n = 4$ case where we give an explicit factorisation of the L-operator in § 3.2.1 and a closed form expression for the intertwiners in § 4.2.2 needed to build the R-matrix $\mathcal{R}(u)$. Since the analogy between the undeformed and deformed cases is strong we will often use the same notation for convenience.

Chapter 2

The Symmetry Algebras

The purpose of this chapter is to introduce the two families of symmetry algebras considered in this thesis, with a heuristic approach towards their representation theory. We begin with the (complex) Lie algebra $\mathfrak{sl}_n$ and its universal enveloping algebra $U(\mathfrak{sl}_n)$ in § 2.1 and then introduce its q-deformed universal enveloping algebra $U_q(\mathfrak{sl}_n)$ in § 2.2.

2.1 The Lie Algebra $\mathfrak{sl}_n$

We begin with an abstract description of the complex Lie algebra $\mathfrak{sl}_n$ (for $n > 1$). Let us for convenience denote $\mathfrak{g} = \mathfrak{sl}_n$ for the remainder of this section, unless otherwise specified. Then, $\mathfrak{g}$ is finite dimensional and simple and is hence determined completely by its $(n-1) \times (n-1)$ Cartan matrix

$$a = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{2.1}$$

as follows:

Definition 2.1. $\mathfrak{sl}_n$ is the complex Lie algebra generated by the $3(n-1)$ elements

e_i, f_i, h_i for $i = 1, \ldots, n-1$ subject to the Chevalley-Serre relations [25]

$$[h_i, h_j] = 0, \tag{2.2a}$$
$$[e_i, f_j] = \delta_{ij} h_i, \tag{2.2b}$$
$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{2.2c}$$
$$(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0, \quad (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) = 0, \quad \text{for } i \neq j, \tag{2.2d}$$

for all i, j, where δ_{ij} is the Kronecker delta function, a_{ij} denotes the entries of (2.1), and ad_x is the Lie algebra homomorphism defined by $\mathrm{ad}_x(y) = [x, y]$.

If we define $\mathfrak{g}_0$ to be the (abelian) subalgebra generated by elements h_i and $\mathfrak{g}_\pm$ to be the subalgebra generated by elements e_i and f_i respectively then $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ with the properties $[\mathfrak{g}_0, \mathfrak{g}_\pm] = \mathfrak{g}_\pm$ and $[\mathfrak{g}_+, \mathfrak{g}_-] = \mathfrak{g}_0$. This is known as a triangular decomposition for $\mathfrak{g}$ and is of particular use in its studying representation theory. In a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ the subalgebra $\mathfrak{g}_0$ (known as the Cartan subalgebra) becomes a simultaneously diagonalisable space of operators allowing V to be decomposed into its h_i-eigenspaces. In particular, the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, defined by $x \mapsto \mathrm{ad}_x$, gives rise to the "root space" decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \mathbb{C}^{n-1} \setminus \{0\}} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h_i, x] = \alpha_i x, \text{ for all } i\}. \tag{2.3}$$

The spaces $\mathfrak{g}_\alpha$ are known as root spaces. Only finitely many spaces $\mathfrak{g}_\alpha$ are non-zero and furthermore, one has that each $\mathfrak{g}_\alpha$ is at most 1-dimensional and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ [25].

To obtain an explicit realisation of this decomposition we now construct a spanning set for $\mathfrak{g}$. One method of doing this is to take the union over all sets $W\{e_i, f_i, h_i \mid i = 1, \ldots, n-1\}$ where W is a finite composite of maps ad_{e_i} and ad_{f_j}. A convenient basis is the set $\{E_{ij}\}$, defined iteratively as follows

$$E_{ii} - E_{i+1,i+1} = h_i, \quad \sum_{i=1}^{n} E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i, \tag{2.4a}$$
$$E_{ij} = [E_{i,j-1}, E_{j-1,j}], \quad \text{for } j > i+1, \tag{2.4b}$$
$$E_{ij} = [E_{i,i-1}, E_{i-1,j}], \quad \text{for } j < i-1, \tag{2.4c}$$

where in light of the condition $\sum_{i=1}^{n} E_{ii} = 0$ we should discard one of the E_{ii} (conventionally one discards E_{nn}). The resulting basis $\{E_{ij}\}$ for $\mathfrak{g}$ is known as the Cartan-Weyl basis. This provides a mnemonic description of the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$; $\mathfrak{g}_0 = \text{span}\{E_{ii}\} = \text{span}\{h_i\}$ are the "diagonal" elements, and $\mathfrak{g}_+ = \text{span}\{E_{ij} \mid j > i\}$ ($\mathfrak{g}_- = \text{span}\{E_{ij} \mid j < i\}$) are the "upper" ("lower") triangular ones.

The defining commutation relations for $\mathfrak{g}$ (2.2a) to (2.2d), expressed in the Cartan-Weyl basis take the compact form

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{jk}. \tag{2.5}$$

Notice in particular that this basis diagonalises $\text{ad}_{E_{ii}}$ (and hence ad_{h_i}) since $[E_{ii}, E_{jk}] = (\delta_{ij} - \delta_{ik})E_{jk}$ so the Cartan-Weyl basis realises the decomposition (2.3) with $E_{jk} \in \mathfrak{g}_{\alpha_{jk}}$ ($(\alpha_{jk})_i := \delta_{ij} - \delta_{ik} - \delta_{i+1,j} + \delta_{i+1,k}$). Observe that for $j < k < l$, each upper triangular E_{jk} has distinct ad_{h_i} eigenvalues ($(\alpha_{jk})_i$), and $[E_{jk}, E_{kl}] = E_{jl} \in \mathfrak{g}_{\alpha_{jk}+\alpha_{kl}}$ verifying the claims made previously about the root spaces $\mathfrak{g}_\alpha$, with analogous results holding for lower triangular elements.

Before studying the representation theory of $\mathfrak{g}$, it will be helpful to introduce its universal enveloping algebra (UEA). In fact this can be defined for any Lie algebra.

Definition 2.2. For $\mathfrak{g}$ a (finite-dimensional) complex Lie algebra, define the universal enveloping algebra $U(\mathfrak{g})$ to be the complex, associative, unital algebra constructed as the quotient

$$U(\mathfrak{g}) := T\mathfrak{g}/I, \quad T\mathfrak{g} = \bigoplus_{n=0}^{\infty} \left(\mathfrak{g}^{\otimes n}\right). \tag{2.6}$$

Here $T\mathfrak{g}$ denotes the tensor algebra of $\mathfrak{g}$ where it is understood that $\mathfrak{g}^{\otimes 0} = \mathbb{C}$ and $I \subset T\mathfrak{g}$ is the two sided ideal generated by $[x, y] - (x \otimes y - y \otimes x) \in T\mathfrak{g}$ for all $x, y \in \mathfrak{g}$.

A representation $\rho : \mathfrak{g} \to \text{End}(V)$ defines a representation of $U(\mathfrak{g})$ by iterating the formula $\rho(x \otimes y) = \rho(x)\rho(y)$. Accompanying Definition 2.2 is an important theorem:

Theorem 2.3. *(Poincaré-Birkhoff-Witt) [5, 28] The canonical map from $\mathfrak{g}$ into $U(\mathfrak{g})$ is an inclusion. Furthermore, given an ordered basis $\{X_i \mid i = 1, \ldots, N\}$ for $\mathfrak{g}$ the set*

$$\{X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m} \mid m \in \mathbb{Z}_{\geq 0}, i_1 \leq i_2 \leq \cdots \leq i_m\} \subset U(\mathfrak{g}), \tag{2.7}$$

is a basis for $U(\mathfrak{g})$ known as a Poincaré-Birkhoff-Witt (PBW) basis.

For this thesis, the relevant takeaway of this result is that the UEA $U(\mathfrak{g})$ is a larger (associative and unital) algebra, which contains $\mathfrak{g}$ and allows for elements of $\mathfrak{g}$ to be "multiplied" in a way that realises the Lie-bracket as a commutator:

$$[x, y] = xy - yx, \quad [e_j^n, f_i] = \delta_{ij} n e_i^{n-1} h_i. \tag{2.8}$$

On the right we have included a useful sample calculation in the $\mathfrak{g} = \mathfrak{sl}_n$ case. Note that we omit the tensor product symbols for multiplication in $U(\mathfrak{g})$.

Now we begin the study of the representation theory of $\mathfrak{g} = \mathfrak{sl}_n$. We start with the observation that whilst (2.5) is an equality in $\mathfrak{g}$, which has so far only been defined abstractly, it is satisfied if we replace E_{ij} with e_{ij} the $n \times n$ matrix unit. This is no coincidence. In the defining representation $T : \mathfrak{g} \to \mathrm{End}(\mathbb{C}^n)$ elements E_{ij} are realised as traceless $n \times n$ matrices by the formula

$$T(E_{ij}) = e_{ij} - \frac{\delta_{ij}}{n} I_n, \tag{2.9}$$

where I_n is the $n \times n$ identity matrix. The term $-\frac{1}{n} I_n$ does not affect commutation relation (2.5) and is necessary to ensure the traceless condition holds for diagonal elements $T(E_{ii})$.

The structure of the defining representation (2.9) will serve as a prototype for a large class of representations of particular utility in this thesis. The idea is that all information about the representation is encoded by a single vector $v = (\,0 \,\dots\, 0\; 1\,)^T$. This claim is based on three observations:

1. $T(h_i)v = -\delta_{in} v$.
2. $T(E_{ij})v = 0$ for all $i > j$.
3. $T(E_{in})v = (\,0 \,\dots\, 0\; \overset{i}{1}\; 0 \,\dots\, 0\,)^T$ for $i < n$ and so in particular $\mathbb{C}^n = T(U(\mathfrak{g}_+))v$.

A large class of representations for $\mathfrak{g}$ that are essentially determined by a single vector can be constructed by abstracting the above properties. From now for convenience on we will interchangably use $\mathfrak{g}$-module terminology and notation when discussing representations.

Definition 2.4. A $\mathfrak{g}$-module V is a lowest weight module if it contains $v \in V$ such that $h_i v = m_i v$ (for $m_i \in \mathbb{C}$), $E_{ij} v = 0$ for $i > j$ and $U(\mathfrak{g}_+)v = V$. The $n-1$-tuple [1] $\boldsymbol{m} = (m_1, \dots, m_{n-1}) \in \mathbb{C}^{n-1}$ is known as the lowest weight of V and $v \in V$ is known as the lowest weight vector of V.

[1]It is more precise to regard $\boldsymbol{m}$ as an element of the dual space $\mathfrak{g}_0^*$ such that $hv = \boldsymbol{m}(h)v$ for any $h \in \mathfrak{g}_0$. In this language we have written $\boldsymbol{m}$ in component form with respect to the dual basis of $\{h_1, \dots, h_{n-1}\}$. For our purposes it will be appropriate to regard $\boldsymbol{m}$ as such.

In order to construct lowest weight modules with arbitrary lowest weight $\boldsymbol{m} \in \mathbb{C}^{n-1}$ we now introduce Verma modules. [2]

Definition 2.5. For $\boldsymbol{m} \in \mathbb{C}^{n-1}$ the Verma module $M(\boldsymbol{m})$ is the quotient of $U(\mathfrak{g})$ by the left ideal generated by E_{ij} for $i > j$ and $h_i - m_i.1$.

By construction $M(\boldsymbol{m})$ is a lowest weight module, with lowest weight vector 1: the quotient ensures that $E_{ij}.1 = 0$ and $h_i.1 = m_i.1$. To see that $U(\mathfrak{g}_+).1 = M(\boldsymbol{m})$ we choose a PBW basis for $U(\mathfrak{g})$ based on an ordering of the Cartan-Weyl basis $\{E_{ij}\}$ in which all upper triangular elements appear before all diagonal elements which in turn appear before all lower triangular elements. Then since the lower triangular elements annihilate $1 \in M(\boldsymbol{m})$ and the h_i (and hence the E_{ii}) act by scalar multiplication on 1, we can ignore any PBW basis elements containing any of these elements when considering $U(\mathfrak{g}).1 = M(\boldsymbol{m})$ to obtain $U(\mathfrak{g}_+).1 = M(\boldsymbol{m})$ as desired.

Remark 2.6. The Verma module $M(\boldsymbol{m})$ is infinite dimensional. It is irreducible for $\boldsymbol{m} \in \mathbb{C}^{n-1}$ such that $m_i \notin \mathbb{Z}_{\leq 0}$ and reducible otherwise. It is the maximal lowest weight $\mathfrak{g}$-module in the sense that if V is another lowest weight $\mathfrak{g}$-module with lowest weight $\boldsymbol{m}$, and lowest weight vector v, then there exists a surjective $\mathfrak{g}$-module map $M(\boldsymbol{m}) \twoheadrightarrow V$ defined uniquely by $1 \mapsto v$. Using this, all lowest weight $\mathfrak{g}$-modules appear as a quotient of some Verma module. Since all irreducible, finite dimsenional $\mathfrak{g}$-modules are lowest weight, all such representations arise this way. [25]

We now arrive at a key point for this section. By choosing an appropriate (in the previously discussed sense) PBW basis for $U(\mathfrak{g})$ we obtain the following basis for $M(\boldsymbol{m})$

$$\{{E_{12}}^{\alpha_{12}}{E_{13}}^{\alpha_{13}} \dots {E_{n-1,n}}^{\alpha_{n-1,n}}.1 \mid \alpha_{ij} \in \mathbb{Z}_{\geq 0}\}. \tag{2.10}$$

Forgetting all but the vector space structure, this basis can be regarded as a monomial basis for the space of polynomials in the E_{ij} for $i < j$. Remarkably, for almost all lowest weights $\boldsymbol{m}$ the Verma module $M(\boldsymbol{m})$ has a $\mathfrak{g}$-module realisation as the vector space of polynomials in $n(n-1)/2$ variables x_{ij} for $i > j$, where the generators of $\mathfrak{g}$ are realised as differential operators.

Let $\mathcal{V}$ be the (complex) space of polynomials in $n(n-1)/2$ variables. A basis for $\mathcal{V}$ is $\{\prod_{i>j} x_{ij}^{\alpha_{ij}} \mid \alpha_{ij} \in \mathbb{Z}_{\geq 0}\}$. Let us define some elementary operators on $\mathcal{V}$

[2] Verma modules are normally constructed as highest weight modules, that is, we swap the roles of triangular summands $\mathfrak{g}_+$ and $\mathfrak{g}_-$. These viewpoints are reconciled by the Lie algebra automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ defined on generators by $e_i \mapsto f_i, f_j \mapsto e_j$, and $h_i \mapsto -h_i$.

which are used to construct a $\mathfrak{g}$-module structure. Define the partial derivatives $\partial_{lk} = \partial_{x_{lk}}$ by the action $\partial_{lk}(\prod_{i>j} x_{ij}{}^{\alpha_{ij}}) = \alpha_{lk} x_{lk}^{\alpha_{lk}-1}(\prod_{i>j,(ij\neq lk)} x_{ij}^{\alpha_{ij}})$ on a basis element, and define the multiplication operators x_{lk} to be multiplication by the variable x_{lk}. Care is needed in distinguishing between the multiplication operator $x_{lk} \in \mathrm{End}(\mathcal{V})$, and the vector $x_{lk} \in \mathcal{V}$, however, it is cumbersome to use any other notation for the multiplication operator. The commutation rules between these operators are as follows

$$[\partial_{lk}, \partial_{ij}] = [x_{lk}, x_{ij}] = 0, \quad [\partial_{lk}, x_{ij}] = \delta_{il}\delta_{kj}. \tag{2.11}$$

The last identity can be interpreted as the product rule. More generally one can prove the operator equality $[\partial_{lk}, f(\boldsymbol{x})] = \frac{\partial f(\boldsymbol{x})}{\partial x_{lk}}$ where $f(\boldsymbol{x}) \in \mathrm{End}(\mathcal{V})$ is multiplication by the polynomial $f(\boldsymbol{x}) \in \mathcal{V}$. It is also convenient to work with the "homogeneity" operators $N_{lk} = x_{lk}\partial_{lk}$ which are diagonal with respect to the monomial basis for $\mathcal{V}$, $N_{lk}(\prod_{i>j} x_{ij}^{\alpha_{ij}}) = \alpha_{lk}(\prod_{i>j} x_{ij}^{\alpha_{ij}})$. The commutation relations for N_{lk} can be derived from (2.11).

In order to define a representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathcal{V})$, it suffices to give expressions for the Cartan-Weyl basis elements $\rho(E_{ij}) = E_{ij} \in \mathrm{End}(\mathcal{V})$ (using $\mathfrak{g}$-module notation) and verify the commutation relations. Here we give without proof expressions for $E_{ij} \in \mathrm{End}(\mathcal{V})$

$$E_{ij} = \left(Z D(\rho_1, \rho_2, \ldots, \rho_n) Z^{-1}\right)_{ji}, \tag{2.12}$$

where Z and $D(\rho_1, \rho_2, \ldots, \rho_n)$ are lower-triangular and upper-triangular $n \times n$ matrices respectively, defined as follows

$$Z = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \ldots & x_{n,n-1} & 1 \end{pmatrix}, \tag{2.13}$$

$$\tilde{D}(\rho_1, \rho_2, \ldots, \rho_n) = \begin{pmatrix} -\rho_n & \tilde{D}_{12} & \tilde{D}_{13} & \ldots & \tilde{D}_{1n} \\ & -\rho_{n-1} & \tilde{D}_{23} & \ldots & \tilde{D}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & -\rho_2 & \tilde{D}_{n-1,n} \\ & & & & -\rho_1 \end{pmatrix}, \tag{2.14}$$

where elements of $\tilde{D}$ above the diagonal are given by

$$\tilde{D}_{ij} := -\partial_{ji} - \sum_{k=j+1}^{n} x_{kj}\partial_{ki}. \tag{2.15}$$

The parameters $\rho_i \in \mathbb{C}$ entering (2.14) determine the representation ρ uniquely and obey the constraint $\sum_{i=1}^{n} \rho_i = n(n-1)/2$. The expression (2.12) is due to Derkachov and Manashov [11, 9]. We will see in § 3 that (2.12) is much more than a compact way to express basis elements in the representation ρ. For $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n) \in \mathbb{C}^n$ such that $\rho_i - \rho_{i+1} \notin \mathbb{Z}_{<0}$, (2.12) makes $\mathcal{V}$ into a lowest weight module with lowest weight vector $1 \in \mathcal{V}$, with h_i-eigenvalues given by $h_i.1 = (\rho_{n-i} - \rho_{n+1-i} + 1)$. This provides a realisation of the Verma module $M(\boldsymbol{m})$ where $(\boldsymbol{m})_i = (\rho_{n-i} - \rho_{n+1-i} + 1)$. We will use $\mathcal{V}_{\boldsymbol{\rho}}$ for $\boldsymbol{\rho} \in \mathbb{C}^n$ to denote the $\mathfrak{g}$-module defined on $\mathcal{V}$ by the representation ρ with parameters $\rho_i = (\boldsymbol{\rho})_i$.

Remark 2.7. Two such representations $\rho, \rho' : \mathfrak{g} \to \mathrm{End}(\mathcal{V})$ are isomorphic (as $\mathfrak{g}$-modules) if the defining parameters $(\rho_1, \dots, \rho_n)$ and $(\rho'_1, \dots, \rho'_n)$ are related by a permutation [9]. In § 4 we will see this albeit in a rather different form.

Examples 2.8. Here the two simplest cases, $n = 2, 3$, are considered. For $n = 2$ we express the Cartan-Weyl basis elements using (2.12) as follows

$$\begin{pmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{pmatrix} = \begin{pmatrix} -\rho_2 + \partial_x \cdot x & -\partial_x \\ x(\rho_1 - \rho_2) + x\partial_x \cdot x & -\rho_1 - x\partial_x \end{pmatrix} = \begin{pmatrix} m/2 + N_x & -\partial_x \\ x(m + N_x) & -m/2 - N_x \end{pmatrix}, \tag{2.16}$$

where we have omitted the subscript on $x = x_{21}$. On the second line we set $\rho_1 = m/2, \rho_2 = 1 - m/2$ to realise the $\mathfrak{sl}_2$-Verma module $M(m)$ specifed by the single parameter m (provided $m \notin \mathbb{Z}_{\leq 0}$). The standard basis (2.2a) to (2.2d) for $\mathfrak{sl}_2$ is realised as follows

$$e = x(m + N_x), \quad f = -\partial_x, \quad h = m + 2N_x. \tag{2.17}$$

Observe for example the relation (2.2b)

$$[e, f] = [x, -\partial_x]m + [x^2, -\partial_x]\partial_x = m + 2x\partial_x = h. \tag{2.18}$$

So long as m is not a negative integer, e raises the degree of a monomial and thus by repeatedly applying e to 1 we generate all of V. In the case $m = -N \in \mathbb{Z}_{<0}$, then $U(\mathfrak{g}_+).1 = \mathrm{span}\{x^\alpha \mid \alpha \leq N\}$ is an $\mathfrak{sl}_2$-invariant subspace which can be identified with the $N+1$-dimensional, irreducible $\mathfrak{sl}_2$-module. This shows that $\mathcal{V}_{\boldsymbol{\rho}}$ does not provide a realisation of the Verma module $M(-N)$.

For the $n = 3$ case, the $\mathfrak{sl}_3$ module $\mathcal{V}_{\boldsymbol{\rho}}$ is defined by

$$\begin{pmatrix} E_{11} & E_{21} & E_{31} \\ E_{12} & E_{22} & E_{32} \\ E_{13} & E_{23} & E_{33} \end{pmatrix} = \begin{pmatrix} -\rho_3+2+N_{21}+N_{31} & -\partial_{21} & -\partial_{31} \\ x_{21}A+x_{31}\partial_{32} & -\rho_2+1-N_{21}+N_{32} & -\partial_{32}-x_{21}\partial_{31} \\ -x_{32}x_{21}B+x_{31}(A+B+N_{32}) & x_{32}B-x_{31}\partial_{21} & -\rho_1-N_{31}-N_{32} \end{pmatrix}, \tag{2.19}$$

where

$$A = (m_1 + N_{21} + N_{31} - N_{32}), \quad B = (m_2 + N_{32}), \tag{2.20a}$$

$$m_1 = \rho_2 - \rho_3 + 1, \quad m_2 = \rho_1 - \rho_2 + 1. \tag{2.20b}$$

The m_i obey $h_i.1 = m_i.1$. Consider for example the lower diagonal elements $f_1 = -\partial_{21}, f_2 = -\partial_{32} - x_{21}\partial_{31}$ and $E_{31} = [f_2, f_1] = -\partial_{31}$. They satisfy the cubic Serre relation (2.2d)

$$(\mathrm{ad}_{f_1})^2(f_2) = [f_1, [f_1, f_2]] = [-\partial_{21}, \partial_{31}] = 0. \tag{2.21}$$

We now come to another key point regarding the representation theory of $\mathfrak{g} = \mathfrak{sl}_n$. Suppose we have two representations $\rho_i : \mathfrak{g} \to \mathrm{End}(V_i)$ $(i = 1, 2)$. Then one can define some associated representations

$$\rho_{12} : \mathfrak{g} \to \mathrm{End}(V_1 \otimes V_2), \qquad \rho_{12}(x) := \rho_1(x) \otimes I + I \otimes \rho_2(x), \tag{2.22a}$$

$$\varrho_i : \mathfrak{g} \to \mathrm{End}(V_i^*), \qquad \varrho_i(x)(\varphi) := \varphi \circ \rho_i(-x), \tag{2.22b}$$

where V_i^* denotes the dual space of V_i. One may be inclined to view these formulae as general phenomena valid for any algebra but this is not the case. The real engine behind (2.22a) and (2.22b) is that the UEA $U(\mathfrak{g})$ is in fact a *Hopf algebra*. That is, $U(\mathfrak{g})$ is an associative, unital, (complex) algebra equipped with maps

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad \epsilon : U(\mathfrak{g}) \to \mathbb{C}, \quad S : U(\mathfrak{g}) \to U(\mathfrak{g}), \tag{2.23}$$

where Δ and ϵ are algebra homomorphisms and S is a $\mathbb{C}$-linear bijection. The maps Δ and ϵ are known as the coproduct and counit respectively and satisfy respective "coassociativity" and "counitary" properties

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = x = (\mathrm{id} \otimes \epsilon) \circ \Delta(x). \tag{2.24}$$

In the two rightmost equalities above we have identified elements of $\mathbb{C} \otimes U(\mathfrak{g})$ and $U(\mathfrak{g}) \otimes \mathbb{C}$ with elements of $U(\mathfrak{g})$ in the natural way. For $U(\mathfrak{g})$, Δ and ϵ are defined uniquely by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$ for $x \in \mathfrak{g}$.

The map S, known as the antipode, satisfies the compatibility property

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \mu \circ (\mathrm{id} \otimes S) \circ \Delta(x) = \epsilon(x).1, \tag{2.25}$$

for all $x \in U(\mathfrak{g})$ ($\mu : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ is multiplication). One can show that this property defines S uniquely and furthermore, S is multiplication reversing. For $U(\mathfrak{g})$, S is defined uniquely by $S(x) = -x$ for $x \in \mathfrak{g}$.

Now observe that a Hopf-algebra structure is secretly what was used to define the representations (2.22a) and (2.22b):

$$\rho_{12} = (\rho_1 \otimes \rho_2) \circ \Delta, \quad \varrho_i(x)(\varphi) = \varphi \circ \rho_i \circ S(x). \tag{2.26}$$

The multiplicativity of Δ and anti-multiplicativity of S are precisely what make these representations. Note also that the counit ϵ is exactly a 1-dimensional representation of $U(\mathfrak{g})$. Properties (2.24) ensure that for $\mathfrak{g}$-modules V_1, V_2, V_3, the canonical vector space isomorphisms $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$, and $\mathbb{C} \otimes V_1 \simeq V_1 \otimes \mathbb{C} \simeq V_1$ are also $\mathfrak{g}$-module isomorphisms.

We conclude this section by noting that just as one may consider the opposite algebra structure by reversing multiplication, one can consider the "coopposite" structure of a Hopf algebra by replacing the coproduct Δ with its opposite $\Delta^{\mathrm{op}} = \sigma \circ \Delta$ (where σ is defined by $\sigma(x \otimes y) = y \otimes x$). Unfortunately, whilst $U(\mathfrak{g})$ is not a commutative algebra it is *cocommutative*, that is, $\Delta^{\mathrm{op}} = \Delta$. In the next section we shall see that this makes it rather less interesting than its q-deformation $U_q(\mathfrak{sl}_n)$.

2.2 The Quantum Group $U_q(\mathfrak{sl}_n)$

The quantum group $U_q(\mathfrak{sl}_n)$ is a deformation of the universal enveloping algebra of $\mathfrak{sl}_n$. It is defined using the Cartan matrix (2.1) as follows.

Definition 2.9. For $q = e^h \in \mathbb{C} \setminus \{\pm 1, 0\}$, the quantum group $U_q(\mathfrak{sl}_n)$ is the (complex) associative, unital algebra generated[3] by the $3(n-1)$ elements e_i, f_i, h_i for $i = 1, \ldots, n-1$ subject to the following commutation relations,

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} e_j, \tag{2.27a}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} = \delta_{ij} [h_i]_q, \tag{2.27b}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_q e_i^{1-a_{ij}-m} e_j e_i^m = 0, \quad \text{for } i \neq j, \tag{2.27c}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_q f_i^{1-a_{ij}-m} f_j f_i^m = 0, \quad \text{for } i \neq j, \tag{2.27d}$$

where $\left[{n \atop k} \right]_q$ are the q-binomial coefficients (1.21).

[3]Technically we require that e_i, f_i, h_i are topological generators in the h-adic topology so we can make sense of expressions such as $q_i^h = e^{h.h_i}$.

There are some interesting consequences of allowing $q \in \mathbb{C}$ to be a root of unity, however, this thesis will only consider the non root of unity case. In fact q is generally treated as an indeterminate. With this assumption, one uses the Cartan matrix (2.1), to write formulas (2.27c) and (2.27d) more sensibly as

$$[e_i, e_j] = [f_i, f_j] = 0, \quad \text{for } |i-j| > 1, \tag{2.28a}$$

$$e_i^2 e_{i\pm1} - (q+q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0, \tag{2.28b}$$

$$f_i^2 f_{i\pm1} - (q+q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0. \tag{2.28c}$$

Remark 2.10. Definition 2.9 is technically that of h-adic quantum group $U_h(\mathfrak{sl}_n)$. The standard quantum group $U_q(\mathfrak{sl}_n)$ as defined by Jimbo and Drinfeld [12, 16] is normally presented with generators $k_i = q_i^h$ (invertible), e_i, and f_i, whereby (2.27a) and (2.27b) are replaced by the relations

$$[k_i, k_j] = 0, \tag{2.29a}$$

$$k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \tag{2.29b}$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}. \tag{2.29c}$$

Although $U_h(\mathfrak{sl}_n)$ is a larger algebra than $U_q(\mathfrak{sl}_n)$, this distinction will not be relevant for our purposes and so we opt to work with $U_q(\mathfrak{sl}_n)$ as per Definition 2.9 instead. We may still use the notation $k_i = q^{h_i}$ at times.

The idea with this construction is that $U_q(\mathfrak{sl}_n)$ is an algebra which can be "tuned" by the parameter q. It generalises the universal enveloping algebra $U(\mathfrak{sl}_n)$ in the sense that $U(\mathfrak{sl}_n)$ appears as the limit of $U_q(\mathfrak{sl}_n)$ as $q \to 1$ (or equivalently $h \to 0$). The relations (2.27a) are already identical to (2.2a) and (2.2c). Otherwise, consider for example (2.27b) which is not the same as (2.2b). We can rewrite it as follows

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} = \delta_{ij} \frac{\sinh(h h_i)}{\sinh(h)}, \tag{2.30}$$

making it clear that in the limit $q \to 1$ ($h \to 0$) we recover the $\mathfrak{sl}_n$ relation $[e_i, f_j] = \delta_{ij} h_i$. In a similar way (2.27c) and (2.27d) appear as the appropriate generalisations of the Serre relations (2.2d) using the fact that $\lim_{q\to1} \left[{n \atop k} \right]_q = \binom{n}{k}$.

Generalisations in mathematics are rarely of interest unless they create a richer structure for us to study. There is no exception in this case. If a line of questioning similar to that in the end of § 2.1 is pursued, one may wonder if the algebra $U_q(\mathfrak{sl}_n)$ can be equipped with a coproduct, counit or an antipode map satisfying properties (2.24) and (2.25). The answer to all of these questions is yes:

Proposition 2.11. *There is a unique Hopf algebra structure on $U_q(\mathfrak{sl}_n)$ with coproduct Δ, counit ϵ and antipode S such that*

$$\Delta(k_i) = k_i \otimes k_i, \tag{2.31a}$$

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i, \tag{2.31b}$$

$$\epsilon(k_i) = 1, \quad \epsilon(e_i) = \epsilon(f_i) = 0, \tag{2.31c}$$

$$S(k_i) = k_i^{-1}, \quad S(e_i) = -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i. \tag{2.31d}$$

Proof. See [20] §6.1.2. □

Notice in particular that the coproduct Δ, is not cocommutative (e.g. (2.31b)). It is natural then to ask if the coproduct Δ, and its opposite $\Delta^{\mathrm{op}} = \sigma \circ \Delta$ are related. In fact they are related in a remarkable way:

Proposition 2.12. *$U_q(\mathfrak{sl}_n)$ is a quasitriangular Hopf algebra. That is, there exists an invertible element $\mathcal{R} \in U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{sl}_n)$ such that*

$$\Delta^{\mathrm{op}}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \text{for any } x \in U_q(\mathfrak{sl}_n), \tag{2.32a}$$

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.32b}$$

where $\mathcal{R}_{12} = \sum_i x_i \otimes y_i \otimes 1$, $\mathcal{R}_{13} = \sum_i x_i \otimes 1 \otimes y_i$ and $\mathcal{R}_{23} = \sum_i 1 \otimes x_i \otimes y_i$ given $\mathcal{R} = \sum_i x_i \otimes y_i$. $\mathcal{R}$ is known as a universal R-matrix.

Proof. An explicit form for $\mathcal{R}$ is given in [20] §8.3.2. □

It is not worth presenting the explicit form of the universal R-matrix here, since care is needed in interpretting it properly and it will not be used in this thesis. However, an immediate consequence of Proposition 2.12 worth noting is as follows:

Corollary 2.13. *If $\mathcal{R}$ is a universal R-matrix for $U_q(\mathfrak{sl}_n)$ then it solves the quantum Yang-Baxter equation*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.33}$$

Proof. The proof is a direct application of (2.32a) and (2.32b):

$$\begin{aligned}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} &= \mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\mathcal{R}) = (\Delta^{\mathrm{op}} \otimes \mathrm{id})(\mathcal{R})\mathcal{R}_{12} \\ &= (\sigma \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R})\mathcal{R}_{12} = (\sigma \otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.\end{aligned}$$

□

At this stage, one may wonder if the quest to construct solutions of the Yang-Baxter equation is over; Proposition 2.12 and Corollary 2.13 assert the existence of and give an explicit form for a solution to the Yang-Baxter equation with $U_q(\mathfrak{sl}_n)$ symmetry. However, one should recognise that (2.33) is an equation that holds in $U_q(\mathfrak{sl}_n)^{\otimes 3}$, an algebra only defined abstractly. [4] Therefore, equation (2.33) cannot be given a physical meaning in its own right. In order to do this $\mathcal{R}$ should be evaluated in some representation.

This provides two routes by which the story may progress. First is the construction of $U_q(\mathfrak{sl}_n)$ representations. Second and more subtle, is that although Corollary 2.13 and Proposition 2.12 provide a path to a general solution of the YBE, it may not be the best path when working in a given class of representations. Indeed, in some classes of $U_q(\mathfrak{sl}_n)$-representations, the defining relation (1.12) for $\mathcal{R}(u)$ constructed via the programme outlined in § 1.1, may admit illuminating interpretations which allow for a more elegant construction of a solution and a more beautiful form thereof.

With this in mind we begin a study of the representation theory of $U_q(\mathfrak{sl}_n)$, with the later sections of this thesis in mind. For convenience we will now set $\mathcal{A} = U_q(\mathfrak{sl}_n)$. In the previous section we made use of the Cartan-Weyl basis (2.4a) to (2.4c) in studying the representation theory of $\mathfrak{sl}_n$. A q-analog of this set is $\{E_{ij}\} \subset \mathcal{A}$ for $i, j = 1, \ldots, n$ defined as follows [17]

$$E_{ii} - E_{i+1,i+1} = h_i, \quad \sum_{i=1}^{n} E_{ii} = 0, \quad E_{i,i+1} = e_i, \quad E_{i+1,i} = f_i, \tag{2.34a}$$

$$E_{ij} = [E_{i,j-1}, E_{j-1,j}]_q, \quad \text{for } j > i+1, \tag{2.34b}$$

$$E_{ij} = [E_{i,i-1}, E_{i-1,j}]_{q^{-1}}, \quad \text{for } j < i-1, \tag{2.34c}$$

where $[A, B]_q := AB - qBA$ denotes the q-commutator. Since $\mathcal{A}$ generalises $U(\mathfrak{sl}_n)$ instead of $\mathfrak{sl}_n$ the set $\{E_{ij}\}$ (excluding E_{nn}) is not a basis for $\mathcal{A}$, but we can write an associated PBW basis $\{{E_{12}}^{\alpha_{12}} \ldots {E_{n,n-1}}^{\alpha_{n,n-1}} \mid \alpha_{ij} \in \mathbb{Z}_{\geq 0}\}$. Using this we obtain the following q-analog of the triangular decomposition for $\mathfrak{sl}_n$

$$\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_0 \otimes \mathcal{A}_-, \tag{2.35}$$

where $\mathcal{A}_0$ is the subalgebra generated by the diagonal elements E_{ii} (or equivalently the h_i) and $\mathcal{A}_\pm$ is the subalgebra generated by the upper (lower) triangular elements E_{ij} for $i < j$ ($i > j$) respectively.

[4] It also has no spectral parameter dependence as in (1.2). This can be fixed (see [18] § 4.4).

We now discuss some familiar representation theoretic notions for $\mathcal{A}$ [6]: An $\mathcal{A}$-module V has lowest weight $\boldsymbol{m} = (m_1, \ldots, m_{n-1}) \in \mathbb{C}^{n-1}$ if there exists a vector $v \in V$ such that $h_i v = m_i v$, $E_{ij} v = 0$ for all $i > j$, and $\mathcal{A}_+.v = V$. The quantum Verma module $M_q(\boldsymbol{m})$, for arbitrary $\boldsymbol{m} \in \mathbb{C}^{n-1}$, is constructed as the quotient of $\mathcal{A}$ by the left ideal generated by $(h_i - m_i.1)$ for $i = 1, \ldots, n-1$ and the lower triangular elements E_{ij} for $i > j$. It is an infinite dimensional lowest weight module and is maximal in the sense that for any $\mathcal{A}$-module V with lowest weight $\boldsymbol{m}$ there exists a surjective $\mathcal{A}$-module homomorphism $\phi : M_q(\boldsymbol{m}) \twoheadrightarrow V$. $M_q(\boldsymbol{m})$ is irreducible for generic $\boldsymbol{m}$ but is reducible if any component m_i is a non-negative integer. Furthermore, if $\boldsymbol{m} \in \mathbb{Z}^{n-1}_{\leq 0}$ then $M_q(\boldsymbol{m})$ has a finite dimensional irreducible quotient with lowest weight $\boldsymbol{m}$. All finite dimensional, irreducible $\mathcal{A}$-modules are lowest weight modules and hence appear as a quotient of some Verma module.

One may be concerned that the previous paragraph makes essentially no distinction between the representation theory of $U(\mathfrak{sl}_n)$ and $\mathcal{A}$. Ultimately, this serves only to emphasise how similar the two cases truly are as long as q is not a root of unity. What can break in the root of unity case is that distinct h_i-eigenvalues do not correspond to distinct $k_i = q_i^h$-eigenvalues, which is relevant if we are using the standard definiton of $U_q(\mathfrak{sl}_n)$. This gives rise to *cyclic* representations where $e^N, f^N, k^N \propto I$, if q was a primitive N-th root of unity. Provided q is not a root of unity, differences between the two cases often involve simply replacing combinatorial factors with their q-analogs. For example one can inductively prove the following generalisation of (2.8)

$$[e_j^n, f_i] = \delta_{ij}[n]_q e^{n-1} \frac{q^{n-1+h_i} - q^{-n+1-h_i}}{q - q^{-1}} \quad \Rightarrow \quad (f_i e_i^n).v = [n]_q [m_i + n - 1]_q e_i^{n-1} v, \tag{2.36}$$

where on the right hand side we are supposing that v is the lowest weight of some $\mathcal{A}$-module V.

By using an iterative process to construct representations of $\mathcal{A}$ on $\mathcal{V}$ it has been shown that the $\mathcal{A}$-Verma module $M(\boldsymbol{m})$ has a similar realisation, as in the undeformed case, as a space of polynomials in $n(n-1)/2$ variables where the generators of $\mathcal{A}$ act as q-difference operators [10] (provided that no component of $\boldsymbol{m}$ is a non-negative integer). The details of this construction will not be given here, however, let us summarise the elementary difference operators used to construct these representations.

Let $\mathcal{V}$ be the (complex) space of polynomials in the $n(n-1)/2$ variables x_{ij}

for $1 \leq j < i \leq n$ and define the invertible q-shift operator $\mathcal{T}_{ij}$ $(i > j)$ by

$$\mathcal{T}_{ij} f(x_{21}, \ldots, x_{ij}, \ldots, x_{n,n-1}) = f(x_{21}, \ldots, qx_{ij}, \ldots, x_{n,n-1}), \tag{2.37}$$

where $f(\boldsymbol{x}) \in \mathcal{V}$ is an arbitrary polynomial. By checking the action on a monomial eigenbasis for $\mathcal{T}_{ij}$ one may write the shift operator in the exponential form $q^{N_{ij}} = e^{hN_{ij}}$ where N_{ij} is the homogeneity operator from § 2.1.

Remark 2.14. Generally, operator exponentials should be cause for concern due to their complicated commutation relations. However, in this case the homogeneity operators N_{ij}, are a family of commuting operators on $\mathcal{V}$, so as long as the multiplication operators x_{ij} do not appear in exponents (which they will not) one can compose exponential operators by simply adding their indices $q^{N_{ij}}.q^{N_{lk}} = q^{N_{ij}+N_{lk}} = q^{N_{lk}}.q^{N_{ij}}$.

Now the q-anologue of the derivative operator ∂_{ij} can be defined using the shift operator (2.37). Define $D_{ij} \in \mathrm{End}(\mathcal{V})$ as

$$D_{ij} := \frac{1}{x_{ij}}.\frac{\mathcal{T}_{ij} - \mathcal{T}_{ij}^{-1}}{q - q^{-1}} = \frac{1}{x_{ij}}.\frac{q^{N_{ij}} - q^{-N_{ij}}}{q - q^{-1}} = \frac{1}{x_{ij}}[N_{ij}]_q, \tag{2.38}$$

where $\frac{1}{x_{ij}}$ denotes the one sided inverse of the multiplication operator x_{ij}. [5] In the last equality we have extended the definition of the q-number (1.21) to allow operator arguments; so long as we restrict to a family of commuting operators we are free to use all results from Appendix A.1, since these only require additivity of indices under multiplication. Applying (2.38) to an arbitrary monomial gives $D_{ij} \prod_{l>k} x_{lk}^{\alpha_{lk}} = [\alpha_{ij}]_q x_{ij}^{\alpha_{ij}-1} \prod_{l>k,l,k\neq i,j} x_{lk}^{\alpha_{lk}}$ from which ones sees that $\partial_{ij} = \lim_{q\to 0} D_{ij}$.

A collection of helpful composition and commutation relations amongst operators $q^{N_{ij}}, D_{ij}$, and the mutliplication operator x_{ij} is given below

$$D_{ij}^m = \frac{1}{x_{ij}^m}[N_{ij}]_q[N_{ij} - 1]_q \ldots [N_{ij} - (m-1)]_q, \quad \text{for } m \geq 1 \tag{2.39a}$$

$$[D_{ij}, x_{lk}]_{q^{\pm 1}} = \delta_{il}\delta_{lk} q^{\mp N_{ij}}, \tag{2.39b}$$

$$P(N_{ij}) x_{lk}^m = x_{lk}^m P(N_{ij} + \delta_{il}\delta_{jk} m), \qquad q^{\alpha N_{ij}} P(x_{lk}) = P(q^{\delta_{il}\delta_{jk}\alpha} x_{lk}) q^{\alpha N_{ij}}, \tag{2.39c}$$

for $m \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$ where $P(\ )$ is an operator constructed as some power series in its argument.

[5]The term $q^{N_{ij}} - q^{-N_{ij}}$ gives 0 when evaluated on any term $\propto x_{ij}^0$ which ensures D_{ij} is well defined.

Unfortunately there is no known closed form expression analogous to (2.12) to (2.15) which expresses the generators of $\mathcal{A}$, realised in End($\mathcal{V}$), in compact form. The $\mathcal{A}$ module structure on $\mathcal{V}$ [10] is, however, parameterised by the same n parameters ρ_i, such that $h_i.1 = m_i.1$ where $m_i = (\rho_{n-i} - \rho_{n+1-i} + 1)$, and $\sum \rho_i = n(n-1)/2$. We will denote the $\mathcal{A}$-module on $\mathcal{V}$ defined by parameters $\rho_i = (\boldsymbol{\rho})_i$ as $\mathcal{V}_{\boldsymbol{\rho}}$ as in the undeformed case. Let us examine the two simplest cases.

Examples 2.15. In the $n = 2$ case we construct a representation on $\mathcal{V}$, the space of polynomials in the single variable $x_{21} = x$, with lowest weight $m \in \mathbb{C}$. We will construct this representation using a method outlined in [26]. Start with the Verma module (now as a highest weight module) $M(m)$ with highest weight vector v, i.e. $e.v = (h + m.\mathrm{id}) = 0$. Now extend scalars to take values in the completion of $\mathcal{V}$ (power series) and consider the "generating" vector $w := e_{q^{-2}}(-x(1-q^{-2})f)v$ where the q-exponential e_q is as per (A.12). The idea is that for any $A \in U_q(\mathfrak{sl}_2)$ we can represent the action of A on w as a differential operator a_x acting on the variable x. Importantly this reverses the order of operation

$$(AB)w = (b_x a_x)(w). \tag{2.40}$$

We can fix this by supposing that the $U_q(\mathfrak{sl}_2)$ generators start with their opposite commutation relations

$$[e, f] = -\frac{q^h - q^{-h}}{q - q^{-1}}, \quad q^h e = q^{-2} e q^h, \quad q^h f = q^2 f q^h, \tag{2.41}$$

then the differential realisations of these generators will satisfy the correct $U_q(\mathfrak{sl}_2)$ commutation relations.

Now let us use the series realisation of $e_{q^{-2}}$ to calculate

$$\begin{aligned} q^h w &= \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-x)^n}{(q^{-2};q^{-2})_n} q^h f^n v = \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-x)^n}{(q^{-2};q^{-2})_n} q^{2n} f^n q^h v \\ &= q^m \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-q^2 x)^n}{(q^{-2};q^{-2})_n} f^n v = q^{m+2N_x} w. \end{aligned} \tag{2.42}$$

We now use the commutation rule (2.36) (with the roles of e and f swapped due to them obeying (2.41)) to calculate

$$
\begin{aligned}
e.w &= \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-x)^n}{(q^{-2};q^{-2})_n} e f^n v \\
&= \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-x)^n}{(q^{-2};q^{-2})_n} f^n e v - \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n(-x)^n}{(q^{-2};q^{-2})_n} [n]_q f^{n-1} \frac{q^{n-1+h}-q^{-n+1-h}}{q-q^{-1}} v \\
&= \frac{x}{q-q^{-1}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n q^n (q^m(-qx)^n - q^{-m}(-q^{-1}x)^n)}{(q^{-2};q^{-2})_n} f^n v \\
&= xq^{N_x}[m+N_x]_q.w. \qquad (2.43)
\end{aligned}
$$

A similar but less complicated calculation shows that f is realised as $-\frac{1}{x}[N_x]_q q^{-N_x+1}$. To summarize we have the following expressions for generators of $U_q(\mathfrak{sl}_2)$ realised as difference operators in $\mathrm{End}(\mathcal{V})$

$$
e = xq^{N_x}[N_x+m]_q, \quad f = -\frac{1}{x}[N_x]_q q^{-N_x+1} = -D_x q^{-N_x+1}, \qquad (2.44a)
$$

$$
E_{11} = -E_{22} = m/2 + N_x, \quad h = m + 2N_x. \qquad (2.44b)
$$

Observe for example the commutation relation

$$
\begin{aligned}
[e,f] &= xq^{N_x}[N_x+m]_q \left(-\frac{1}{x}[N_x]_q q^{-N_x+1}\right) - \left(-\frac{1}{x}[N_x]_q q^{-N_x+1}\right) xq^{N_x}[N_x+m]_q \\
&= -[N_x+m-1]_q[N_x]_q + [N_x+1]_q[N_x+m]_q = [2N_x+m]_q,
\end{aligned}
$$

where the last equality is due to (A.4). The vector $1 \in \mathcal{V}$ satisfies $f.1 = 0$ and $h.1 = m$ and we notice that for $m \notin \mathbb{Z}_{\leq 0}$, $(e)^j.1 \propto x^j$ is non-zero and so $U_q(\mathfrak{sl}_2)_+.1 = \mathcal{V}$. This fails for $m \in \mathbb{Z}_{\leq 0}$ as $(e)^{1-m}$ will annhilate 1. In this case $U_q(\mathfrak{sl}_2)_+.1 = \mathrm{Span}\{x^j \mid j \leq -m\} \subset \mathcal{V}$ is a $U_q(\mathfrak{sl}_2)$ invariant subspace, isomorphic to an irreducible, $(1-m)$-dimensional representation of $U_q(\mathfrak{sl}_2)$. Therefore, we see analogously to the $\mathfrak{sl}_2$ case that $\mathcal{V}_{(\rho_1,1-\rho_1)}$ realises the Verma module $M_q(m)$ if $m = 2\rho_1 \notin \mathbb{Z}_{\leq 0}$, but not otherwise.

An important observation is that the expressions (2.44a) and (2.44b) for generators are not unique. There is freedom due to the fact that we can rescale our variables $x := \lambda x'$ for a constant $\lambda \in \mathbb{C}$ without changing homogeniety operators $N_x = N_{x'}$. In fact we can even rescale our variables by operators so long as our new variables still all commute with each other. For example, we can rewrite (2.44a) and (2.44b) in the rescaled variable $x \mapsto xq^{-N_x}$ to give

$$
\tilde{e} = x[N_x+m]_q, \quad \tilde{f} = -\frac{1}{x}[N_x]_q = -D_x, \qquad (2.45a)
$$

$$
\tilde{E}_{11} = -\tilde{E}_{22} = m/2 + N_x, \quad \tilde{h} = m + 2N_x. \qquad (2.45b)
$$

Perhaps the best way to see that this is a valid way to rewrite (2.44a) and (2.44b) is that it can be realised as the similarity transformation $\tilde{E}_{ij} = q^{-\frac{(N_x-1)}{2}N_x} E_{ij} q^{\frac{(N_x-1)}{2}N_x}$. To see this it is enough to check the relations

$$q^{-\frac{(N_x-1)}{2}N_x}(N_x)q^{\frac{(N_x-1)}{2}N_x} = N_x,$$
$$q^{-\frac{(N_x-1)}{2}N_x} x q^{\frac{(N_x-1)}{2}N_x} = xq^{-\frac{N_x}{2}(N_x+1-(N_x-1))} = xq^{-N_x}.$$

For the $n = 3$ case we construct a lowest weight representation on $\mathcal{V}$, the space of polynomials in x_{21}, x_{31}, x_{32}. This can be done by the same method as in the $n = 2$ case except now with "generating vector"

$$w = e_{q^{-2}}(-x_{21}(1-q^{-2})E_{21})e_{q^{-2}}(-x_{31}(1-q^{-2})E_{31})e_{q^{-2}}(-x_{32}(1-q^{-2})E_{32})v, \tag{2.46}$$

where v is the highest weight vector of the (highest weight) $U_q(\mathfrak{sl}_3)$ Verma module $M(\boldsymbol{m})$. We will use the following transformed versions of generators which appear in [26]

$$f_1 = -D_{21}, \quad f_2 = -D_{32}q^{N_{21}-N_{31}} - x_{21}D_{31}, \tag{2.47a}$$
$$E_{31} = [f_2, f_1]_{q^{-1}} = -D_{31}q^{-(N_{21}+1)}, \tag{2.47b}$$
$$e_1 = x_{31}D_{32}q^{N_{32}-N_{31}-m_1} + x_{21}[A]_q, \tag{2.47c}$$
$$e_2 = x_{32}[B]_q - x_{31}D_{21}q^{2N_{32}+m_2}, \tag{2.47d}$$
$$E_{13} = [e_1, e_2]_q = -x_{21}x_{32}[B]_q q^A + x_{31}[A+B+N_{32}]_q q^{N_{21}+1}, \tag{2.47e}$$
$$E_{11} = -\rho_3 + 2 + N_{21} + N_{31}, \quad E_{22} = -\rho_2 + 1 - N_{21} + N_{32}, \tag{2.47f}$$
$$E_{33} = -\rho_1 - N_{31} - N_{32}, \tag{2.47g}$$
$$h_1 = 2N_{21} + N_{31} - N_{32} + m_1, \quad h_2 = -N_{21} + N_{31} + 2N_{32} + m_2, \tag{2.47h}$$

where A, B, and the ρ_i are as per (2.20a) and (2.20b). The expressions in [26] are transformed to ours by the similarity transformations $x_{ij} \mapsto x_{ij}q^{-N_{ij}}$ (and also $E_{31}, E_{13} \mapsto q^{-1}E_{31}, qE_{13}$ respectively, due to differing conventions). We check for example the relation

$$[e_1, f_1] = [D_{21}, x_{31}D_{32}q^{N_{32}-N_{31}-m_1} + x_{21}[A]_q] = ([N_{21}+1]_q[A]_q - [A-1]_q[N_{21}]_q)$$
$$= [N_{21}+A]_q = [2N_{21}+N_{31}-N_{32}+m_1]_q = [h_1]_q.$$

Note that in both the $n = 2$ and $n = 3$ examples the expressions for generators (2.44a) and (2.44b), and (2.47a) to (2.47h) return to the undeformed expressions (2.16) and (2.19) in the limit as $q \to 1$. This is because the structure of the representations are essentially the same. To go from the q-deformed expression to the undeformed is easy. For example, in (2.47a) to (2.47h) if we replace all q-number expressions with their argument ($[A]_q \mapsto A$), and take all q-exponent terms equal to 1 we recover the undeformed generators.

Unfortunately, going from the undeformed expressions to deformed ones is not as easy. By comparing the expressions (2.16) and (2.19) with (2.44a) and (2.44b), and (2.47a) to (2.47h), one can see that all expressions for diagonal elements E_{ii} are the same. Otherwise, for E_{ij} such that $i \neq j$, one may hypothesise that by factoring the undeformed expressions so that they are a sum of terms of the form $X(N+c)$, where the X are distinct monomials in the variables x_{ij} (perhaps containing negative powers), N are linear combinations of the N_{ij}, and $c \in \mathbb{C}$, then we can determine the form of the deformed expressions by performing the replacement $(N + c) \to [N + c]_q q^a$. The problem with this is that one is forced to introduce arbitrary exponents q^a, where a is some combination of the N_{ij} (and a constant). The terms q^a then need to be fixed in order to satisfy the deformed algebra relations (2.27a) to (2.27d). Furthermore, as demonstrated, these exponents are not uniquely determined. For small cases this approach is feasible, however, it quickly becomes an unmanageable computation.

Chapter 3

L-operators

The purpose of this chapter is to introduce the L-operators with $\mathcal{A} = U(\mathfrak{sl}_n)$ or $U_q(\mathfrak{sl}_n)$ symmetry, which act in the tensor product of $\mathbb{C}^n$ (the defining representation of $\mathcal{A}$) with the $\mathcal{A}$-module $\mathcal{V}_{\boldsymbol{\rho}}$ of polynomials in $n(n-1)/2$ variables.

The defining relation for these L-operators is an RLL-relation with the defining R-matrix for $\mathcal{A}$. The defining R-matrix is an $n^2 \times n^2$ matrix $R(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ which solves the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n). \quad (3.1)$$

Here the subscripts on an operator denote the spaces on which it acts non-trivially, e.g $R_{12}(u) = R(u) \otimes I$.

The L-operator is written as $L(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}})$ and the defining RLL-relation then takes the form

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}}), \quad (3.2)$$

where $L_1(u) = L(u) \otimes I$ and $L_2(u) = I \otimes L(u)$.

Our method for constructing these L-operators will be to introduce universal L-operators $\tilde{L}(u) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}$ which satisfy the RLL-relation (1.9). The desired L-operators $L(u) \in \mathrm{End}(\mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}})$ are then obtained by evaluating the second factor of $\tilde{L}(u)$ in the representation $\rho : \mathcal{A} \to \mathcal{V}_{\boldsymbol{\rho}}$.

3.1 Undeformed Case

In this section we give the defining R-matrix and L-operator for the undeformed universal enveloping algebra $\mathcal{A} = U(\mathfrak{sl}_n)$. The defining R-matrix $R(u) \in$

$\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is the following simple solution to the Yang-Baxter equation

$$R(u) = uI + P, \tag{3.3}$$

where P is the permutation matrix $P(v \otimes w) = w \otimes v$.

In order to construct a universal L-operator for $\mathcal{A}$ we now introduce the split Casimir operator $\hat{C} \in \mathcal{A} \otimes \mathcal{A}$. For a basis $\{X_a\}$ of $\mathfrak{sl}_n$ we denote by g^{ab}, components of the inverse of the (symmetric) Killing form matrix. The quadratic casimir $C_2 \in \mathcal{A}$ is a central element defined to be the basis independent expression $C_2 = g^{ab} X_a.X_b$ where we are using implicit summation convention. The split Casimir operator $\hat{C} \in \mathcal{A} \otimes \mathcal{A}$ is defined to be

$$\hat{C} = g^{ab} X_a \otimes X_b = g^{ab} X_b \otimes X_a. \tag{3.4}$$

An important property of (3.4) is that it can be built from the quadratic Casimir element as follows [15]

$$\hat{C} = \frac{1}{2} \left(\Delta(C_2) - C_2 \otimes 1 - 1 \otimes C_2 \right). \tag{3.5}$$

An immediate consequence of this is that $[\Delta(\mathcal{A}), \hat{C}] = 0$, where Δ is the coproduct (2.23). Using the notation $\hat{C}_{12} = g^{ab} X_a \otimes X_b \otimes 1$ and likewise for $\hat{C}_{13}$, and $\hat{C}_{23}$ it can be seen that

$$[\hat{C}_{12} + \hat{C}_{13}, \hat{C}_{23}] = g^{ab} X_a \otimes \Big(\Delta(X_b)\hat{C}_2 - \hat{C}_2 \Delta(X_b) \Big) = 0. \tag{3.6}$$

By permuting tensor factors in (3.6) and exploiting the symmetry of (3.4) one can arrive at other equivalent expressions such as $[\hat{C}_{12}, \hat{C}_{13} + \hat{C}_{23}] = [\hat{C}_{12} + \hat{C}_{23}, \hat{C}_{13}] = 0$.

The split Casimir element for $\mathfrak{sl}_n$ can be written in the form [7]

$$\hat{C} = \sum_{i,j=1}^{n} \tilde{E}_{ij} \otimes \tilde{E}_{ji}, \tag{3.7}$$

where the $\tilde{E}_{ij}$ is the Cartan-Weyl basis for $\mathfrak{gl}(n)$ and is related to the basis (2.4a) to (2.4c) by $\tilde{E}_{ij} = E_{ij} + \delta_{ij} \frac{1}{n} I$ (viewed as an equality in $U(\mathfrak{sl}_n)$). In the defining representation of $\mathfrak{gl}(n)$ one takes $\tilde{E}_{ij} \mapsto e_{ij}$, where e_{ij} denotes the matrix unit. Now consider the element $C(u) = uI + \hat{C} \in \mathcal{A} \otimes \mathcal{A}$. Evaluating both tensor factors in the defining representation T, we recover the universal R-matrix (3.3)

$$(T \otimes T)C(u) = uI + e_{ij} \otimes e_{ji} = uI + P. \tag{3.8}$$

With this we can now prove the following:

Proposition 3.1. *Define*

$$\tilde{L}(u) = (T \otimes \mathrm{id})(C(u)) = uI + e_{ij} \otimes \tilde{E}_{ji} \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{A}, \tag{3.9}$$

where $T : \mathcal{A} \to \mathrm{End}(\mathbb{C}^n)$ *is the defining representation. Then* $\tilde{L}(u)$ *is a universal* L*-operator for the defining* R*-matrix* (3.3)*, that is, it satisfies* (1.9)*.*

Proof. By explicit calculation one can check that terms with quadratic and cubic dependence on spectral parameters cancel directly out of (1.9). Thus using (3.8) the difference between the left and right hand sides of (1.9) reduces to

$$\begin{aligned} LHS - RHS = &(T \otimes T \otimes \mathrm{id}) \left(v[\hat{C}_{12}, \hat{C}_{13}] + u[\hat{C}_{12}, \hat{C}_{23}] + (u-v)[\hat{C}_{13}, \hat{C}_{23}] \right) \\ &+ P_{12} \left((T \otimes T \otimes \mathrm{id})(\hat{C}_{13}\hat{C}_{23}) \right) - \left((T \otimes T \otimes \mathrm{id})(\hat{C}_{23}\hat{C}_{13}) \right) P_{12} \\ = &(T \otimes T \otimes \mathrm{id}) \left(v[\hat{C}_{12} + \hat{C}_{23}, \hat{C}_{13}] + u[\hat{C}_{12} + \hat{C}_{13}, \hat{C}_{23}] \right), \end{aligned}$$

which vanishes as a consequence of the identity (3.6). □

This proposition gives a universal L-operator with $\mathfrak{sl}_n$ symmetry. It is rewritten in terms of the Cartan-Weyl elements for $\mathfrak{sl}_n$ as $\tilde{L}(u) = (u - 1/n)I + e_{ij} \otimes E_{ji}$. By absorbing the $-1/n$ into u ($u \mapsto u + 1/n$) we obtain the following simple expression for the $\tilde{L}(u)$

$$\tilde{L}(u) = uI + e_{ij} \otimes E_{ji}. \tag{3.10}$$

Viewed as an $\mathcal{A}$ valued $n \times n$ matrix (3.10) describes a matrix with $u + E_{ii}$ on the i-th diagonal and E_{ji} in the i, j-th entry for distinct i and j.

Universality of the L-operator (3.10) means that the second tensor factor can be evaluated in any representation. Let us now evaluate the second tensor factor in the representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$ on the space of polynomials in $n(n-1)/2$ variables from § 2.1. Using the formula (2.12) this evaluation yields

$$\begin{aligned} L(u; \rho_1, \ldots, \rho_n) := (\mathrm{id} \otimes \rho)(L(u)) &= uI + Z\tilde{D}(\rho_1, \ldots, \rho_n)Z^{-1} \\ &= Z(\tilde{D}(u; \rho_1, \ldots, \rho_n))Z^{-1}, \end{aligned} \tag{3.11}$$

where $\tilde{D}(u; \rho_1, \ldots, \rho_n) := uI + D(\rho_1, \ldots, \rho_n)$. Notice that the representation parameters ρ_i and spectral parameter u only appear on the diagonal of the central factor $\tilde{D}$ where they are absorbed into the combinations $u_i := u - \rho_i$. We will therefore write $L(u_1, \ldots, u_n) = L(u; \rho_1, \ldots, \rho_n)$ and $\tilde{D}(u_1, \ldots, u_n) = \tilde{D}(u; \rho_1, \ldots, \rho_n)$ or $L(\boldsymbol{u})$ and $\tilde{D}(\boldsymbol{u})$ for short.

Remark 3.2. The equation (3.11) realises the expression (2.12) as not only a compact way to express generators of an $\mathfrak{sl}_n$ representation, but as an L-operator (evaluated at $u = 0$), a solution of the Yang-Baxter equation in the form (3.2).

Examples 3.3. Let us construct explicitly the L-operators (3.11) for the $n = 2, 3$ cases. Using Examples 2.8 we can construct these immediately. In the $n = 2$ case we have (using $x_{21} = x$)

$$L(u_1, u_2) = \begin{pmatrix} u_2 + 1 + N_x & -\partial_x \\ x(u_2 - u_1 + 1 + N_x) & u_1 - N_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} u_2 & -\partial_x \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}, \tag{3.12}$$

where $u_2 - u_1 + 1 = \rho_1 - \rho_2 + 1 = m$ is the lowest weight of the representation $h.1 = m.1$. In the $n = 3$ case we have

$$L(u_1, u_2, u_3) = \begin{pmatrix} u_3 + 2 + N_{21} + N_{31} & -\partial_{21} & -\partial_{31} \\ x_{21}A + x_{31}\partial_{32} & u_2 + 1 - N_{21} + N_{32} & -\partial_{32} - x_{21}\partial_{31} \\ -x_{32}x_{21}B + x_{31}(A + B + N_{32}) & x_{32}B - x_{31}\partial_{21} & u_1 - N_{31} - N_{32} \end{pmatrix}, \tag{3.13}$$

where A and B are as per (2.20a) and are expressed in terms of the u_i as

$$A = (u_3 - u_2 + 1 + N_{21} + N_{31} - N_{32}), \quad B = (u_2 - u_1 + 1 + N_{32}). \tag{3.14}$$

The lowest weight $\boldsymbol{m} = (m_1, m_2) \in \mathbb{C}^2$, such that $h_i.1 = m_i.1$, is recovered from the u_i as $m_1 = u_3 - u_2 + 1, m_2 = u_2 - u_1 + 1$.

3.2 Deformed Case

In this section we give the defining R-matrix and the universal L-operator for the algebra $\mathcal{A} = U_q(\mathfrak{sl}_n)$. The defining R-matrix $R(u) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is given explicitly by [17]

$$R(u) = P.\hat{R}(u), \quad \hat{R}(u) = q^u R + q^{-u} R^{-1}, \tag{3.15a}$$

$$(R)_{\boldsymbol{ij}} = \delta_{i_1,j_1}\delta_{i_2,j_2}(1 + (q-1)\delta_{i_1,i_2}) + (q - q^{-1})\delta_{i_1,j_2}\delta_{i_2,j_1}\sigma_{i_1,i_2}, \tag{3.15b}$$

where $P \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is the permutation matrix and $\sigma_{i,j}$ is given by

$$\sigma_{i_1,i_2} = \begin{cases} 1, & i_1 < i_2, \\ 0, & \text{else}. \end{cases} \tag{3.16}$$

We are indexing rows (columns) of R by pairs $\boldsymbol{i} = (i_1, i_2)$ ($\boldsymbol{j} = (j_1, j_2)$) for clarity. The YBE for $R(u)$ can be proved by making use of the following facts for the matrix R (3.15b)

$$(R^{-1} - q^{-1}I)(R^{-1} + qI) = 0, \quad R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{3.17}$$

We present without proof here the universal L-operator for $\mathcal{A}$ introduced by Jimbo [17]

$$\tilde{L}(u) = \sum_{ij} e_{ij} \otimes \hat{E}_{ji}(u), \tag{3.18a}$$

$$\hat{E}_{ij}(u) = \begin{cases} q^{u+(E_{ii}+E_{jj}-1)/2} E_{ij}, & j > i, \\ (q^{-1})^{u+(E_{ii}+E_{jj}-1)/2} E_{ij}, & i > j, \\ [u + E_{ii}]_q, & i = j, \end{cases} \tag{3.18b}$$

where E_{ij} are the q-analogs of the Cartan-Weyl elements (2.34a) to (2.34c). Viewing $\tilde{L}(u)$ as an $\mathcal{A}$-valued $n \times n$ matrix observe that it admits the decomposition

$$\tilde{L}(u) = q^u L^+ - q^u L^-, \tag{3.19}$$

where L^+ (L^-) is upper (lower) triangular and independent of u. Using the forms (3.15a) and (3.19) one can check that the defining relation for $\tilde{L}(u)$ (1.9) is equivalent to the relations

$$L_1^{\pm} L_2^{\pm} R = R L_2^{\pm} L_1^{\pm}, \quad L_1^+ L_2^- R = R L_2^- L_1^+. \tag{3.20}$$

Remark 3.4. Notice that for $i \neq j$ we have $\hat{E}_{ij}(u) \to E_{ij} \in \mathfrak{sl}_n$ as $q \to 1$ and on the diagonals we have $\hat{E}_{ii}(u) \to (u + E_{ii}) \in U(\mathfrak{sl}_n)$. From this it is clear that one recovers the undeformed L-operator (3.10) from (3.18a) and (3.18b) in the limit as $q \to 1$.

Remark 3.5. This whole picture can be flipped on its head. Suppose one starts with a matrix $M \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$. Then, by $U(M)$ denote the algebra generated by elements L_{ij}^+ for $i \leq j$ and L_{ij}^- for $i \geq j$ such that $L_{ii}^+ L_{ii}^- = L_{ii}^- L_{ii}^+ = 1$, and when packaged into the upper (lower) triangular matrices $(L^{\pm})_{ij} = L_{ij}^{\pm}$ respectively, the relations (3.19) are satisfied with R replaced by M. Then $U(M)$ can be given a Hopf algebra structure with coproduct Δ, counit ϵ, and antipode S defined by

$$\Delta(L_{ij}^{\pm}) = \sum_k L_{ik}^{\pm} \otimes L_{kj}^{\pm}, \quad \epsilon(L_{ij}^{\pm}) = \delta_{ij}, \quad S(L^{\pm}) = (L^{\pm})^{-1}, \tag{3.21}$$

where $L^{\pm}$ are invertible as triangular matrices with units on their diagonals. Performing this process with the specific choice of matrix $M = R$ as per (3.15b) one recovers the algebra $U_q(\mathfrak{sl}_n)$. [1] In other words, the $U_q(\mathfrak{sl}_n)$ algebra relations are not only sufficient for the relation (1.9) to hold for $R(u)$ and $L(u)$ as per (3.15a) and (3.18a) respectively, they are necessary! See [20] §§ 8.4, 8.5 for more details.

As before let us now evaluate the second tensor factor of the universal L-operator in the representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$ on the space of polynomials in $n(n-1)/2$ from § 2.2. As noted in § 2.2, there is no known closed form expression for the generators of this representation analagous to (2.12) and hence no such expression for the L-operator. However, we note that this L-operator admits the same parameterisation by variables $(u_1, \ldots, u_n) \in \mathbb{C}$ $(u_i = u - \rho_i)$ as in the undeformed case. For some concrete expressions we consider the $n = 2$ and 3 cases.

Examples 3.6. For $n = 2$ we use the expressions (2.45a) and (2.45b) to construct the L-operator

$$\begin{aligned} L(u_1, u_2) &= \begin{pmatrix} [u + E_{11}]_q & q^{u+(E_{11}+E_{22}-1)/2} f \\ q^{-u+(-E_{11}-E_{22}+1)/2} e & q[u + E_{22}]_q \end{pmatrix} \\ &= \begin{pmatrix} [u_2 + 1 + N_x]_q & -q^{u-1/2} D_x \\ q^{-(u+1/2)} x [u_2 - u_1 + 1 + N_x]_q & [u_1 - N_x]_q \end{pmatrix}, \end{aligned} \tag{3.22}$$

where $u_2 - u_1 + 1 = m$ is the lowest weight of the representation. Notice that one can absorb the dependence of u into the variable x by the shift $x \mapsto xq^{u+1/2}$ so that u appears only in diagonal entries. This is a feature unique to the $n = 2$ case.

[1] Technically one recovers a very related completion of $U_q(\mathfrak{sl}_n)$ known as $U_q^{\text{ext.}}(\mathfrak{sl}_n)$.

For $n = 3$ we use the expressions (2.47a) to (2.47h) to construct the L-operator

$$L(u_1,u_2,u_3) = \begin{pmatrix} [u+E_{11}]_q & q^{u+(E_{11}+E_{22}-1)/2}f_1 & q^{u+(E_{11}+E_{33}-1)/2}E_{31} \\ q^{-u+(-E_{11}-E_{22}+1)/2}e_1 & [u+E_{22}]_q & q^{u+(E_{22}+E_{33}-1)/2}f_2 \\ q^{-u+(-E_{11}-E_{33}+1)/2}E_{31} & q^{-u+(-E_{22}-E_{33}+1)/2}e_1 & [u+E_{33}]_q \end{pmatrix},$$

$$= \begin{pmatrix} [u_3+2+N_{21}+N_{31}]_q & -q^{(u_3+u_2+2+N_{31}+N_{32})/2}D_{21} & -q^{(u_3+u_1-1-N_{21}-N_{32})/2}D_{31} \\ \begin{matrix} q^{-(u_3+u_2+2+N_{31}+N_{32})/2} \\ \times\left(x_{21}[A]_q+x_{31}D_{32}q^{N_{32}-N_{31}-m_1}\right) \end{matrix} & [u_2+1-N_{21}+N_{32}]_q & \begin{matrix} -q^{(u_1+u_2-N_{21}-N_{31})/2} \\ \times\left(D_{32}q^{N_{21}-N_{31}}+x_{21}D_{31}\right) \end{matrix} \\ \begin{matrix} q^{-(u_3+u_1+1+N_{21}-N_{32})/2}\left(-x_{21}x_{32}[B]_q q^A \right. \\ \left. +x_{31}[A+B+N_{32}]_q q^{N_{21}+1}\right) \end{matrix} & \begin{matrix} q^{-(u_1+u_2-N_{21}-N_{31})/2} \\ \times\left(x_{32}[B]_q-x_{31}D_{21}q^{2N_{32}+m_2}\right) \end{matrix} & [u_1-N_{31}-N_{32}]_q \end{pmatrix}, \tag{3.23}$$

where $m_1 = u_3 - u_2 + 1, m_2 = u_2 - u_1 + 1$ are components of the lowest weight $h_i.1 = m_i.1$, and A and B are as per (3.14).

Remarkably, the L-operators for the $n = 2$ and $n = 3$ cases exhibit a factorisation analagous to that in the undeformed case. For the $n = 2$ case let us first perform the shift $x \mapsto xq^{u+1/2}$. Then observe

$$L(u_1,u_2) = \begin{pmatrix} [u_2+1+N_x]_q & -D_x \\ x[u_2-u_1+1+N_x]_q & [u_1-N_x]_q \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ q^{u_1}x & 1 \end{pmatrix} \begin{pmatrix} [u_2]_q q^{-N_x-1} & -D_x \\ 0 & [u_1]_q q^{N_x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -xq^{u_2} & 1 \end{pmatrix} \tag{3.24a}$$

$$= \begin{pmatrix} 1 & 0 \\ q^{u_1}x & [u_1]_q \end{pmatrix} \begin{pmatrix} q^{-N_x-1} & -D_x \\ 0 & q^{N_x} \end{pmatrix} \begin{pmatrix} [u_2]_q & 0 \\ -xq^{u_2} & 1 \end{pmatrix}. \tag{3.24b}$$

We present the two separate forms (3.24a) and (3.24b) as the first is part of (what appears to be) a general pattern and the second will be helpful for a later calculation.

The $n = 3$ L-operator exhibits a factorisation $L(u_1,u_2,u_3) = Z_1\tilde{D}Z_2^{-1}$, where

$$\tilde{D} = \begin{pmatrix} q^{N_{31}-N_{21}}[u_3]_q & \begin{matrix} -q^{(u_3+u_2+2+N_{31}+N_{32})/2} \\ \times\left(D_{21}+q^{N_{31}-N_{32}}\,x_{32}D_{31}\right) \end{matrix} & -q^{(u_3+u_1-1-N_{21}-N_{32})/2}D_{31} \\ 0 & q^{N_{21}-N_{32}-1}[u_2]_q & -q^{(u_1+u_2+N_{21}-3N_{31})/2}D_{32} \\ 0 & 0 & q^{-N_{31}+N_{32}}[u_1]_q \end{pmatrix}, \tag{3.25a}$$

$$Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ q^{(u_2-u_3+N_{32}-N_{31})/2}x_{21} & 1 & 0 \\ q^{(N_{21}+3N_{32}-u_3-3u_1+1)/2}x_{31} & q^{(u_1-u_2+N_{31}-N_{21})/2}x_{32} & 1 \end{pmatrix}, \tag{3.25b}$$

$$Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ q^{(N_{31}-N_{32}+u_3-u_2)/2}x_{21} & 1 & 0 \\ q^{-u_3+(N_{32}-N_{21}-u_1-u_3-1)/2}x_{31} & q^{(u_2-u_1+N_{21}+3N_{31})/2}x_{32} & 1 \end{pmatrix}. \tag{3.25c}$$

Now by performing the similarity transformations $x_{21} \mapsto q^{(u_3+u_2)/2}x_{21}, x_{31} \mapsto q^{(u_3+u_1)/2}x_{31},\ x_{32} \mapsto q^{(u_1-u_2)/2}x_{32}$ observe that the dependence on parameters u_1, u_2, u_3 can be factored as follows

$$\tilde{D}(u_2) = \begin{pmatrix} q^{N_{31}-N_{21}} & -q^{(2+N_{31}+N_{32})/2}\left(D_{21} + q^{N_{31}-N_{32}}x_{32}D_{31}\right) & -q^{(-1-N_{21}-N_{32})/2}D_{31} \\ 0 & q^{N_{21}-N_{32}-1}[u_2]_q & -q^{u_2+(N_{21}-3N_{31})/2}D_{32} \\ 0 & 0 & q^{-N_{31}+N_{32}} \end{pmatrix}, \tag{3.26a}$$

$$Z_1(u_1,u_2) = \begin{pmatrix} 1 & 0 & 0 \\ q^{u_2+(N_{32}-N_{31})/2}x_{21} & 1 & 0 \\ q^{-u_1+(N_{21}+3N_{32}+1)/2}x_{31} & q^{u_1-u_2+(N_{31}-N_{21})/2}x_{32} & [u_1]_q \end{pmatrix}, \tag{3.26b}$$

$$(Z_2(u_3))^{-1} = \begin{pmatrix} [u_3]_q & 0 & 0 \\ -q^{u_3+(N_{31}-N_{32})/2}x_{21} & 1 & 0 \\ -q^{-u_3+(N_{32}-N_{21}-1)/2}x_{31} + q^{u_3+2N_{31}+(N_{21}-N_{32}+1)/2}x_{21}x_{32} & -q^{(N_{21}+3N_{31})/2}x_{32} & 1 \end{pmatrix}. \tag{3.26c}$$

Using the fact that $[u_i]_q \to u_i$ and $D_{ij} \to \partial_{ij}$ as $q \to 1$ one can see that the factorisations (3.24a) to (3.25c) return to the factorisations for the undeformed L-operators (3.11) in this limit. In other words these factorisations appear to be appropriate q-analogs of the undeformed factorisation for the $n = 2, 3$ cases. This suggests that the factorisation (3.11) maye have q-analogs for all n. However, as explained in the end of § 2.2 producing a q-deformed expression from an undeformed expression is difficult, even if one has a good hunch as to what form it should take. Nonetheless, this will be our approach in § 3.2.1 to construct a $U_q(\mathfrak{sl}_4)$ L-operator.

3.2.1 A Factorised $U_q(\mathfrak{sl}_4)$ L-operator

The goal of this subsection is to explicitly construct an L-operator for $U_q(\mathfrak{sl}_4)$ acting in the tensor product of $\mathbb{C}^4$, and $\mathcal{V}^{(4)}$, the space of polynomials in the 6 variables, x_{ij} for $1 \le j < i \le 4$. Our approach is to start with a factorisation reminiscent of formulas (3.24a), and (3.25a) to (3.25c), which generalises the undeformed factorisation (3.11), and use this to find a general form for entries of the L-operator. In doing so we introduce some arbitrary q exponents. By taking the viewpoint mentioned in Remark 3.5, it is enough to check that this defines a representation of the algebra $U_q(\mathfrak{sl}_4)$. Therefore, by extracting the generating set e_i, f_i, h_i $(i = 1, 2, 3)$ from the L-operator we can fix exponents by demanding that they satisfy the Chevalley-Serre relations (2.27a) to (2.27d).

Equations (3.24a), and (3.25a) to (3.25c) suggest we start with a factorisation of the form

$$L(\boldsymbol{u}) = Z(\boldsymbol{a})\tilde{D}(\boldsymbol{b})Z(\boldsymbol{c})^{-1}, \tag{3.27a}$$

$$Z(\boldsymbol{a}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21}q^{a_{21}} & 1 & 0 & 0 \\ x_{31}q^{a_{31}} & x_{32}q^{a_{32}} & 1 & 0 \\ x_{41}q^{a_{41}} & x_{42}q^{a_{42}} & x_{43}q^{a_{43}} & 1 \end{pmatrix}, \tag{3.27b}$$

$$\tilde{D}(\boldsymbol{b}) = \begin{pmatrix} [u_4]_q q^{b_{11}} & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} \\ 0 & [u_3]_q q^{b_{22}} & \tilde{D}_{23} & \tilde{D}_{24} \\ 0 & 0 & [u_2]_q q^{b_{33}} & \tilde{D}_{34} \\ 0 & 0 & 0 & [u_1]_q q^{b_{44}} \end{pmatrix}, \tag{3.27c}$$

$$\tilde{D}_{ji} = -D_{ij}q^{b_{ij}} - \sum_{k=i+1}^{4} x_{ki}D_{kj}q^{b_{ijk}}, \quad \text{for } i > j, \tag{3.27d}$$

where all exponents $a_{ij}, b_{ij}, b_{ijk}, c_{ij}$ are linear combinations of homogeneity operators N_{ij} (and a constant term). There are in total 26 exponents introduced, each with 7 components so a brute force method is not feasible, at least not without significant simplification. Our approach will therefore be to impose constraints on the exponents as we go to obtain pleasing forms that mirror the $n = 2, 3$ cases, with the hope that our intuition is rewarded. To that end, we point out that because the exponents have only linear dependence on the homogeneity operators, they will have simple commutation relations with the multiplication operators x_{ij}. Let us denote by $(I)_{ij}$ the constant term $[I, x_{ij}] = (I)_{ij}x_{ij}$ so that in particular

$$q^I.x_{ij} = x_{ij}.q^{I+(I)_{ij}}, \tag{3.28}$$

where I is one of the exponents $a_{ij}, b_{ij}, b_{ijk}, c_{ij}$.

Using the triangularity of the matrices (3.27b) and (3.27c) we obtain the following formula for elements of the matrix product (3.27a)

$$(L(\boldsymbol{u}))_{ij} = \sum_{k=1}^{i}(Z(\boldsymbol{a}))_{ik}\left(\sum_{l=\max k,j}^{4} \tilde{D}_{kl}Z(\boldsymbol{c}))_{lj}\right), \tag{3.29}$$

where we are using the notation $\tilde{D}_{kk} = [u_{5-k}]_q q^{b_{kk}}$.

Our immediate goal is to extract the generating set $\{e_i, f_i, h_i \mid i = 1,2,3\}$ from $L(\boldsymbol{u})$. By comparing with the form (3.18a) and (3.18b) we see this only necessitates calculation of the near diagonal entries $(L(\boldsymbol{u}))_{ij}$ for $|i-j| \leq 1$, of which there are 10. We begin with the three above diagonal entries $L(\boldsymbol{u})_{i,i+1}$ since these are the simplest. Explicit calculation gives

$$\begin{aligned} L(\boldsymbol{u})_{12} =& -\big(D_{21}q^{b_{21}} + x_{32}D_{31}(q^{b_{213}} - q^{b_{31}+c_{32}+(b_{31})_{32}}) + x_{42}D_{41}(q^{b_{214}} - q^{b_{41}+c_{42}+(b_{41})_{42}}) \\ &+ x_{43}D_{41}\left(q^{b_{314}} - q^{b_{41}+c_{43}+(b_{41})_{43}}\right)x_{32}q^{c_{32}}\big), \qquad (3.30a) \\ L(\boldsymbol{u})_{23} =& -\big(D_{32}q^{b_{32}} + x_{43}D_{42}(q^{b_{324}} - q^{b_{42}+c_{43}+(b_{42})_{43}}) \\ &+ x_{21}q^{a_{21}}\left(D_{31}q^{b_{31}} + x_{43}D_{41}(q^{b_{314}} - q^{b_{14}+c_{43}+(b_{14})_{43}})\right)\big), \qquad (3.30b) \\ L(\boldsymbol{u})_{34} =& -\left(D_{43}q^{b_{43}} + x_{32}D_{42}q^{b_{42}+a_{32}-(a_{32})_{42}} + x_{31}D_{41}q^{b_{41}+a_{31}-(a_{31})_{41}}\right). \qquad (3.30c) \end{aligned}$$

The repeated appearance of the factor $(q^{b_{ijk}} - q^{b_{kj}+c_{ki}+(b_{kj})_{ki}})$ for $k > i > j$ suggests we should eliminate the four exponents b_{ijk} by imposing $b_{ijk} = b_{kj} + c_{ki} + (b_{kj})_{ki}$. Doing so simplifies the above expressions to

$$\begin{aligned} L(\boldsymbol{u})_{12} = -D_{21}q^{b_{21}}, \quad L(\boldsymbol{u})_{23} = -D_{32}q^{b_{32}} + x_{21}D_{31}q^{b_{31}+a_{21}-(a_{21})_{31}}, \qquad (3.31a) \\ L(\boldsymbol{u})_{34} = -D_{43}q^{b_{43}} - x_{32}D_{42}q^{b_{42}+a_{32}-(a_{32})_{42}} - x_{31}D_{41}q^{b_{41}+a_{31}-(a_{31})_{41}}, \qquad (3.31b) \end{aligned}$$

which appear to be more inline with the $n = 3$ case where $(L(u_1,u_2,u_3))_{2,1} \propto D_{21}$. The constraint on the b_{ijk} appears further justified when we calculate diagonal entries. For example explicit calculation of $L(\boldsymbol{u})_{33}$ gives

$$\begin{aligned} L(\boldsymbol{u})_{33} =& [u_2]_q q^{b_{33}} - x_{31}q^{a_{31}}D_{31}q^{b_{31}} - x_{31}q^{a_{31}}D_{41}(x_{43}q^{b_{314}} - q^{b_{41}}x_{43}q^{c_{43}}) \\ &- x_{32}q^{a_{32}}D_{32}q^{b_{32}} - x_{32}q^{a_{32}}D_{42}(x_{43}q^{b_{324}} - q^{b_{42}}x_{43}q^{c_{43}}) + D_{43}q^{b_{43}}x_{43}q^{c_{43}} \\ =& [u_2]_q q^{b_{33}} - [N_{31}]_q q^{b_{31}+a_{31}-(a_{31})_{31}} - [N_{32}]_q q^{b_{32}+a_{32}-(a_{32})_{32}} \\ &+ [N_{43}+1]_q q^{b_{43}+(b_{43})_{43}+c_{43}}, \qquad (3.32) \end{aligned}$$

where in the last line we have used the constraint on b_{ijk}. Similar calculations using the same constraints give the other diagonal entries

$$\begin{aligned} L(\boldsymbol{u})_{11} =& [u_4]_q q^{b_{11}} + [N_{21}+1]_q q^{b_{21}+(b_{21})_{21}+c_{21}} + [N_{31}+1]_q q^{b_{31}+(b_{31})_{31}+c_{31}} \\ &+ [N_{41}+1]_q q^{b_{41}+(b_{41})_{41}+c_{41}}, \qquad (3.33a) \\ L(\boldsymbol{u})_{22} =& [u_3]_q q^{b_{22}} - [N_{21}]_q q^{a_{21}+(a_{21})_{21}+b_{21}} + [N_{32}+1]_q q^{b_{32}+(b_{32})_{32}+c_{32}} \\ &+ [N_{42}+1]_q q^{b_{42}+(b_{42})_{42}+c_{42}}, \qquad (3.33b) \\ L(\boldsymbol{u})_{44} =& [u_1]_q q^{b_{44}} - [N_{41}]_q q^{a_{41}+(a_{41})_{41}+b_{41}} - [N_{42}]_q q^{a_{42}+(a_{42})_{42}+b_{42}} \\ &- [N_{43}]_q q^{a_{43}+(a_{43})_{43}+b_{43}}. \qquad (3.33c) \end{aligned}$$

By comparing with the form (3.18b) the diagonal entries should be q-number expressions. In order to achieve this, let us suppose that for (3.32) to (3.33c) all exponents are as per (A.3) (for some ordering of the summands) so that we obtain the pleasing form

$$L(\boldsymbol{u})_{ii} = \left[u_{5-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{4} (N_{ji}+1) \right]_q . \tag{3.34}$$

As discussed in Appendix A.1 the result (A.3) cannot uniquely specify the exponents. One can however imagine using (A.3) to eliminate the b_{ii}.

From (3.34) one can read off the following expression for Cartan elements in the representation defined by (3.27a)

$$E_{ii} = -\rho_{5-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{4} (N_{ji}+1), \tag{3.35a}$$

$$h_1 = E_{11} - E_{22} = m_1 + 2N_{21} + N_{31} - N_{32} + N_{41} - N_{42}, \tag{3.35b}$$

$$h_2 = E_{22} - E_{33} = m_2 - N_{21} + N_{31} + 2N_{32} + N_{42} - N_{43}, \tag{3.35c}$$

$$h_3 = E_{33} - E_{44} = m_3 - N_{31} - N_{32} + N_{41} + N_{42} + 2N_{43}, \tag{3.35d}$$

where $\rho_i = u - u_i$ and $m_i = u_{5-i} - u_{4-i} + 1$. Note that the relation $\sum E_{ii} = 0$ is satisfied if we impose representation parameter condition $\sum_i \rho_i = 6$. Furthermore, it is clear that we have $[h_i, h_j] = 0$ for all i, j.

Finally by using the same constraints on the b_{ijk} and by presumptively applying (A.3) we obtain the following expressions for the elements $L(\boldsymbol{u})_{i+1,i}$

$$L(\boldsymbol{u})_{21} = x_{21} \left[u_4 - u_3 + 1 + \sum_{l=2}^{4} N_{l1} - \sum_{l=3}^{4} N_{l2} \right]_q q^{d_{21}} + \sum_{l=3}^{4} x_{l1} D_{l2} q^{b_{l2}+c_{l1}+(b_{l2})_{l1}}, \tag{3.36a}$$

$$\begin{aligned} L(\boldsymbol{u})_{32} = {} & x_{32}[u_3 - u_2 + 1 + N_{32} + N_{42} - N_{43}]_q q^{d_{32}} \\ & + x_{42} D_{43} q^{b_{43}+c_{42}+(b_{43})_{42}} - x_{31} D_{21} q^{b_{21}+a_{31}-(a_{31})_{21}}, \end{aligned} \tag{3.36b}$$

$$L(\boldsymbol{u})_{43} = x_{43}[u_2 - u_1 + 1 + N_{43}]_q q^{d_{43}} - \sum_{k=1}^{2} x_{4k} D_{3k} q^{b_{3k}+a_{4k}-(a_{4k})_{3k}}. \tag{3.36c}$$

By comparing with the expression (3.18b) we can read off that the remaining generators in the representation defined by (3.27a) are

$$e_i = q^{\Delta_{i,i+1}} L(\boldsymbol{u})_{i+1,i}, \quad f_i = q^{-\Delta_{i,i+1}} L(\boldsymbol{u})_{i,i+1} \tag{3.37a}$$

$$\Delta_{ij} = u + (E_{ii} + E_{jj} - 1)/2. \tag{3.37b}$$

Since the elements h_i (3.35b) to (3.35d) are linear combinations of the homogeneity operators, they will commute with all exponents q^a and so the relations (2.27a) can be checked directly using only the multiplication operator dependence of elements (3.37a). This leaves only the relations (2.27b) and (2.28a) to (2.28c) to be checked, of which there are fifteen distinct relations for $n = 4$. These relations can be implemented digitally. Although the q-exponents are not determined uniquely, the following delightful solution was found

$$f_1 = -D_{21}, \quad f_2 = -D_{32}q^{N_{21}-N_{31}} - x_{21}D_{31}, \tag{3.38a}$$

$$f_3 = -D_{43}q^{N_{31}+N_{32}-N_{41}-N_{42}} - x_{32}D_{42}q^{N_{31}-N_{41}} - x_{31}D_{41}, \tag{3.38b}$$

$$e_1 = x_{21}\left[m_1 + \sum_{l=2}^{4} N_{l1} - \sum_{l=3}^{4} N_{l2}\right]_q + \left(x_{31}D_{32}q^{-N_{31}+N_{32}} + x_{41}D_{42}\right)q^{-m_1-N_{41}+N_{42}}, \tag{3.38c}$$

$$e_2 = x_{32}[m_2 + N_{32} + N_{42} - N_{43}]_q + x_{42}D_{43}q^{-m_2-N_{42}+N_{43}} - x_{31}D_{21}q^{m_2+2N_{32}+N_{42}-N_{43}}, \tag{3.38d}$$

$$e_3 = x_{43}[m_3 + N_{43}]_q - \left(x_{41}D_{31}q^{-N_{32}+N_{42}} + x_{42}D_{32}\right)q^{m_3+2N_{43}}. \tag{3.38e}$$

We can now calculate the remaining Cartan-Weyl elements E_{ij} (for $|i-j| > 1$) by formulas (2.34b) and (2.34c) and write the $U_q(\mathfrak{sl}_4)$ L-operator $L(u_1, u_2, u_3, u_4)$ using (3.18a) and (3.18b). In light of these pleasing expressions (3.38a) to (3.38e) and their relation to (2.47a), (2.47c), (2.47d), (2.47f) and (2.47g), the author cannot resist engaging in speculation:

Conjecture A. *The following defines a representation of $U_q(\mathfrak{sl}_n)$ on $\mathcal{V}^{(n)}$, the space of polynomials in the $n(n-1)/2$-variables x_{ij} for $1 \le j < i \le n$.*

$$E_{ii} = -\rho_{n+1-i} - \sum_{j=1}^{i-1} N_{ij} + \sum_{j=i+1}^{n} (N_{ji} + 1), \tag{3.39a}$$

$$f_i = -D_{i+1,i}q^{\sum_{j=1}^{i-1} N_{ij} - \sum_{j=1}^{i-1} N_{i+1,j}} - \sum_{j=1}^{i-1} x_{ij}D_{i+1,j}q^{\sum_{k=1}^{j-1} N_{ik} - \sum_{k=1}^{j-1} N_{i+1,k}}, \tag{3.39b}$$

$$\begin{aligned} e_i = {} & x_{i+1,i}\left[m_i + \textstyle\sum_{j=i+1}^{n} N_{ji} - \sum_{j=i+2}^{n} N_{j,i+1}\right]_q + q^{-m_i}\sum_{j=i+2}^{n} x_{ji}D_{j,i+1}q^{\sum_{k=j}^{n} N_{k,i+1} - \sum_{k=j}^{n} N_{k,i}} \\ & - q^{m_i+2N_{i+1,i}}\sum_{j=1}^{i-1} x_{i+1,j}D_{ij}q^{\sum_{k=i+2}^{n}(N_{ki}-N_{k,i+1}) - \sum_{k=j+1}^{i-1}(N_{i+1,k}-N_{i,k})}, \end{aligned} \tag{3.39c}$$

for $i = 1, \ldots, n-1$ with parameters ρ_i and m_i related by $m_i = \rho_{n-i} - \rho_{n+1-i} + 1$ and $\sum_{i=1}^{n} \rho_i = n(n-1)/2$. The vector $1 \in \mathcal{V}$ satisfies $h_i.1 = m_i$.

Let us now examine some interesting features of the representation defined by (3.35b) and (3.35d), and (3.38a) to (3.38e). In the final paragraph of § 2.2 we discussed the difficultly of obtaining a representation of $U_q(\mathfrak{sl}_n)$ from a representation of $\mathfrak{sl}_n$ on the space $\mathcal{V}^{(n)}$. We also proposed the following method for obtaining the general form of (non-diagonal) q-deformed Cartan-Weyl elements $E_{ij} \in \mathrm{End}(\mathcal{V})$ from the undeformed expression (2.12): rearrange undeformed expressions into sums of terms of the form $X(N+c)$ and then perform the replacement $X(N+c) \mapsto X[N+c]_q q^{\alpha}$, where the X are distinct monomials in the multiplication operators that allow negative exponents, N are combinations of the N_{ij}, and $c \in \mathbb{C}$. By examining the product (3.27a) to (3.27d) and all calculations in this subsection, it is apparent that the form obtained for generators $E_{i,i\pm 1}$ here ((3.38a) to (3.38e)), agrees with the form obtained via the previously proposed method.

Interestingly however, if we now use formulas (2.34b) and (2.34c) to calculate the remaining Cartan-Weyl elements we will see that they have a form which does not arise by this method. For an example of this we calculate the element E_{42}, with result

$$\begin{aligned} E_{42} &= [f_3, f_2]_{q^{-1}} \\ &= -D_{42} q^{-1+N_{21}-N_{32}-N_{41}} - x_{21} D_{41} q^{-(1+N_{31})} + (q-q^{-1}) x_{31} D_{41} D_{32} q^{N_{21}-N_{31}-1}. \end{aligned} \tag{3.40}$$

In the classical, limit the rightmost term will vanish as a result of the prefactor $(q-q^{-1})$. Therefore, E_{42} as per (3.40) cannot arise from an undeformed expression by the proposed method since the vanishing term is undetected by the undeformed expression. Terms with a $(q-q^{-1})$ prefactor also appear in the element E_{24} but nowhere else.

It is unclear whether the appearance of these terms is an artefact of the choice[2] of exponents in (3.38a) to (3.38e), however, exponents which do not exhibit this behaviour have not yet been found. It is currently suspected that such terms are an inherent part of representations of $U_q(\mathfrak{sl}_n)$ on $\mathcal{V}^{(n)}$ for $n > 3$ which implies that the method proposed in § 2.2 for obtaining the general form of q-deformed Cartan-Weyl elements fails in this case.

Another consequence of the $(q-q^{-1})$ terms appearing is that the factorisation we started with (3.27a) to (3.27d) needs to be modified. This is because it gives no mechanism by which the rightmost term in (3.40), which is second order in the

[2]More precisely choice of similarity class of exponents (e.g. (2.44a) and (2.44b), and (2.45a) and (2.45b) define similar representations).

D_{ij}, can arise. The following factorised form for the L-operator $L(\boldsymbol{u})$ constructed using the formula (3.18a) to (3.18b), and the representation (3.34), and (3.38a) to (3.38e), was obtained by implementing a row and column reduction procedure:

$$L(\boldsymbol{u}) = \tilde{Z}_1(\boldsymbol{u})\tilde{D}(\boldsymbol{u})(\tilde{Z}_2(\boldsymbol{u}))^{-1}, \tag{3.41}$$

where

$$\tilde{Z}_1(\boldsymbol{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \begin{matrix} q^{\Delta_{42}-\Delta_{41}+1/2+N_{21}}\times \\ (x_{21}-\lambda x_{31}D_{32}q^{N_{21}+1}) \end{matrix} & 1 & 0 & 0 \\ q^{\Delta_{43}-\Delta_{41}+3/2+N_{21}+N_{31}}x_{31} & \begin{matrix} q^{\Delta_{43}-\Delta_{42}+1/2}\times \\ q^{-N_{21}+N_{31}+N_{32}}x_{32} \end{matrix} & 1 & 0 \\ \begin{matrix} q^{-\Delta_{43}-\Delta_{31}+C+3/2}\times \\ q^{N_{21}-N_{32}+N_{42}+N_{43}}x_{41} \end{matrix} & \begin{matrix} q^{-\Delta_{43}-\Delta_{32}+C+1/2}\times \\ q^{-N_{21}-N_{31}+N_{43}}x_{42} \end{matrix} & q^{u_1-\Delta_{43}-N_{31}-N_{32}}x_{43} & 1 \end{pmatrix}, \tag{3.42}$$

$$\tilde{Z}_2(\boldsymbol{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q^{u+E_{11}-1-\Delta_{21}-N_{21}}x_{21} & 1 & 0 & 0 \\ \begin{matrix} q^{\Delta_{21}-\Delta_{32}-3/2}\times \\ q^{-N_{21}-N_{31}}x_{31} \end{matrix} & \begin{matrix} q^{-\Delta_{32}+E_{22}-1+N_{21}-N_{31}-N_{32}}\times \\ (x_{32}-\lambda x_{21}D_{31}q^{N_{31}+1}) \end{matrix} & 1 & 0 \\ \begin{matrix} q^{-\Delta_{21}-\Delta_{24}-A+5/2}\times \\ q^{N_{31}+N_{41}}x_{41} \end{matrix} & \begin{matrix} q^{-\Delta_{43}-\Delta_{32}-B+3/2}\times \\ q^{-N_{31}+N_{41}+N_{42}}x_{42} \end{matrix} & q^{u_2-\Delta_{43}+N_{41}+N_{42}}x_{43} & 1 \end{pmatrix}, \tag{3.43}$$

$$\tilde{D}(\boldsymbol{u}) = \begin{pmatrix} \begin{matrix} [u_4]_q\times \\ q^{-N_{21}-N_{31}+N_{41}-1} \end{matrix} & \begin{matrix} -q^{\Delta_{21}}(q^{-2u_3+N_{41}-N_{42}}x_{42}D_{41}+ \\ q^{N_{31}-N_{32}}D_{21}+x_{32}D_{31})q^{-(N_{31}+N_{32}+2)} \end{matrix} & \begin{matrix} -q^{\Delta_{31}}(q^{N_{32}+N_{41}+N_{42}-N_{43}}x_{43}D_{41} \\ +D_{31})q^{-(N_{21}+1)} \end{matrix} & \begin{matrix} -q^{\Delta_{41}}D_{41}\times \\ q^{-(N_{21}-N_{31}+2)} \end{matrix} \\ 0 & \begin{matrix} [u_3]_q\times \\ q^{N_{21}-N_{32}+N_{42}} \end{matrix} & \begin{matrix} -q^{\Delta_{32}}(q^{N_{42}-N_{43}}x_{43}D_{42} \\ +D_{32})q^{N_{21}+N_{31}} \end{matrix} & \begin{matrix} -q^{\Delta_{42}}D_{42}\times \\ q^{-1+N_{21}-N_{32}-N_{41}} \end{matrix} \\ 0 & 0 & \begin{matrix} [u_2]_q\times \\ q^{N_{31}+N_{32}-N_{43}-1} \end{matrix} & \begin{matrix} -q^{\Delta_{43}}D_{43}\times \\ q^{N_{31}+N_{32}-N_{41}-N_{42}} \end{matrix} \\ 0 & 0 & 0 & \begin{matrix} [u_1]_q\times \\ q^{-N_{41}-N_{42}+N_{43}} \end{matrix} \end{pmatrix}, \tag{3.44}$$

with Δ_{ij} are as per (3.37b). Here we are using the shorthand $\lambda = (q - q^{-1})$ and

$$A = m_1 + \sum_{l=2}^{4} N_{l1} - \sum_{l=3}^{4} N_{l2}, \quad B = m_2 + N_{32} + N_{42} - N_{43}, \quad C = m_3 + N_{43}, \tag{3.45}$$

which are the arguments of the q-number terms in the e_i (3.38c) to (3.38e).

The dependence of each factor on the parameters u_i can be obtained from the dependence of terms E_{ii}, Δ_{ij} and A, B, C. It is hoped that there exists a similarity transformation $x_{ij} \mapsto q^{\lambda_{ij}} x_{ij}$ for $\lambda_{ij} \in \mathbb{C}$, such that the dependence on u_1 (u_4)

can be absorbed into the rightmost factor $\tilde{Z}_2$ (leftmost factor $\tilde{Z}_1$) analagously to in (3.26a) to (3.26c), and (3.24b). Such a transformation has not yet been found.

Despite the fact that the factorisation (3.41) to (3.44) limits to the undeformed factorisation (3.11) as a result of the $(q - q^{-1})$-terms vanishing, the reader may be disturbed by the fact that we began with a factorisation of a different form (3.27a). Let us point out as a consolation that it was not checked that the assumptions made about q-exponents entering the factors in (3.27a) could be consistently imposed. In the absence of a factorisation of this form, formula (3.27a) can at most be viewed as a short-cut to obtaining the general form of elements $E_{ij} \in \mathrm{End}(\mathcal{V})$ for $|i - j| \leq 1$.

Chapter 4

Permutation Operators & the Yang-Baxter Equation

The purpose of this chapter is to construct R-matrices for the algebras $\mathcal{A} = U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$, which act in the tensor square [1] of $\mathcal{A}$-modules $\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}}$. These R-matrices are written as $\mathcal{R}(u) \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$ and satisfy the defining relation

$$\mathcal{R}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}(u-v) \in \mathrm{End}(\mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}}), \qquad (4.1)$$

where $L_1(u) = L(u; \boldsymbol{\rho}) \otimes I_{\mathcal{V}_{\boldsymbol{\rho}}}$ and $L_2(v) = L(v; \boldsymbol{\sigma}) \otimes I_{\mathcal{V}_{\boldsymbol{\sigma}}}$ are built from the L-operators constructed in §§ 3.1 and 3.2. The R-matrix $\mathcal{R}(u)$ depends on the representations $\mathcal{V}_{\boldsymbol{\rho}}$, and $\mathcal{V}_{\boldsymbol{\sigma}}$, and so will in general depend on the defining parameters $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$.

As pointed out in § 1.1 this relation (4.1) differs from the defining RLL-relation (3.2) for the L-operator $L(u)$, in that it occurs in the endomorphism ring $\mathrm{End}(\mathbb{C}^n \otimes \mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$ which contains two copies of the space $\mathcal{V}$. Here both L-operators L_i act in the same copy of $\mathbb{C}^n$ with the subscript denoting which copy of $\mathcal{V}$ they are acting non-trivially in.

In order to construct a solution of (4.1) we follow the approach of [8, 11, 26]. The first step is to rewrite (4.1) in terms of the operator $\hat{\mathcal{R}} = \mathcal{P} \circ \mathcal{R}$ where $\mathcal{P} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ is the permutation operator $\mathcal{P}(f(\boldsymbol{x}) \otimes g(\boldsymbol{y})) = g(\boldsymbol{x}) \otimes f(\boldsymbol{y})$. Here we are using $\boldsymbol{x}$ ($\boldsymbol{y}$) to denote the variables in the first (second) copy of $\mathcal{V}$ respectively. In terms of $\hat{\mathcal{R}}$ the relation (4.1) is equivalent to

$$\hat{\mathcal{R}}(u-v)L_1(\boldsymbol{u})L_2(\boldsymbol{v}) = L_1(\boldsymbol{v})L_2(\boldsymbol{u})\hat{\mathcal{R}}(u-v), \qquad (4.2)$$

[1] $\mathcal{V}_{\boldsymbol{\rho}} = \mathcal{V}_{\boldsymbol{\sigma}} = \mathcal{V}$ are equal as vector spaces but not as $\mathcal{A}$-modules. We include the subscript to emphasise that $\mathcal{R}(u)$ depends on the representation.

where $(\boldsymbol{u})_i = u_i$ and $(\boldsymbol{v})_i = v_i$ are the defining parameters $u_i = u - \rho_i$ and $v_j = v - \sigma_j$ for the two L-operators. In this form equation (4.2) admits a simple interpretation; $\hat{\mathcal{R}}(u-v)$ commutes with the product of two L-operators by swapping their parameters $(u_1, \ldots, u_n, v_1, \ldots, v_n) \mapsto (v_1, \ldots, v_n, u_1, \ldots, u_n)$. One may then consider a more general question: given a permutation $s \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}) \simeq S_{2n}$ [2] is there an operator $\mathcal{S} \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$ which commutes with the product $L_1(\boldsymbol{u})L_2(\boldsymbol{v})$ by performing the permutation s on the $2n$-tuple of parameters $(\boldsymbol{u}, \boldsymbol{v})$?

To answer this question recall that the symmetric group S_N is generated by the $N-1$ elementary transpositions $s_i := (i\, i+1)$ for $i = 1, \ldots, N-1$. Thus by writing $s \in S_{2n}$ as a word in the s_i for $i = 1, \ldots, 2n-1$ a sufficient condition for the existence of the desired operator $\mathcal{S}$, is the existence of $2n-1$ operators $\mathcal{S}_i$ which satisfy

$$\mathcal{S}_j L_1(u_1, \ldots, u_n) L_2(\boldsymbol{v}) = L_1(u_1, \ldots, \overset{j-1}{\overset{\downarrow}{u_j}}, u_{j-1}, \ldots, u_n) L_2(\boldsymbol{v}) \mathcal{S}_j, \tag{4.3a}$$

$$\mathcal{S}_{n+j} L_1(\boldsymbol{u}) L_2(v_1, \ldots, v_n) = L_1(\boldsymbol{u}) L_2(v_1, \ldots, \overset{j-1}{\overset{\downarrow}{v_j}}, v_{j-1}, \ldots, v_n) \mathcal{S}_{n+j}, \tag{4.3b}$$

$$\mathcal{S}_n L_1(u_1, \ldots, u_n) L_2(v_1, \ldots, v_n) = L_1(u_1, \ldots, u_{n-1}, v_1) L_2(u_n, v_2, \ldots, v_n) \mathcal{S}_j, \tag{4.3c}$$

for $j = 1, \ldots, n-1$. Note that operators $\mathcal{S}_j$ ($\mathcal{S}_{n+j}$) only affect the first (second) factor in the matrix product. As such it is natural to look for solutions to relations (4.3a) and (4.3b) by solving instead the relation $\mathcal{T}_j L(\boldsymbol{u}) = L(s_j \boldsymbol{u}) \mathcal{T}_j$ where $\mathcal{T}_j \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$. Then we can build operators $\mathcal{S}_j, \mathcal{S}_{n+j} \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}})$ as $\mathcal{S}_j = \mathcal{T}_j \otimes I_{\mathcal{V}_{\boldsymbol{\sigma}}}$ and $\mathcal{S}_{n+j} = I_{\mathcal{V}_{\boldsymbol{\rho}}} \otimes \mathcal{T}'_j$ where $\mathcal{T}'_j = (\mathcal{T}_j)|_{\boldsymbol{x} \to \boldsymbol{y}, \boldsymbol{u} \to \boldsymbol{v}} \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\sigma}})$. Thus solving relations (4.3a) to (4.3c) is reduced to finding $n-1$ "intertwining" operators $\mathcal{T}_j$, and a single "exchange" operator $\mathcal{S}_n$.

Provided these relations are solved, then the desired operator $\mathcal{S}$ performing any permutation of the $2n$ parameters can be realised. In particular, the relation (4.2) can be solved modulo one detail. In equation (4.2) the operator $\hat{\mathcal{R}}(u-v)$ is only allowed to depend on the difference in spectral parameters $u-v$ (as well as representation parameters). Whilst the operators $\mathcal{S}_i$ should in general depend on all parameters $(\boldsymbol{u}, \boldsymbol{v})$, we will construct examples with the dependence:

$$\mathcal{S}_j(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{S}_j(u_j - u_{j+1}), \quad \mathcal{S}_{n+j}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{S}_{n+j}(v_j - v_{j+1}), \quad \mathcal{S}_n(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{S}_n(u_n - v_1), \tag{4.4}$$

[2] In practice we will identify $\mathrm{Perm}(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(m)})$ with S_{mn} for notational convenience where we understand $s \in S_{mn}$ as the permutation $s(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(m)}) = (u_{s(1)}, \ldots u_{s(mn)})$ where the $(jn+i)$-th components of $(\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(m)})$ is $u_i^{(j+1)}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, m-1$.

for $j = 1, \dots, n-1$. Since we have $u_i = u - \rho_i$, and $v_j = v - \sigma_j$, the spectral parameters can only enter the operators $\mathcal{S}_i$ in the combination $u - v$. Thus (4.2) can be solved by an appropriate combination of the operators $\mathcal{S}_i$ performing the permutation $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{v}, \boldsymbol{u})$.

Given a permutation $s \in S_{2n}$ and a decomposition into transpositions $s = s_{i_k} s_{i_{n-1}} \dots s_{i_1}$, we can therefore write an operator $\mathcal{S}$ with the desired commutation relations with the product $L_1(\boldsymbol{u}) L_2(\boldsymbol{v})$ using the operators $\mathcal{S}_i$. However, since s does not have a unique decomposition into transpositions, a second decomposition say $s = s_{j_l} s_{j_{l-1}} \dots s_{j_1}$ provides two equal candidates for the operator $\mathcal{S}$:

$$\mathcal{S} = S_{i_k}(s_{i_{k-1}} \dots s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \dots \mathcal{S}_{i_3}(s_{i_2} s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \mathcal{S}_{i_2}(s_{i_1}(\boldsymbol{u}, \boldsymbol{v})) \mathcal{S}_{i_1}(\boldsymbol{u}, \boldsymbol{v}), \tag{4.5a}$$

$$\mathcal{S}' = S_{j_l}(s_{j_{l-1}} \dots s_{j_2} s_{j_1}(\boldsymbol{u}, \boldsymbol{v})) \dots \mathcal{S}_{j_2}(s_{j_2} s_{j_1}(\boldsymbol{u}, \boldsymbol{v})) \mathcal{S}_{j_2}(s_{j_1}(\boldsymbol{u}, \boldsymbol{v})) \mathcal{S}_{j_1}(\boldsymbol{u}, \boldsymbol{v}). \tag{4.5b}$$

We will see that not only do these two operators realise the same permutation $s \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v})$ of parameters entering the product $L_1(\boldsymbol{u}) L_2(\boldsymbol{v})$, but they are in fact the same operator $\mathcal{S} = \mathcal{S}'$. A pleasing consequence of this result is that the Yang-Baxter equation in braid form

$$\hat{\mathcal{R}}_{12}(v-w) \hat{\mathcal{R}}_{23}(u-w) \hat{\mathcal{R}}_{12}(u-v) = \hat{\mathcal{R}}_{23}(u-v) \hat{\mathcal{R}}_{12}(u-w) \hat{\mathcal{R}}_{23}(v-w) \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\sigma}} \otimes \mathcal{V}_{\boldsymbol{\tau}}), \tag{4.6}$$

is solved by the R-matrix $\hat{\mathcal{R}}(u-v)$ built out of the operators $\mathcal{S}_i$ to solve (4.2). This relation is equivalent to the YBE (1.2) for the untransformed R-matrix $\mathcal{R}(u) = \mathcal{P} \circ \hat{\mathcal{R}}(u)$.

The rest of this chapter is organised as follows: in § 4.1 we give without proof formulas for the operators $\mathcal{S}_i$ in the undeformed $\mathcal{A} = U(\mathfrak{sl}_n)$ case for generic n, and study the $n = 3$ case. In § 4.2.1 we derive all operators $\mathcal{S}_i$ in the deformed case $\mathcal{A} = U_q(\mathfrak{sl}_2)$ following [8], and in § 4.2.2 derive the intertwiners $\mathcal{T}_i$ in the $U_q(\mathfrak{sl}_4)$ case. In § 4.3 we prove that the $U_q(\mathfrak{sl}_4)$-intertwiners constructed in § 4.2.2 satisfy analogs of the S_4 group relations and, with the assumption that exchange operator $\mathcal{S}_4$ can be constructed, derive the Yang-Baxter equation (4.6) for $\hat{\mathcal{R}}$ as a consequence.

4.1 Undeformed Case

Recall that the L-operator $L(\boldsymbol{u})$ for $\mathcal{A} = U(\mathfrak{sl}_n)$ (3.11) is given by

$$L(\boldsymbol{u}) = Z(\tilde{D}(\boldsymbol{u})) Z^{-1}, \tag{4.7}$$

where Z is the lower triangular matrix (2.13) and $\tilde{D}(\boldsymbol{u})$ is the upper triangular matrix is given by

$$\tilde{D}(\boldsymbol{u}) = \begin{pmatrix} u_n & \tilde{D}_{12} & \tilde{D}_{13} & \dots & \tilde{D}_{1n} \\ & u_{n-1} & \tilde{D}_{23} & \dots & \tilde{D}_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & u_2 & \tilde{D}_{n-1,n} \\ & & & & u_1 \end{pmatrix}, \tag{4.8}$$

$$\tilde{D}_{ji} = -\partial_{ij} - \sum_{k=i+1}^{n} x_{ki}\partial_{kj}. \tag{4.9}$$

The expressions for intertwiners $\mathcal{T}_j \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$ which satisfy the definining relation $\mathcal{T}_j L(\boldsymbol{u}) = L(s_j\boldsymbol{u})\mathcal{T}_j$ are taken from [9]

$$\mathcal{T}_j(u_j - u_{j+1}) = (-\tilde{D}_{n-1-j,n-j})^{u_j-u_{j+1}}. \tag{4.10}$$

As mentioned the operators $\mathcal{S}_j, \mathcal{S}_{n+j}$ can now be built from the intertwiners (4.10) as $\mathcal{S}_j = \mathcal{T}_j \otimes I_{\mathcal{V}_{\boldsymbol{\sigma}}}, \mathcal{S}_{n+j} = I_{\mathcal{V}_{\boldsymbol{\rho}}} \otimes \mathcal{T}_j|_{\boldsymbol{x}\to\boldsymbol{y},\boldsymbol{u}\to\boldsymbol{v}}$.

Remark 4.1. If $u_j - u_{j+1} = \rho_{j+1} - \rho_j = 1 - m_{n-j}$ is a positive integer, then the formula (4.10) is simply repeated application of a differential operator which is perfectly well defined on the space of polynomials $\mathcal{V}_{\boldsymbol{\rho}}$. However, we would like to allow $u_j - u_{j+1}$ to take arbitrary complex values in which case (4.10) is extended to non-integer values using fractional calculus. This will mean that $\mathcal{T}_j$ is not in general an operator on a space of a polynomials, so we will instead write $\mathcal{T}_j \in \mathrm{End}\left(\hat{\mathcal{V}}_{\boldsymbol{\rho}}\right)$ where $\mathcal{V}_{\boldsymbol{\rho}} \subset \hat{\mathcal{V}}_{\boldsymbol{\rho}}$ is the space of formal power series in the x_{ij} (centred at $x_{ij} = 1$). The key fact that makes this work is that these series provide an adequate realisation of (principal valued) complex power functions x^α for $\alpha \notin \mathbb{Z}_{<0}$, for our purposes (see Appendix B). We will use the same extended space $\hat{\mathcal{V}}_{\boldsymbol{\rho}}$ in the q-deformed case and will often denote it as just $\mathcal{V}_{\boldsymbol{\rho}}$ for simplicity.

Remark 4.2. Notice that the intertwiners $\mathcal{T}_j$ are independent of the spectral parameter u since $u_j - u_{j+1} = \rho_{j+1} - \rho_j$. This is no coincidence. By setting $u = 0$ in the L-operator (4.7) we recover the compact expression for a differential representation of $\mathfrak{sl}_n$ (2.12). Thus, the operators $\mathcal{T}_j$ can be given a purely representation theoretic interpretation as a family of $\mathcal{A}$-module isomorphisms $\mathcal{T}_j : \hat{\mathcal{V}}_{\boldsymbol{\rho}} \to \hat{\mathcal{V}}_{s_j\boldsymbol{\rho}}$. In particular this demonstrates the claim made in Remark 2.7, at least if one is happy to work in the extended space $\hat{\mathcal{V}}_{\boldsymbol{\rho}}$.

The expression for the exchange operator $\mathcal{S}_n$ which commutes with the product $L_1(\boldsymbol{u})L_2(\boldsymbol{v})$ by swapping the parameters u_n and v_1 is also taken from [9]. It is the following multiplication operator in the variables x_{ij} and y_{ij}

$$\mathcal{S}_n(u_n - v_1) = \left(((Z^{(y)})^{-1}Z^{(x)})_{N1}\right)^{u_n - v_1}, \tag{4.11}$$

where $Z^{(x)}$ is the matrix Z (2.13) and $Z^{(y)}$ is the same matrix after the replacement $x_{ij} \mapsto y_{ij}$. Whilst the entry $((Z^{(y)})^{-1}Z^{(x)})_{N1}$ is a polynomial expression in the mutliplication operators, $\mathcal{S}_n$ does not act on a space of polynomials for non-positive integer powers $u_n - v_1$. Therefore, $\mathcal{S}_n$ must be considered an operator on the extended space $\mathcal{V}_{\boldsymbol{\rho}} \otimes \mathcal{V}_{\boldsymbol{\tau}} \subset \hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\tau}}$.

Example 4.3. In this example we will study the permutation operators $\mathcal{S}_i$ for $i = 1, 2, \ldots, 5$, for the $\mathfrak{sl}_3$ L-operator (3.13). According to (4.10) the intertwiners are

$$\mathcal{T}_2(u_2 - u_3) = (\partial_{21} + x_{32}\partial_{31})^{u_2 - u_3}, \quad \mathcal{T}_1(u_1 - u_2) = (\partial_{32})^{u_1 - u_2}. \tag{4.12}$$

We will start by considering the simpler intertwiner $\mathcal{T}_1$ in the case where $u_1 - u_2 = \rho_2 - \rho_1$ is a positive integer. Here it is helpful to consider the representation theoretic role of $\mathcal{T}_1$ as an $\mathcal{A}$-module map $\mathcal{T}_1 : \mathcal{V}_{\boldsymbol{\rho}} \to \mathcal{V}_{s_1\boldsymbol{\rho}}$. The representation $\mathcal{V}_{\boldsymbol{\rho}}$ is also described completely by its lowest weight $\boldsymbol{m} \in \mathbb{C}^2$ with components $m_i = \rho_{4-i} - \rho_{3-i} + 1$. Under the parameter swap $(\rho_1, \rho_2, \rho_3) \mapsto (\rho_2, \rho_1, \rho_3)$ the lowest weight becomes $(m_1, m_2) \mapsto (m_1 + m_2 + 1, 2 - m_2)$. The defining relation for $\mathcal{T}_1$ is therefore equivalent to the relations $\mathcal{T}_1 E_{ij}(\rho_1, \rho_2, \rho_3) = E_{ij}(\rho_2, \rho_1, \rho_3)\mathcal{T}_1$ or $\mathcal{T}_1 E_{ij}(m_1, m_2) = E_{ij}(m_1 + m_2 - 1, 2 - m_2)\mathcal{T}_1$ for all i, j, where $E_{ij}(\boldsymbol{m}) = E_{ij}(\boldsymbol{\rho})$ are the Cartan-Weyl basis elements evaluated in the representation $\mathcal{V}_{\boldsymbol{\rho}}$ (2.19). Furthermore, since the elements E_{ij} for $|i - j| \leq 1$ determine all other E_{ij} by (2.4a) to (2.4c), it suffices to check the defining relations for these elements only. We will check this by checking a simpler set of commutation relations for ∂_{32}.

Firstly, since neither of E_{21} and E_{32} depend on the lowest weight $\boldsymbol{m}$ or the mutliplication operator x_{32} we have

$$\partial_{32}E_{21}(m_1, m_2) = E_{21}(m_1, m_2)\partial_{32} = E_{21}(m_1 - 1, m_2 + 2)\partial_{32}, \tag{4.13a}$$

$$\partial_{32}E_{32}(m_1, m_2) = E_{32}(m_1, m_2)\partial_{32} = E_{32}(m_1 - 1, m_2 + 2)\partial_{32}. \tag{4.13b}$$

Next, by using the commutation relation $\partial_{32}N_{32} = (1 + N_{32})\partial_{32}$ and the fact that $E_{33}(\boldsymbol{\rho})$ depends only on ρ_1 we have

$$\partial_{32}E_{33}(\boldsymbol{\rho}) = \partial_{32}(-\rho_1 - N_{31} - N_{32}) = (-\rho_1 - 1 - N_{21} - N_{32})\partial_{32} = E_{33}(\rho_1 + 1, \rho_2 - 1, \rho_3)\partial_{32}. \tag{4.14}$$

Similarly one finds

$$\partial_{32}E_{22}(\boldsymbol{\rho}) = E_{22}(\rho_1+1,\rho_2-1,\rho_3)\partial_{32}, \quad \partial_{32}E_{11}(\boldsymbol{\rho}) = E_{11}(\rho_1+1,\rho_2-1,\rho_3)\partial_{32}. \tag{4.15}$$

Finally for the elements E_{12} and E_{23} we observe

$$\begin{aligned}\partial_{32}E_{12}(\boldsymbol{m}) &= \partial_{32}\left(x_{21}(m_1+N_{21}+N_{31}-N_{32})+x_{31}\partial_{32}\right)\\ &= \left(x_{21}(m_1-1+N_{21}+N_{31}-N_{32})+x_{31}\partial_{32}\right)\partial_{32} \qquad (4.16)\\ &= E_{12}(m_1-1,m_2+2)\partial_{32},\\ \partial_{32}E_{23}(\boldsymbol{m}) &= \partial_{32}\left(x_{32}(m_2+N_{32})+x_{31}\partial_{21}\right)\\ &= \left(x_{32}(m_2+2+N_{32})+x_{31}\partial_{21}\right)\partial_{32}+\left(\partial_{32}(x_{32}(m_2+N_{32}))-x_{32}(m_2+2+N_{32})\partial_{32}\right)\\ &= E_{23}(m_1-1,m_2+2)\partial_{32}+m_2. \qquad (4.17)\end{aligned}$$

Due to these simple commutation relations one sees that for k a positive integer we have

$$\partial_{32}^k E_{ii}(\boldsymbol{\rho}) = E_{ii}(\rho_1+k,\rho_2-k,\rho_3)\partial_{32}^k, \tag{4.18a}$$

$$\partial_{32}^k E_{ij}(m_1,m_2) = E_{ij}(m_1-k,m_2+2k)\partial_{32}^k+\delta_{i2}\delta_{j3}k(m_2+k-1)\partial_{32}^{k-1}, \tag{4.18b}$$

where we have used $\sum_{m=0}^{k-1}(m_2+2m) = k(m_2+k-1)$ in the last line. Taking $k=\rho_2-\rho_1 = 1-m_2$ we see that in both cases ∂_{32}^k satisfies the defining relation for the intertwiner $\mathcal{T}_1$. We would now like to extend ∂_{32}^k to non-integer values of k in such a way that the results (4.18a) and (4.18b) still hold.

We start with an alternate expression for ∂_{32}^k for k a positive integer

$$\partial_{32}^k = (x_{32}^{-1}N_{32})^k = x_{32}^{-k}(N_{32}-k+1)\dots(N_{32}-1)N_{32} := x_{32}^{-k}\Phi_k(N_{32}). \tag{4.19}$$

The relation (4.18a) is checked using the commutation rule $x_{32}^{-k}N_{32} = (N_{32}+k)x_{32}^{-k}$. The relation (4.18b) can be checked using the same rule in combination with the following two relations

$$x_{32}^{-k}\Phi_k(N_{32})\partial_{32} = \partial_{32}x_{32}^{-k}\Phi_k(N_{32}), \tag{4.20a}$$

$$\begin{aligned}x_{32}^{-k}\Phi_k(N_{32})(x_{32}(m_2+N_{32})) &= (x_{32}(m_2+2k+N_{32}))x_{32}^{-k}\Phi_k(N_{32})\\ &\quad + k(m_2+k-1)x_{32}^{-k+1}\Phi_{k-1}(N_{32}). \qquad (4.20b)\end{aligned}$$

Both of these results can be obtained by using the properties

$$N_{32}\Phi_k(N_{32}-1) = (N_{32}-k)\Phi_k(N_{32}), \quad (N_{32}-k+1)\Phi_{k-1}(N_{32}) = \Phi_k(N_{32}), \tag{4.21}$$

of the finite product Φ_k. For example we check

$$\begin{aligned}
&x_{32}^{-k}\Phi_k(N_{32})(x_{32}(m_2+N_{32})) = \\
&= x_{32}^{-k+1}((m_2+k-1)\Phi_k(N_{32}+1)+(N_{32}-k+1)\Phi_k(N_{32}+1)) \\
&= x_{32}^{-k+1}\left((m_2+k-1)(k+(N_{32}-k+1))\Phi_{k-1}(N_{32})+(N_{32}+1)\Phi_k(N_{32})\right) \\
&= x_{32}^{-k+1}\left(k(m_2+k-1)\Phi_{k-1}(N_{32})+(N_{32}+1)\Phi_k(N_{32})+(m_2+k-1)\Phi_k(N_{32})\right) \\
&= k(m_2+k-1)x^{-k+1}\Phi_{k-1}(N_{32})+x_{32}(m_2+N_{32}+2k)x_{32}^{-k}\Phi_k(N_{32}),
\end{aligned}$$

where in the third line we used $\Phi_k(N_{32}+1) = (N_{32}-k+2)\Phi_{k-1}(N_{32}+1) = (N_{32}+1)\Phi_{k-1}(N_{32})$. To check (4.20a) it is helpful to write $\partial_{32} = x_{32}^{-1}N_{32}$.

One can therefore imagine extending ∂_{32}^k to non-positive integer k by using the rightmost expression in (4.19) and replacing $\Phi_k(N_{32})$ with a generalisation of the finite product which retains the properties (4.21). An appropriate generalisation is obtained using the Gamma function

$$\Phi_k(N_{32}) := \frac{\Gamma(N_{32}+1)}{\Gamma(N_{32}-k+1)}, \tag{4.22}$$

whereby properties (4.21) are obtained as a consequence of the property $\Gamma(\xi+1) = \xi\Gamma(\xi)$. One may be worried about using this definition (4.22), however we point for practical purposes that it is diagonal with respect to power functions with well defined complex eigenvalue,

$$\Phi_k(N_{32})x_{32}^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}x_{32}^{\alpha}, \tag{4.23}$$

provided $\alpha \notin \mathbb{Z}_{<0}$. The eigenvalues are zero at poles of the denominator $\alpha-k \in \mathbb{Z}_{<0}$. Combining (4.19) and (4.22) we obtain the following well known expression for the fractional derivative of a power function [27]

$$\partial_{32}^k x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}x^{\alpha-k}, \quad \text{for } \alpha \notin \mathbb{Z}_{<0}. \tag{4.24}$$

Remark 4.4. In [9] the fractional derivatives are realised as integral operators on the (complex) function space $L_2(\boldsymbol{x})$. This approach is more rigorous but does not appear to extend to the q-deformed case so we will not treat it here.

The second intertwiner $\mathcal{T}_2$ (4.12) is best dealt with by performing the change of variables $x_{21} = r_{21}, x_{32} = r_{32}, x_{31} = r_{31} + r_{21}r_{32}$, whereby

$$\partial_{21}^{(r)} = \partial_{21}^{(x)} + x_{32}\partial_{31}^{(x)} = -\tilde{D}_{21}^{(x)}, \quad \partial_{32}^{(r)} = \partial_{32}^{(x)} + x_{21}\partial_{31}^{(x)}, \quad \partial_{31}^{(r)} = \partial_{31}^{(x)}, \tag{4.25a}$$

$$N_{21}^{(r)} = N_{21}^{(x)} + \left(\tfrac{x_{32}x_{21}}{x_{31}}\right)N_{31}^{(x)}, \quad N_{32}^{(r)} = N_{32}^{(x)} + \left(\tfrac{x_{32}x_{21}}{x_{31}}\right)N_{31}^{(x)}, \tag{4.25b}$$

$$N_{31}^{(r)} = N_{31}^{(x)} - \left(\tfrac{x_{32}x_{21}}{x_{31}}\right)N_{31}^{(x)}, \tag{4.25c}$$

and the L-operator (3.13) becomes

$$L(u_1,u_2,u_3)=\begin{pmatrix} u_3+2+N_{21}^{(r)}+N_{31}^{(r)} & -\partial_{21}^{(r)}+r_{32}\partial_{31}^{(r)} & -\partial_{31}^{(r)} \\ r_{21}A'+r_{31}\partial_{32}^{(r)} & u_2+1-N_{21}^{(r)}+N_{32}^{(r)} & -\partial_{32}^{(r)} \\ r_{32}r_{21}A'+r_{31}(A'+B'+N_{21}) & r_{32}B'-r_{31}\partial_{21}^{(r)} & u_1-N_{31}^{(r)}-N_{32}^{(r)} \end{pmatrix}, \tag{4.26}$$

where

$$A'=m_1+N_{21}^{(r)},\quad B'=m_2+N_{32}^{(r)}+N_{31}^{(r)}-N_{21}^{(r)}. \tag{4.27}$$

In some sense this change of variable switches the roles of the variables r_{21} and x_{32}. Importantly the expression $\partial_{21}^{(x)}+x_{32}\partial_{31}^{(x)}=-\tilde{D}_{21}^{(x)}$ becomes the partial derivative $\partial_{21}^{(r)}$. This allows us to perform a similar analysis as in the case of $\mathcal{T}_1$ to show that $\mathcal{T}_2$ satisfies its defining relation

$$\left(\partial_{21}^{(r)}\right)^{u_2-u_3}L(u_1,u_2,u_3)=L(u_1,u_3,u_2)\left(\partial_{21}^{(r)}\right)^{u_2-u_3}, \tag{4.28}$$

where $\left(\partial_{21}^{(r)}\right)^{k}$ is extended to non-integer values of k in the same way

$$\left(\partial_{21}^{(r)}\right)^{k}=r_{21}^{-k}\frac{\Gamma(N_{21}^{(r)}+1)}{\Gamma(N_{21}^{(r)}-k+1)}=x_{21}^{-k}\frac{\Gamma\left(N_{21}+\left(\frac{x_{21}x_{32}}{x_{31}}\right)N_{31}+1\right)}{\Gamma\left(N_{21}+\left(\frac{x_{21}x_{32}}{x_{31}}\right)N_{31}-k+1\right)}. \tag{4.29}$$

Note that the quantity $\frac{x_{21}x_{32}}{x_{31}}$ commutes with all Cartan elements $E_{ii}(\boldsymbol{\rho})$ so the commutation relations between (4.29) and these entries are determined by the leftmost multiplication operator x_{21}^{-k} only.

Remark 4.5. This change of variables approach is part of a more general phenomenon. Indeed, in the $\mathfrak{sl}_n$ case if one performs the change of variables

$$x_{k+1,k}=r_{k+1,k},\quad x_{m,k+1}=r_{m,k+1},\quad x_{m,k}=r_{m,k}+r_{k+1,k}r_{m,k+1}, \tag{4.30}$$

for $k+1<m\leq n$ whilst leaving all other variables unchanged, then the operator $\tilde{D}_{k+1,k}$ becomes the partial derivative $\partial_{k+1,k}^{(r)}$ [9].

The operators $\mathcal{S}_i$ for $i=1,2,4,5$ can be built from the intertwiners $\mathcal{T}_j$ for $j=1,2$ in the prescribed way. Let us point out that since the construction of intertwiners is a representation theoretic problem they only involve the representation parameters $\boldsymbol{\rho}$, and not the spectral parameter dependent combinations $\boldsymbol{u}$. The same will not be true for exchange operator $\mathcal{S}_3$.

To study the exchange operator $\mathcal{S}_3$ it is useful first to consider the factorisation (3.11). Using this factorisation the matrix product $L_1(u_1,u_2,u_3)L_2(v_1,v_2,v_3)$ may be written as

$$
\begin{aligned}
L_1(\boldsymbol{u})L_2(\boldsymbol{v}) &= Z^{(x)}\tilde{D}^{(x)}(u_1,u_2,u_3)(Z^{(x)})^{-1}Z^{(y)}\tilde{D}^{(y)}(v_1,v_2,v_3)(Z^{(y)})^{-1} \\
&= Z_1^{(x)}(u_1)\tilde{D}^{(x)}(u_2)Z_2^{(x)}(u_3)Z_1^{(y)}(v_1)\tilde{D}^{(y)}(v_2)Z_2^{(y)}(v_3),
\end{aligned} \tag{4.31}
$$

where in the second line we have moved the u_1 (u_3) dependence of the factor $\tilde{D}^{(x)}(u_1,u_2,u_3)$ to the left (right) adjacent factor by taking

$$
\tilde{D}(u_2) = \begin{pmatrix} 1 & -\partial_{21} - x_{32}\partial_{31} & -\partial_{31} \\ 0 & u_2 & -\partial_{32} \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{D}(\boldsymbol{u}) = d_1(u_1)\tilde{D}(u_2)d_2(u_3), \tag{4.32a}
$$

$$
Z_1^{(x)}(u_1) = Z^{(x)}d_1(u_1), \quad Z_2^{(x)} = d_2(u_3)(Z^{(x)})^{-1}, \tag{4.32b}
$$

$$
d_1(a) = \mathrm{diag}(1,1,a), \qquad d_2(a) = \mathrm{diag}(a,1,1), \tag{4.32c}
$$

and likewise for the factor $\tilde{D}^{(y)}(v_1,v_2,v_3)$. In doing this we are able to sandwich the dependence of the variables u_3 and v_1 into the central two factors $Z_2^{(x)}(u_3)Z_1^{(y)}(v_1)$. This suggests what we already knew, that the exchange operator should be multiplication so that it commutes (element-wise) with the two outermost factors in the product (4.31), simplifying its defining relation. Indeed, under this assumption alone its defining relation is reduced to

$$
\begin{aligned}
&\left(\tilde{D}(u_2)^{(x)}\right)^{-1}\mathcal{S}_3(u_3-v_1)\tilde{D}(u_2)^{(x)}Z_2^{(x)}(u_3)Z_1^{(y)}(v_1) \\
&= Z_2^{(x)}(v_1)Z_1^{(y)}(u_3)\tilde{D}(v_2)^{(y)}\mathcal{S}_3(u_3-v_1)\left(\tilde{D}(v_2)^{(y)}\right)^{-1}.
\end{aligned} \tag{4.33}
$$

To proceed we use the explicit form of the exchange operator given by (4.11):

$$
\mathcal{S}_3(u_3-v_1) = \left(\begin{pmatrix} 1 & 0 & 0 \\ y_{21} & 1 & 0 \\ y_{31} & y_{32} & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix}\right)_{31}^{u_3-v_1} = (x_{31}-y_{31}-y_{32}(x_{21}-y_{21}))^{u_3-v_1}. \tag{4.34}
$$

Using the facts $[\partial_{32}^{(x)},\mathcal{S}_3(a)] = [\partial_{21}^{(y)} + y_{32}\partial_{31}^{(y)},\mathcal{S}_3(a)] = 0$ one finds

$$
\begin{aligned}
\left(\tilde{D}(u_2)^{(x)}\right)^{-1}\mathcal{S}_3(a)\tilde{D}(u_2)^{(x)} &= \left(\tilde{D}(u_2)^{(x)}\right)^{-1}\left(\tilde{D}(u_2)^{(x)}\mathcal{S}_3(a) + \begin{pmatrix} 0 & [\partial_{21}^{(x)}+x_{32}\partial_{31}^{(x)},\mathcal{S}_3(a)] & [\partial_{31}^{(x)},\mathcal{S}_3(a)] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} \mathcal{S}_3(a) & a(x_{32}-y_{32})\mathcal{S}_3(a-1) & a\mathcal{S}_3(a-1) \\ 0 & \mathcal{S}_3(a) & 0 \\ 0 & 0 & \mathcal{S}_3(a) \end{pmatrix},
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
\tilde{D}(v_2)^{(y)}\mathcal{S}_3(a)\left(\tilde{D}(v_2)^{(y)}\right)^{-1} &= \left(\mathcal{S}_3(a)\tilde{D}(v_2)^{(y)} + \begin{pmatrix} 0 & 0 & -[\partial_{31}^{(y)},\mathcal{S}_3(a)] \\ 0 & 0 & -[\partial_{32}^{(y)},\mathcal{S}_3(a)] \\ 0 & 0 & 0 \end{pmatrix}\right)\left(\tilde{D}(v_2)^{(y)}\right)^{-1} \\
&= \begin{pmatrix} \mathcal{S}_3(a) & 0 & a\mathcal{S}_3(a-1) \\ 0 & \mathcal{S}_3(a) & a(x_{21}-y_{21})\mathcal{S}_3(a-1) \\ 0 & 0 & \mathcal{S}_3(a) \end{pmatrix}.
\end{aligned} \tag{4.36}
$$

Using the formulas (4.35) and (4.36), the relation (4.33) is then checked by using basic identities for the power function $\mathcal{S}_3(a)$. For example the leading diagonal of (4.33) is the relation

$$v_1\mathcal{S}_3(u_3 - v_1) = \\ = u_3\mathcal{S}_3(u_3 - v_1) - (u_3 - v_1)(x_{31} - y_{31} - y_{32}(x_{21} - y_{21}))\mathcal{S}_3(u_3 - v_1 - 1). \quad (4.37)$$

Before this section concludes let us emphasise the importance of the factorisation (3.11) for constructing the exchange operator. We will soon see that the factorisation (3.24b) leads to an appropriate q-analog of this analysis as per [8]. The factorisation (3.26a) to (3.26c) is used for the same purpose in [26]. It is hoped that the factorisation (3.42)*to* (3.44) can provide a path to constructing the exchange operator in the $U_q(\mathfrak{sl}_4)$ case.

4.2 Deformed Case

In this section our first goal is to construct the permutation operators $\mathcal{S}_i \in \mathrm{End}(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}})$ for the $U_q(\mathfrak{sl}_2)$ L-operator in § 4.2.1 following the approach of [8, 26]. Then in § 4.2.2 we will find that the simpler $U_q(\mathfrak{sl}_2)$ case provides helpful insight into how the approach taken in Example 4.3, can be generalised to the $U_q(\mathfrak{sl}_4)$ case where we will construct only the intertwiners $\mathcal{T}_i$.

4.2.1 Permutation operators for $U_q(\mathfrak{sl}_2)$

In the $n = 2$ case the L-operator is given by the expression (3.24a). Construction of the permutation operators $\mathcal{S}_1, \mathcal{S}_2$, and $\mathcal{S}_3$ reduces to the construction of a single intertwiner $\mathcal{T}_1$ and the exchange operator $\mathcal{S}_2$.

We start with the intertwiner $\mathcal{T}_1$ with the defining relation

$$\mathcal{T}_1 L(u_1, u_2) = L(u_2, u_1)\mathcal{T}_1. \quad (4.38)$$

Comparing diagonal entries of the above relation one determines $\mathcal{T}_1 q^{N_x} = q^{u_1 - u_2 + N_x}\mathcal{T}_1$. This completely fixes the multiplication operator dependence of $\mathcal{T}_1$ and so we obtain the ansatz

$$\mathcal{T}_1 = x^{-(u_1 - u_2)}\Phi(N_x), \quad (4.39)$$

where $\Phi(N_x)$ is some undetermined function. This form is analagous to (4.19).

Comparing off-diagonal entries of the relation (4.38) we obtain the conditions

$$\mathcal{T}_1 D_x = D_x \mathcal{T}_1, \quad \mathcal{T}_1 x[u_2 - u_1 + 1 + N_x]_q = x[u_1 - u_2 + 1 + N_x]_q \mathcal{T}_1. \quad (4.40)$$

Using the ansatz (4.39) one finds the following as a necessary and sufficient condition for the relations (4.40) to hold:

$$\Phi(N_x+1)[u_2-u_1+1+N_x]_q = [N_x+1]_q\Phi(N_x). \tag{4.41}$$

One should immediately recognise this condition as a q-analog of the leftmost relation in (4.21). It is solved in similar fashion

$$\Phi(N_x) = q^{-(u_1-u_2)N_x}\frac{(q^{2(N_x+1-(u_1-u_2))};q^2)}{(q^{2(N_x+1)};q^2)}, \tag{4.42}$$

where $(a;q)$ is the q-Pochhammer function (A.7), and the property (4.41) is obtained as a consequence of the identity $(1-a)(qa;q)=(a;q)$. The formula (4.42) is valid for $|q|<1$, however, we can extend it to $|q|>1$ using the q-exponential function (A.12). From now on we will work in the $|q|<1$ case with the understanding that all results can be extended to $|q|>1$.

Remark 4.6. Up to a constant factor, the function Φ (4.42) can be rewritten with the q-gamma function (A.17) as

$$\Phi(N_x) = \frac{\Gamma_{q^2}(N_x+1)}{\Gamma_{q^2}(N_x+1-(u_1-u_2))}.$$

Using the result (A.18) this allows us to obtain the rational limit $q\to 1$ easily and make connection with the undeformed $\mathfrak{sl}_2$ case (see [8] § 4.2).

Remark 4.7. In the case where $(u_1-u_2)=k$ is a positive integer, the ratio of q-Pochhammer functions reduces to a finite product. Using this fact, the whole expression (4.39) becomes

$$\mathcal{T}_1(k) = x^{-k}q^{-kN_x}(1-q^{2(N_x-k+1)})...(1-q^{2(N_x-1)})(1-q^{2(N_x)}) \propto D_x^k, \tag{4.43}$$

where the proportionality holds using (2.39a). In this sense we see that $\mathcal{T}_1$ is an extension of D_x^k to non-integer k.

To construct the exchange operator $\mathcal{S}_2$ we will make use of the factorisation (3.24b). Using this the product $L_1(u_1,u_2)L_2(v_1,v_2)$ can be written as

$$L_1(u_1,u_2)L_2(v_1,v_2) = Z_1(u_1)^{(x)}\tilde{D}^{(x)}Z_2(u_2)^{(x)}Z_1(v_1)^{(y)}\tilde{D}^{(y)}Z_2(v_2)^{(y)}, \tag{4.44}$$

where $Z_1(u_1)^{(x)},\tilde{D}^{(x)},Z_2(u_2)^{(x)}$ are the matrix factors in (3.24b) in their respective order, and likewise for $Z_1(v_1)^{(y)},\tilde{D}^{(y)},Z_2(v_2)^{(y)}$. If it is assumed that $\mathcal{S}_2$ is a multiplication operator in the variables x,y, it will commute (element-wise) with the two outermost factors of the above (4.44). Then the defining relation for $\mathcal{S}_2$ simplifies to

$$(\tilde{D}^{(x)})^{-1}\mathcal{S}_2\tilde{D}^{(x)}Z_2(u_2)^{(x)}Z_1(v_1)^{(y)} = Z_2(v_1)^{(x)}Z_1(u_2)^{(y)}\tilde{D}^{(y)}\mathcal{S}_2(\tilde{D}^{(y)})^{-1}. \quad (4.45)$$

Writing $\mathcal{S}_2 = \mathcal{S}_2(x,y)$ this relation can be written explicitly as

$$\begin{pmatrix} \mathcal{S}_2(qx,y) & \frac{1}{x}\frac{\mathcal{S}_2(qx,y)-\mathcal{S}_2(q^{-1}x,y)}{q-q^{-1}} \\ 0 & \mathcal{S}_2(q^{-1}x,y) \end{pmatrix} \begin{pmatrix} [u_2]_q & 0 \\ (q^{-u_2+v_1}y-x)q^{u_2} & [v_1]_q \end{pmatrix} \quad (4.46)$$

$$= \begin{pmatrix} [v_1]_q & 0 \\ (q^{u_2-v_1}y-x)q^{v_1} & [u_2]_q \end{pmatrix} \begin{pmatrix} \mathcal{S}_2(x,q^{-1}y) & \frac{1}{y}\frac{\mathcal{S}_2(x,q^{-1}y)-\mathcal{S}_2(x,qy)}{q-q^{-1}} \\ 0 & \mathcal{S}_2(x,qy) \end{pmatrix}. \quad (4.47)$$

Now let us consider some particular entries of this relation. The top right entry is the relation

$$\frac{1}{x}(\mathcal{S}_2(qx,y) - \mathcal{S}_2(q^{-1}x,y)) = \frac{1}{y}(\mathcal{S}_2(x,q^{-1}y) - \mathcal{S}_2(x,qy)). \quad (4.48)$$

Using (4.48) both diagonal relations are reduced to

$$q^{u_2}S_2(q^{-1}x,y) - q^{-u_2}S_2(qx,y) = q^{v_1}S_2(x,qy) - q^{-v_1}S_2(x,q^{-1}y). \quad (4.49)$$

This is solved by the ansatz $\mathcal{S}_2(x,y) = x^{u_2-v_1}\Phi(y/x)$. One can then check with this ansatz that both off-diagonal entries reduce to the relation

$$\left(1 - q^{-u_2+v_1}\frac{y}{x}\right)\Phi\left(q^{-1}\frac{x}{y}\right) = \left(1 - q^{u_2-v_1}\frac{y}{x}\right)\Phi\left(q\frac{x}{y}\right). \quad (4.50)$$

This relation is again solved using the q-Pochhammer function (A.7)

$$\Phi(\xi) = \frac{(\xi q^{1-u_2+v_1}; q^2)}{(\xi q^{1+u_2-v_1}; q^2)}. \quad (4.51)$$

This concludes the construction of permutation operators for the $U_q(\mathfrak{sl}_2)$ case.

Remark 4.8. Let us consider the case when $u_2 - v_1 = k \in \mathbb{Z}_{>0}$ is a positive integer. In this case the ratio of q-Pochhammer becomes a finite product and we obtain the following expression for $\mathcal{S}_2(k)$:

$$\begin{aligned}\mathcal{S}_2(k) &= x^k\left(1 - \frac{y}{x}q^{k-1}\right)\left(1 - \frac{y}{x}q^{k-3}\right)\dots\left(1 - \frac{y}{x}q^{3-k}\right)\left(1 - \frac{y}{x}q^{1-k}\right) \\ &= \left(x - yq^{k-1}\right)\left(x - yq^{k-3}\right)\dots\left(x - yq^{3-k}\right)\left(x - yq^{1-k}\right). \end{aligned} \quad (4.52)$$

This expression limits to $(x-y)^k$ as $q \to 1$ which is the expression for the undeformed exchange operator (4.11) in the $n = 2$ case. In this sense $\mathcal{S}(u_2 - v_1)$ appears as an appropriate q-analog of the power function $(x-y)^{u_2-v_1}$.

4.2.2 Intertwiners for $U_q(\mathfrak{sl}_4)$

Now we begin the construction of the three intertwiners $\mathcal{T}_j$ for $j = 1, 2, 3$ in the $\mathcal{A} = U_q(\mathfrak{sl}_4)$ case. Our approach will mimic that of [26] by writing $\mathcal{T}_j = (D_j)^{u_j - u_{j+1}}$, where the D_i are q-deformed versions of the expressions $-\tilde{D}_{i,i+1}$ which appear in the underformed intertwiners (4.10). Then by considering first the case $u_j - u_{j+1} \in \mathbb{Z}_{>0}$, we reduce the problem to imposing appropriate commutation relations between the D_i and the generators e_i, f_i ((3.38a) to (3.38e)), and h_i ((3.35a) to (3.35d)) which we will use to fix previously undetermined exponents. Then by analogy with the $n = 2$ case we extend expressions D_i^k to non-integer powers.

We begin by writing $\mathcal{T}_j = (D_j)^{u_j - u_{j+1}}$ where

$$D_3 = D_{43} q^{d_{43}}, \quad D_2 = D_{32} q^{d_{32}} + x_{43} D_{42} q^{d_{42}}, \tag{4.53a}$$

$$D_1 = D_{21} q^{d_{21}} + x_{32} D_{31} q^{d_{31}} + x_{42} D_{41} q^{d_{41}}, \tag{4.53b}$$

and the d_{ij} are linear combinations of the homogeneity operators N_{ij}.

Now by viewing $\mathcal{T}_j$ as an $\mathcal{A}$-module map $\hat{\mathcal{V}}_{\boldsymbol{\rho}} \to \hat{\mathcal{V}}_{s_j \boldsymbol{\rho}}$ and describing the module $\hat{\mathcal{V}}_{\boldsymbol{\rho}}$ in terms of its lowest weight $\boldsymbol{m} \in \mathbb{C}^3$ with components $m_i = u_{5-i} - u_{4-i} + 1$, we see that the defining relations for the $\mathcal{T}_j$ are equivalent to the commutation relations

$$(D_1)^{1-m_3} g(\boldsymbol{m}) = g(m_1, m_2 + m_3 - 1, 2 - m_3)(D_1)^{1-m_3}, \tag{4.54a}$$

$$(D_2)^{1-m_2} g(\boldsymbol{m}) = g(m_1 + m_2 - 1, 2 - m_2, m_3 + m_2 - 1)(D_2)^{1-m_2}, \tag{4.54b}$$

$$(D_3)^{1-m_1} g(\boldsymbol{m}) = g(2 - m_1, m_2 + m_1 - 1, m_3)(D_3)^{1-m_1}, \tag{4.54c}$$

where $g(\boldsymbol{m})$ are any of the generators e_i, f_i, h_i in the representation $\mathcal{V}_{\boldsymbol{\rho}}$ ((3.35a) to (3.35d), and (3.38a) to (3.38e)). Notice that (4.54a) to (4.54c) can be written succinctly as $(D_{4-i})^{1-m_i} g(\boldsymbol{m}) = g(\boldsymbol{m} + (1 - m_i)\boldsymbol{a}_i)(D_{4-i})^{1-m_i}$ where $\boldsymbol{a}_i$ denotes the i-th row of the Cartan matrix (2.1).

Let us first suppose that $k_i := 1 - m_i$ is a positive integer in which case we hope to derive (4.54a) to (4.54c) as a consequence of the simpler relations

$$D_{4-i} g(\boldsymbol{m}) = g(\boldsymbol{m} + \boldsymbol{a}_i) D_{4-i}, \quad D_{4-i} e_j(\boldsymbol{m}) = e_j(\boldsymbol{m} + \boldsymbol{a}_i) D_{4-i} + \delta_{4-i,j} [m_i]_q q^{\alpha_i}, \tag{4.55}$$

for $i = 1, 2, 3$, where the α_i are constants. Note that the expressions $g(\boldsymbol{m})$ in (4.55) now refer only to the generators $h_i, f_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$. These relations are q-analogs of (4.13a) to (4.17).

To solve (4.55) we first observe that the generators $h_i \in \mathrm{End}(\mathcal{V}_{\boldsymbol{\rho}})$ ((3.35a) to (3.35d)) depend only on the homogeneity operators, N_{ij}. As such their commutation relations with the D_j are fixed by the multiplication operator dependence of the latter. One can verify that they satisfy (4.55) making use of the fact that h_i only involves the component m_i. Secondly, since the f_i ((3.38a) and (3.38b)) are independent of the lowest weight the appropriate condition for these generators is simply $[D_i, f_k] = 0$. These nine relations can be implemented digitally to fix the exponents d_{ij} uniquely. In doing so we obtain

$$D_3 = D_{43}, \quad D_2 = D_{32} q^{N_{43}-N_{42}} + x_{43} D_{42}, \tag{4.56a}$$

$$D_1 = D_{21} q^{N_{32}+N_{42}-N_{31}-N_{41}} + x_{32} D_{31} q^{N_{42}-N_{41}} + x_{42} D_{41}. \tag{4.56b}$$

One can now verify the remaining nine relations in (4.55) involving generators e_i ((3.38c) to (3.38e)), making use of the fact that each generator e_i depends only on the component m_i. In doing so we find that $\alpha_i = 0$ for all i.

The relations (4.55) now immediately imply that each $(D_{4-i})^{k_i}$ satisfies its defining relations (4.54a) to (4.54c) when $1 - m_i := k_i \in \mathbb{Z}_{>0}$, excluding the relation involving the generator e_i. In this case one can use q-arithmetic (A.5) in combination with the rightmost relation in (4.55) to inductively prove

$$(D_{4-i})^l e_i(\boldsymbol{m}) = e_i(m_i + 2l)(D_{4-i})^l + [m_i + l - 1]_q [l]_q (D_{4-i})^{l-1}. \tag{4.57}$$

We obtain the desired relation at $l = 1 - m_i$ as a result of the rightmost term vanishing.

We have seen how an integer power of the q-derivative is generalised to an abritrary power in the $n = 2$ case (4.42), so by analogy we write

$$\mathcal{T}_3(u_3 - u_4) = (x_{43}^{-1} q^{-N_{43}})^{(u_3-u_4)} \frac{(q^{2(N_{43}+1-(u_3-u_4))}; q^2)}{(q^{2(N_{43}+1)}; q^2)}, \tag{4.58}$$

as a generalisation of $(D_3)^{(u_3-u_4)}$ to non-integer power.

To generalise expressions $(D_2)^k$ and $(D_1)^k$ to non-integer powers, we first propose an efficient method for verifying formulas (4.54a) to (4.54c) in the integer case and we will see that this has a natural generalisation. We will discuss only the D_2 case for brevity. Let us start with the alternate expression

$$D_2 = -\frac{1}{x_{32}} \frac{q^{N_{43}-N_{42}-N_{32}}}{(q - q^{-1})} \left(1 - q^{2N_{32}} \boldsymbol{X}\right),$$

$$\boldsymbol{X} := \left(1 + \tfrac{x_{43}\, x_{32}}{x_{42}} (q^{N_{42}} - q^{-N_{42}}) q^{N_{42}-N_{32}-N_{43}-3}\right), \tag{4.59}$$

which in combination with the fact $[\boldsymbol{X}, x_{32}^{-1} q^{N_{43}-N_{42}-N_{32}}] = 0$ allows us to write the following compact expression for $(D_2)^k$

$$\begin{aligned}(D_2)^k &= A^{-k}\left(x_{32}^{-1} q^{N_{43}-N_{42}-N_{32}}\right)^k (1-q^{2(N_{32}-(k-1))}\boldsymbol{X})(1-q^{2(N_{32}-(k-2))}\boldsymbol{X})\dots(1-q^{2N_{32}}\boldsymbol{X}) \\ &:= A^{-k}\left(x_{32}^{-1} q^{N_{43}-N_{42}-N_{32}}\right)^k \Phi_k(q^{2N_{32}}\boldsymbol{X}), \qquad (4.60)\end{aligned}$$

where $A = (q^{-1} - q)$. We will drop the factor A^{-k} for simplicity since it does not affect the defining relations (4.54a) to (4.54c).

Now the commutation relations between $(D_2)^k$ and generators (3.35b) to (3.35d), and (3.38a) to (3.38e) can be verified using commutation relations with the quantity $\boldsymbol{X}$ and properties of the finite product Φ_k. Note that the ratio $\frac{x_{43}\,x_{32}}{x_{42}}$ commutes with all Cartan elements h_i ((3.35b) to (3.35d)), so the commutation relation between h_i and the expression (4.60) is determined by the leftmost factor $\propto x_{32}^{-k}$.

The relations involving generators f_i, e_i are somewhat more involved. Let us give a sample calculation. Since f_2 is independent of the m_i the appropriate commutation relation with $(D_2)^k$ relation is simply $[(D_2)^k, f_2] = 0$. Using the dependence of (4.60) one can see that this reduces to $[(D_2)^k, D_{32}] = 0$. This fact is easily deduced using (4.56a), however, our goal is to verify with the form (4.60) which will provide a non-integer generalisation. Our first step is to note the identity

$$\boldsymbol{X}\left(x_{32}^{-1}[N_{32}+\alpha]_q\right) = x_{32}^{-1} q^{1-(N_{32}+\alpha)} + \left(x_{32}^{-1}[N_{32}+\alpha-1]_q\right) q\boldsymbol{X}, \qquad (4.61)$$

for $\alpha \in \mathbb{C}$ a constant, [3] which in turn implies the identity

$$(1-q^{2(N_{32}+\alpha)}\boldsymbol{X})x_{32}^{-1}[N_{32}+\alpha]_q = x_{32}^{-1}[N_{32}+\alpha-1]_q q^{-1}(1-q^{2(N_{32}+\alpha)}\boldsymbol{X}). \quad (4.62)$$

Using (4.62) we can obtain the following simple commutation relation between D_{32} and the product Φ_k

$$\Phi_k(q^{2N_{32}}\boldsymbol{X})D_{32} = \left(x_{32}^{-1}[N_{32}-k]_q q^{-k}\right)\Phi_k(q^{2N_{32}}\boldsymbol{X}), \qquad (4.63)$$

which results in the desired commutation relation with (4.60) after commuting $\left(x_{32}^{-1}[N_{32}-k]_q q^{-k}\right)$ with the leftmost factor. The remaining commutation relations can be derived in similar fashion.

[3] We may even allow α to be a combination of N_{ij} which commutes with $\frac{x_{43}\,x_{32}}{x_{42}}$. Equation (4.62) will also require that α commutes with x_{32}.

Our task is now to generalise the finite product Φ_k. This is done using the q-Pochhammer function

$$\Phi_{(u_2-u_3)}(q^{2N_{32}}\boldsymbol{X}) = \frac{(q^{2(N_{32}+1-(u_2-u_3))}\boldsymbol{X};q^2)}{(q^{2(N_{32}+1)}\boldsymbol{X};q^2)}. \tag{4.64}$$

For the sake of example, let us now use the series realisation (A.11) to verify that the relation (4.63) still holds. To do this we will need the following identity proved inductively from (4.61)

$$(q^{2(N_{32}+1)}\boldsymbol{X})^l D_{32} = x_{32}^{-1}[N_{32}-l]_q q^{-l}(q^{2(N_{32}+1)}\boldsymbol{X})^l + x_{32}^{-1}q^{-(2l-3)+N_{32}}\frac{(1-q^{2l})}{(1-q^2)}(q^{2(N_{32}+1)}\boldsymbol{X})^{l-1}. \tag{4.65}$$

With this we now calculate

$$\begin{aligned}
\Phi_{(u_2-u_3)}(q^{2N_{32}}\boldsymbol{X})D_{32} &= \sum_{n=0}^{\infty}\frac{(q^{-2(u_2-u_3)};q^2)_n}{(q^2;q^2)_n}(q^{2(N_{32}+1)}\boldsymbol{X})^n D_{32} = \\
&= \sum_{n=0}^{\infty}\frac{(q^{-2(u_2-u_3)};q^2)_n}{(q^2;q^2)_n}\left(x_{32}^{-1}[N_{32}-n]_q q^{-n}(q^{2(N_{32}+1)}\boldsymbol{X})^n + x_{32}^{-1}q^{-(2n-3)+N_{32}}\frac{(1-q^{2n})}{(1-q^2)}(q^{2(N_{32}+1)}\boldsymbol{X})^{n-1}\right) \\
&= \sum_{n=0}^{\infty}\frac{(q^{-2(u_2-u_3)};q^2)_n}{(q^2;q^2)_n}x_{32}^{-1}q^{-n}\left([N_{32}-n]_q + q^{-n+1+N_{32}}\frac{(1-q^{2(n-(u_2-u_3))})}{(1-q^2)}\right)(q^{2(N_{32}+1)}\boldsymbol{X})^n \\
&= \sum_{n=0}^{\infty}\frac{(q^{-2(u_2-u_3)};q^2)_n}{(q^2;q^2)_n}x_{32}^{-1}q^{-n}\left(q^{n-(u_2-u_3)}[N_{32}-(u_2-u_3)]_q\right)(q^{2(N_{32}+1)}\boldsymbol{X})^n \\
&= x_{32}^{-1}[N_{32}-(u_2-u_3)]_q q^{-(u_2-u_3)}\Phi_{(u_2-u_3)}(q^{2N_{32}}\boldsymbol{X}).
\end{aligned} \tag{4.66}$$

A similar treatment of the operator D_1 yields

$$\begin{aligned}
D_1 &= (x_{21}^{-1}q^{N_{32}+N_{42}-N_{21}-N_{31}-N_{41}})(1-q^{2N_{21}}\boldsymbol{Y}), \\
\boldsymbol{Y} &= \begin{pmatrix} 1+\frac{x_{21}x_{32}}{x_{31}}(q^{N_{31}}-q^{-N_{31}})q^{-N_{32}+N_{31}-N_{21}-3} \\ +\frac{x_{21}x_{42}}{x_{41}}(q^{N_{41}}-q^{-N_{41}})q^{-N_{42}+N_{41}-N_{32}+N_{31}-N_{21}-3} \end{pmatrix},
\end{aligned} \tag{4.67}$$

where we note $[\boldsymbol{Y}, x_{21}^{-1}q^{N_{32}+N_{42}-N_{21}-N_{31}-N_{41}}] = 0$. This form provides the following non-integer generalisation of D_1^k

$$\mathcal{T}_1(u_1-u_2) = (x_{21}^{-1}q^{N_{32}+N_{42}-N_{21}-N_{31}-N_{41}})^{(u_1-u_2)}\frac{(q^{2(N_{21}+1-(u_1-u_2))}\boldsymbol{Y};q^2)}{(q^{2(N_{21}+1)}\boldsymbol{Y};q^2)}. \tag{4.68}$$

4.3 Coxeter Relations & Yang Baxter Equation for $U_q(\mathfrak{sl}_4)$

So far we have been able to write the $U_q(\mathfrak{sl}_4)$ intertwiners $\mathcal{T}_i(u_i-u_{i+1})$ $(i=1,2,3)$ which satisfy the defining relations $\mathcal{T}_i(u_i-u_{i+1})L(\boldsymbol{u}) = L(s_i\boldsymbol{u})\mathcal{T}_i(u_i-u_{i+1})$, where

$L(\boldsymbol{u}) \in \mathrm{End}(\mathbb{C}^4 \otimes \mathcal{V}_{\boldsymbol{\rho}})$ is the L-operator constructed in § 3.2.1 and $s_i \in S_4$ is the elementary transposition $(i, i+1)$. Using the fact that the symmetric group S_4 is generated by the elementary tranpositions, we can now write an operator $\mathcal{S}(\boldsymbol{u}) \in \mathrm{End}(\hat{\mathcal{V}}_{\boldsymbol{\rho}})$ which realises any permutation $s \in S_4$ of the 4 parameters $(\boldsymbol{u})_i = u_i$, that is,

$$\mathcal{S}(\boldsymbol{u})L(\boldsymbol{u}) = L(s\boldsymbol{u})\mathcal{S}(\boldsymbol{u}). \tag{4.69}$$

The operator $\mathcal{S}(\boldsymbol{u})$ can be explicitly written as

$$S(\boldsymbol{u}) = \mathcal{T}_{i_N}(s_{i_{N-1}} \dots s_{i_1}\boldsymbol{u})\mathcal{T}_{i_{N-1}}(s_{i_{N-2}} \dots s_{i_1}\boldsymbol{u}) \dots \mathcal{T}_{i_2}(s_{i_1}\boldsymbol{u})\mathcal{T}_{i_1}(\boldsymbol{u}), \tag{4.70}$$

where $s = s_{i_N} s_{i_{N+1}} \dots s_{i_2} s_{i_1}$ is a decomposition of s into elementary transpositions. Note that s does not have a unique decomposition into transpositions; two words in the s_i are equal in S_n if and only if they are related to each other by a finite sequence of Coxeter relations

$$s_i^2 = \mathrm{id}, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1. \tag{4.71}$$

This poses a very serious question: given two decompositions of s into elementary transpositions, how are the associated operators constructed by (4.70) related? Fortunately, this question has a nice answer:

Proposition 4.9. *The operator $\mathcal{S} \in \mathrm{End}(\hat{\mathcal{V}}_{\boldsymbol{\rho}})$ given by* (4.70) *is independent of the decomposition of s into transpositions. That is, $s_i \mapsto \mathcal{T}_i(\boldsymbol{u})$ defines a representation of the symmetric group Perm$(\boldsymbol{u}) \simeq S_4$ on $\hat{\mathcal{V}}_{\boldsymbol{\rho}}$ provided we account for the cumulative effect of permutations of $\boldsymbol{u}$, e.g. $s_i s_j \mapsto \mathcal{T}_i(s_j\boldsymbol{u})\mathcal{T}_j(\boldsymbol{u})$.*

Before we prove this let us rewrite the intertwiners from § 4.2 in the helpful form

$$\mathcal{T}_i(\alpha) = (\Lambda_i)^\alpha \Phi_{(\alpha)}(q^{2N_{i+1,i}}\boldsymbol{X}_i), \tag{4.72}$$

$$\Phi_{(\alpha)}(\boldsymbol{Z}) = \frac{(q^{2(1-\alpha)}\boldsymbol{Z}; q^2)}{(q^2\boldsymbol{Z}; q^2)} = \sum_{n=0}^{\infty} \frac{(q^{-2\alpha}; q^2)_n}{(q^2; q^2)_n}(q^2\boldsymbol{Z})^n, \tag{4.73}$$

where the $(\Lambda_i)^\alpha$ are the prefactors

$$\Lambda_3 = x_{43}^{-1} q^{-N_{43}}, \quad \Lambda_2 = x_{32}^{-1} q^{N_{43}-N_{42}-N_{32}}, \tag{4.74a}$$

$$\Lambda_1 = x_{21}^{-1} q^{N_{32}+N_{42}-N_{21}-N_{31}-N_{41}}, \tag{4.74b}$$

and the arguments $\boldsymbol{X}_i$ are $\boldsymbol{X}_3 = 1, \boldsymbol{X}_2 = \boldsymbol{X}, \boldsymbol{X}_1 = \boldsymbol{Y}$. It is important to note in all cases the property $[\Lambda_i, \boldsymbol{X}_i] = 0$. We are now ready to prove Proposition 4.9.

Proof of Proposition 4.9. It suffices to show that the map $G := \langle s_1, s_2, s_3\rangle \to \mathrm{End}(\hat{\mathcal{V}}_{\boldsymbol{\rho}})$ which sends a word $s_{i_N}s_{i_{N+1}}\dots s_{i_2}s_{i_1}$ to $S(\boldsymbol{u})$ as per (4.70), respects the Coxeter relations (4.71). Using the dependence $\mathcal{T}_i(\boldsymbol{u}) = \mathcal{T}_i(u_i - u_{i+1})$ one can see that the leftmost relations, $s_i^2 = \mathrm{id}$, amount to the claims $\mathcal{T}_i(-\alpha)\mathcal{T}_i(\alpha) = \mathrm{id}$. This can proven for all i using the form (4.72)

$$\begin{aligned}\mathcal{T}_i(-\alpha)\mathcal{T}_i(\alpha) &= (\Lambda_i)^{-\alpha}\Phi_{(-\alpha)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)(\Lambda_i)^{\alpha}\Phi_{(\alpha)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)\\ &= \Phi_{(-\alpha)}(q^{2(N_{i+1,i}-\alpha)}\boldsymbol{X}_i)\Phi_{(\alpha)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)\\ &= \frac{(q^{2(N_{i+1,i}+1)}\boldsymbol{X}_i;q^2)}{(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}_i;q^2)}\frac{(q^{2(N_{i+1,i}+1-\alpha)}\boldsymbol{X}_i;q^2)}{(q^{2(N_{i+1,i}+1)}\boldsymbol{X}_i;q^2)}\\ &= \mathrm{id}.\end{aligned}$$

The relation $s_is_j = s_js_i$ for $|i-j| > 1$ has the single case $s_1s_3 = s_3s_1$ in S_4. This amounts to the claim $\mathcal{T}_1(\alpha)\mathcal{T}_3(\beta) = \mathcal{T}_3(\beta)\mathcal{T}_3(\alpha)$. Since $\mathcal{T}_3(\beta)$ (4.58) involves only the multiplication operator x_{43}, and the homogeneity operator N_{43} and neither of these occur in $\mathcal{T}_1(\alpha)$ (4.68) these operators must commute.

The two remaining relations $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ for $i = 1,2$ require the most work. Our approach is to treat them simultaneously as far as possible. Using the dependence of the $\mathcal{T}_i$ one can see that both cases amount to the claim $\mathcal{T}_i(\alpha)\mathcal{T}_{i+1}(\alpha+\beta)\mathcal{T}_i(\beta) = \mathcal{T}_{i+1}(\beta)\mathcal{T}_i(\alpha+\beta)\mathcal{T}_{i+1}(\alpha)$. First we extract all prefactors to the left which requires the following commutation relations

$$\boldsymbol{X}_{i+1}(\Lambda_i)^{\alpha} = (\Lambda_i)^{\alpha}\boldsymbol{X}_{i+1}, \quad \boldsymbol{X}_i(\Lambda_{i+1})^{\alpha} = (\Lambda_{i+1})^{\alpha}(1 - q^{2\alpha} + q^{2\alpha}\boldsymbol{X}_i), \tag{4.75}$$

for $i = 1,2$. Using this in combination with (4.72) we can write

$$\begin{aligned}&\mathcal{T}_i(\alpha)\mathcal{T}_{i+1}(\alpha+\beta)\mathcal{T}_i(\beta) =\\ &= (\Lambda_i)^{\alpha}\Phi_{(\alpha)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)(\Lambda_{i+1})^{\alpha+\beta}\Phi_{(\alpha+\beta)}(q^{2N_{i+2,i+1}}\boldsymbol{X}_{i+1})(\Lambda_i)^{\beta}\Phi_{(\beta)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)\\ &= q^{(\alpha+\beta)\beta}(\Lambda_i)^{\alpha+\beta}(\Lambda_{i+1})^{\alpha+\beta}\Phi_{(\alpha)}(q^{2(N_{i+1,i}-\beta)}(1-q^{2(\alpha+\beta)}+q^{2(\alpha+\beta)}\boldsymbol{X}_i))\\ &\quad\times \Phi_{(\alpha+\beta)}(q^{2N_{i+2,i+1}}\boldsymbol{X}_{i+1})\Phi_{(\beta)}(q^{2N_{i+1,i}}\boldsymbol{X}_i),\end{aligned} \tag{4.76}$$

where in the last line we used $\Lambda_{i+1}\Lambda_i = q\Lambda_i\Lambda_{i+1}$ to reorganise the prefactor. A similar approach yields

$$\begin{aligned}&\mathcal{T}_{i+1}(\beta)\mathcal{T}_i(\alpha+\beta)\mathcal{T}_{i+1}(\alpha) =\\ &= q^{(\alpha+\beta)\beta}(\Lambda_i)^{\alpha+\beta}(\Lambda_{i+1})^{\alpha+\beta}\Phi_{(\beta)}(q^{2(N_{i+2,i+1}-\alpha)}(\boldsymbol{X}_{i+1}))\Phi_{(\alpha+\beta)}(q^{2N_{i+1,i}}(1-q^{2\alpha}+q^{2\alpha}\boldsymbol{X}_i))\\ &\quad\times \Phi_{(\alpha)}(q^{2N_{i+2,i+1}}\boldsymbol{X}_{i+1}).\end{aligned} \tag{4.77}$$

Therefore, one sees that the equality of (4.76) and (4.77) is reduced to

$$\begin{aligned}&\Phi_{(\alpha)}(q^{2(N_{i+1,i}-\beta)}(1+q^{2(\alpha+\beta)}(\boldsymbol{X}_i-1)))\Phi_{(\alpha+\beta)}(q^{2N_{i+2,i+1}}\boldsymbol{X}_{i+1})\Phi_{(\beta)}(q^{2N_{i+1,i}}\boldsymbol{X}_i)\\ &=\Phi_{(\beta)}(q^{2(N_{i+2,i+1}-\alpha)}(\boldsymbol{X}_{i+1}))\Phi_{(\alpha+\beta)}(q^{2N_{i+1,i}}(1+q^{2\alpha}(\boldsymbol{X}_i-1)))\Phi_{(\alpha)}(q^{2N_{i+2,i+1}}\boldsymbol{X}_{i+1}).\end{aligned} \tag{4.78}$$

We will check this in the $i=2$ case where $\boldsymbol{X}_2=\boldsymbol{X}$ and $\boldsymbol{X}_3=1$. This is done by expanding both sides of (4.78) as power series, and reorganising powers using the fact that $\boldsymbol{x}:=(\boldsymbol{X}-1)\propto x_{43}x_{32}$. For the RHS this process yields

$$\begin{aligned}&\Phi_{(\beta)}(q^{2(N_{43}-\alpha)})\Phi_{(\alpha+\beta)}(q^{2N_{32}}(1+q^{2\alpha}(\boldsymbol{X}-1)))\Phi_{(\alpha)}(q^{2N_{43}})=\\ &=\sum_{l,m,n=0}^{\infty}\frac{(q^{-2\beta};q^2)_l}{(q^2;q^2)_l}\frac{(q^{-2(\alpha+\beta)};q^2)_m}{(q^2;q^2)_m}\frac{(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n}(q^{2(N_{43}+1-\alpha)})^l(q^{2(N_{32}+1)}(1+q^{2\alpha}\boldsymbol{x}))^m(q^{2(N_{43}+1)})^n\\ &=\sum_{l,m,n=0}^{\infty}\frac{(q^{-2\beta};q^2)_l}{(q^2;q^2)_l}\frac{(q^{-2(\alpha+\beta)};q^2)_m}{(q^2;q^2)_m}\frac{(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n}q^{-2l\alpha}(q^{2(N_{43}+1)})^{n+l}((1+q^{2(\alpha-n+1)}\boldsymbol{x})q^{2(N_{32}+1)})^m.\end{aligned} \tag{4.79}$$

Now we simplify the rightmost power using the (integer) q-binomial theorem (1.20)

$$\begin{aligned}((1+q^{2(\alpha-n+1)}\boldsymbol{x})q^{2(N_{32}+1)})^m&=\sum_{k=0}^{m}\begin{bmatrix}m\\k\end{bmatrix}_{q^2}(q^{2(\alpha-n+1)}\boldsymbol{x}q^{2(N_{32}+1)})^k(q^{2(N_{32}+1)})^{m-k}\\ &=\sum_{k=0}^{m}\frac{(q^2;q^2)_m q^{k(k-1+2(\alpha-n))}}{(q^2;q^2)_k(q^2;q^2)_{m-k}}(q^2\boldsymbol{x})^k(q^{2(N_{32}+1)})^m.\end{aligned} \tag{4.80}$$

Using (4.80) in (4.79) now gives

$$\begin{aligned}(4.79)=&\sum_{l,m,n=0}^{\infty}\sum_{k=0}^{m}\frac{(q^{-2\beta};q^2)_l}{(q^2;q^2)_l}\frac{(q^{-2(\alpha+\beta)};q^2)_m}{(q^2;q^2)_k(q^2;q^2)_{m-k}}\frac{(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n}q^{k(k-1+2(\alpha+l))-2l\alpha}\\ &\times(q^2\boldsymbol{x})^k(q^{2(N_{43}+1)})^{n+l}(q^{2(N_{32}+1)})^m\\ =&\sum_{i,j,k=0}^{\infty}\Theta_{i,j,k}(q^2\boldsymbol{x})^k(q^{2(N_{43}+1)})^i(q^{2(N_{32}+1)})^{j+k},\end{aligned} \tag{4.81}$$

where the coefficients $\Theta_{i,j,k}$ are given by

$$\Theta_{i,j,k}:=\frac{(q^{-2(\alpha+\beta)};q^2)_{j+k}}{(q^2;q^2)_k(q^2;q^2)_j}q^{k(k-1+2(\alpha+i))-2i\alpha}\sum_{n=0}^{i}\frac{(q^{-2\beta};q^2)_{i-n}}{(q^2;q^2)_{i-n}}\frac{(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n}q^{2n(\alpha-k)}. \tag{4.82}$$

A similar process for the LHS of (4.78) which requires two applications of (1.20) yields

$$\begin{aligned}\Phi_{(\alpha)}(q^{2(N_{3,2}-\beta)}(1+q^{2(\alpha+\beta)}\boldsymbol{x}))\Phi_{(\alpha+\beta)}(q^{2N_{4,3}})\Phi_{(\beta)}(q^{2N_{3,2}}\boldsymbol{X})\\ =\sum_{i,j,k=0}^{\infty}\Omega_{i,j,k}(q^2\boldsymbol{x})^k(q^{2(N_{43}+1)})^i(q^{2(N_{32}+1)})^{j+k},\end{aligned} \tag{4.83}$$

with coefficients $\Omega_{i,j,k}$ given by

$$\begin{aligned}\Omega_{i,j,k}:=\frac{(q^{-2(\alpha+\beta)};q^2)_i}{(q^2;q^2)_i}q^{k(k-1+2\alpha)-2\beta j}\Bigg(\sum_{m=0}^{j}\sum_{l=0}^{k}\frac{(q^{-2\alpha};q^2)_{k+j-(m+l)}}{(q^2;q^2)_{k-l}(q^2;q^2)_{j-m}}\frac{(q^{-2\beta};q^2)_{m+l}}{(q^2;q^2)_l(q^2;q^2)_m}\\ \times q^{2l(i+j-m)+2(m\beta-l\alpha)}\Bigg).\end{aligned} \tag{4.84}$$

Verifying that $\Omega_{i,j,k}=\Theta_{i,j,k}$ requires some theory of basic hypergeometric series. It is proved in Appendix A.2. One may verify the $i=1$ case of (4.78) in similar fashion. □

Now armed with Proposition 4.9 we can realise any permutation $s\in\mathrm{Perm}(\boldsymbol{u})\simeq S_4$ of the parameters entering the L-operator $L(\boldsymbol{u})$, as commutation with some operator $S(\boldsymbol{u})\in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}})$ (4.70). This operator depends only on the permutation s and not on a decomposition of s into transpositions. More generally, we may consider an m-fold product of L-operators

$$\boldsymbol{L}(\boldsymbol{U})=L_1(\boldsymbol{u}^{(1)})L_2(\boldsymbol{u}^{(2)})\dots L_m(\boldsymbol{u}^{(m)})\in\mathrm{End}\left(\mathbb{C}^4\otimes\left(\bigotimes_{i=1}^{m}\mathcal{V}_{\boldsymbol{\rho}^{(i)}}\right)\right) \tag{4.85}$$

where each $L_i(\boldsymbol{u}^{(i)})$ only acts non-trivially on $\mathbb{C}^4\otimes\mathcal{V}_{\boldsymbol{\rho}^{(i)}}$. Note there are now $4m$ parameters $\boldsymbol{U}:=(\boldsymbol{u}^{(1)},\boldsymbol{u}^{(2)},\dots,\boldsymbol{u}^{(m)})$ entering the product $\boldsymbol{L}$. However, Proposition 4.9 only allows us to realise the subgroup

$$\mathrm{Perm}(\boldsymbol{u}^{(1)})\times\cdots\times\mathrm{Perm}(\boldsymbol{u}^{(m)})\simeq S_4\times\cdots\times S_4\hookrightarrow\mathrm{Perm}(\boldsymbol{U})\simeq S_{4m},$$

of permuations of the parameters entering $\boldsymbol{L}$ via the operators

$$\mathcal{S}_{4j+i}(\boldsymbol{u}^{(j)})=\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\overset{\overset{j}{\downarrow}}{\mathcal{T}_i(\boldsymbol{u}^{(j)})}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\in GL\left(\bigotimes_{i=1}^{n}\hat{\mathcal{V}}_{\boldsymbol{\rho}^{(i)}}\right), \tag{4.86}$$

which satisfy $\mathcal{S}_{4j+i}(\boldsymbol{u}^{(j)})\boldsymbol{L}(\boldsymbol{U})=\boldsymbol{L}(s_{4j+i}\boldsymbol{U})\mathcal{S}_{4j+i}(\boldsymbol{u}^{(j)})$ for $j=1,\dots,m$ and $i=1,2,3$.

To realise the full symmetric group S_{4m} we need the exchange operators $\mathcal{S}_4(u_4 - v_1)$ ($\mathcal{S}_n$ in general (4.3c)), mentioned at the start of this chapter and which we saw for the $\mathfrak{sl}_n$ and $U_q(\mathfrak{sl}_2)$ cases. These allow adjacent L-operators to exchange parameters via the operators

$$\mathcal{S}_{4i} = \mathrm{id}_{\mathcal{V}_{\boldsymbol{\rho}^{(1)}}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{V}_{\boldsymbol{\rho}^{(i-1)}}} \otimes \overset{i,i+1}{\downarrow}{\mathcal{S}_4(u_4^{(i)} - u_1^{(i+1)})} \otimes \mathrm{id}_{V_{\boldsymbol{\rho}^{(i+2)}}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{V}_{\boldsymbol{\rho}^{(m)}}} \in \mathrm{End}\left(\bigotimes_{i=1}^{n} \hat{\mathcal{V}}_{\boldsymbol{\rho}^{(i)}} \right). \tag{4.87}$$

Then for $s = s_{i_N} \dots s_{i_1} \in S_{4m}$ the operator

$$\mathcal{S}(\boldsymbol{U}) = \mathcal{S}_{i_N}(s_{i_{N-1}} \dots s_{i_1} \boldsymbol{U}) \dots \mathcal{S}_{i_2}(s_{i_1} \boldsymbol{U}) \mathcal{S}_{i_1}(\boldsymbol{U}), \tag{4.88}$$

realises the desired permutation $\mathcal{S}(\boldsymbol{U})\boldsymbol{L}(\boldsymbol{U}) = \boldsymbol{L}(s\boldsymbol{U})\mathcal{S}(\boldsymbol{U})$. In order for (4.88) to provide a realisation of S_{4m} which is independent of how we decompose $s \in S_{4m}$ into transpositions, we further require that $\mathcal{S}_4(u_4 - v_1)$ satisfies the relevant Coxeter relations for $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}) \simeq S_8$,

$$\mathcal{S}_4(-\alpha)\mathcal{S}_4(\alpha) = \mathrm{id}, \quad \mathcal{S}_j(\beta)\mathcal{S}_4(\alpha) = \mathcal{S}_4(\alpha)\mathcal{S}_j(\beta), \quad \text{for } |j-4| > 1, \tag{4.89a}$$

$$\mathcal{S}_4(\beta)\mathcal{S}_{4\pm1}(\alpha+\beta)\mathcal{S}_4(\alpha) = \mathcal{S}_{4\pm1}(\alpha)\mathcal{S}_4(\alpha+\beta)\mathcal{S}_{4\pm1}(\beta). \tag{4.89b}$$

These relations are proved in [8] for the $U_q(\mathfrak{sl}_2)$ case and in [9] for the $\mathfrak{sl}_n$ case.

One of the goals of this chapter was to solve the relation (4.2) by using this parameter permutation method. We cannot claim to have accomplished this without writing the exchange operator $\mathcal{S}_4$, however, let us now demonstrate that this operator is the missing piece that allows us to solve (4.2) and the Yang-Baxter equation (4.6).

Theorem 4.10. *Suppose that $\mathcal{S}_4 \in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}})$ is an exchange operator for the $U_q(\mathfrak{sl}_4)$ L-operator* (3.41)*, which also satisfies the Coxeter relations* (4.89a) *and* (4.89b)*. Let $t_i = s_i s_{i+1} s_{i+2} s_{i+3} s_{i+2} s_{i+1} s_i \in Perm(\boldsymbol{u}, \boldsymbol{v}) \simeq S_8$ be a decomposition of the permutation $(i\, i+4) \in S_8$ (for $i = 1, 2, 3, 4$) into transpositions, and let $\mathcal{S}_{t_i}(\boldsymbol{u}, \boldsymbol{v}) \in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}})$ be the associated permutation operator constructed using* (4.88) *on the word t_i. Then for the operator $\hat{\mathcal{R}}(\boldsymbol{u}, \boldsymbol{v}) \in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}})$ defined by*

$$\hat{\mathcal{R}}(\boldsymbol{u}, \boldsymbol{v}) := \mathcal{S}_{t_4}(t_3 t_2 t_1(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_{t_3}(t_2 t_1(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_{t_2}(t_1(\boldsymbol{u}, \boldsymbol{v}))\mathcal{S}_{t_1}(\boldsymbol{u}, \boldsymbol{v}), \tag{4.90}$$

we have

1. *$\hat{\mathcal{R}}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{R}(u - v)$ depends only on the difference in spectral parameters $u - v$ (neglecting dependence on the ρ_i and σ_i),*

2. *$\hat{\mathcal{R}}(u-v)$ satisfies the relation* (4.2),

3. *$\hat{\mathcal{R}}(u-v)$ satisfies the Yang-Baxter equation* (4.6) *in $GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\tau}})$, where we interpret $\hat{\mathcal{R}}(u-v)_{12} \in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\tau}})$ to be the R-matrix $\hat{\mathcal{R}}(u-v) \in GL(\hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}})$ extended with the identity in the third factor. Likewise, $\hat{\mathcal{R}}(u-v)_{23}$ extends the R-matrix $\hat{\mathcal{R}}(u-v) \in GL(\hat{\mathcal{V}}_{\boldsymbol{\sigma}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\tau}})$.*

Proof. 1. Using the dependence of the transposition operators $\mathcal{S}_j$ (4.4) the only combinations which can possibly enter $\mathcal{R}$ are $u_i - u_j, v_i - v_j$ and $u_i - v_j$. Therefore, the spectral parameters can only enter in the combination $u - v$ which occurs in the first factor $\mathcal{S}_{t_1}(\boldsymbol{u}, \boldsymbol{v})$.

2. To prove this it is sufficient to note that $t_4 t_3 t_2 t_1$ is a decomposition into transpositions of the permutation $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (\boldsymbol{v}, \boldsymbol{u}) \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}) \simeq S_8$.

3. To prove this we consider permutations of the 12 parameters $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, entering the triple product

$$L_1(\boldsymbol{u}) L_2(\boldsymbol{v}) L_3(\boldsymbol{w}) \in GL(\mathbb{C}^4 \otimes \hat{\mathcal{V}}_{\boldsymbol{\rho}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\sigma}} \otimes \hat{\mathcal{V}}_{\boldsymbol{\tau}}). \tag{4.91}$$

Let $s^{(12)} \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ be the permutation defined by $s^{(12)}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w})$ and similarly let $s^{(23)} \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ be the permutation defined by $s^{(23)}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$. Then note that $s^{(12)} = t_4^{(12)} t_3^{(12)} t_2^{(12)} t_1^{(12)}$ is a decomposition into transpositions, where $t_i^{(12)}$ is a copy of $t_i \in \mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v})$ associated with the double product $L_1(\boldsymbol{u}) L_2(\boldsymbol{v})$, now considered as an element of $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. Similarly, one has $s^{(23)} = t_4^{(23)} t_3^{(23)} t_2^{(23)} t_1^{(23)}$ where $t_i^{(23)}$ are copies of $t_i \in \mathrm{Perm}(\boldsymbol{v}, \boldsymbol{w})$.

Now we note that permutations $s^{(12)} s^{(23)} s^{(12)}$ and $s^{(23)} s^{(12)} s^{(23)}$ are equal in $\mathrm{Perm}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$. Using the aforementioned decompositions of $s^{(12)}$ and $s^{(23)}$ and the fact that (4.88) does not depend on the choice of decomposition we obtain equality of the two operators

$$\mathcal{R}_{12}(s^{(23)} s^{(12)} \boldsymbol{u}, \boldsymbol{v}) \mathcal{R}_{23}(s^{(12)} \boldsymbol{v}, \boldsymbol{w}) \mathcal{R}_{12}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{R}_{12}(v-w) \mathcal{R}_{23}(u-w) \mathcal{R}_{12}(u-v),$$
$$\mathcal{R}_{23}(s^{(12)} s^{(23)} \boldsymbol{v}, \boldsymbol{w}) \mathcal{R}_{12}(s^{(23)} \boldsymbol{u}, \boldsymbol{v}) \mathcal{R}_{23}(\boldsymbol{v}, \boldsymbol{w}) = \mathcal{R}_{23}(u-v) \mathcal{R}_{12}(u-w) \mathcal{R}_{23}(v-w),$$

which is exactly the Yang-Baxter equation (4.6). □

Conclusion

This thesis has presented a study of the parameter permutation method [8, 9] of constructing solutions to the Yang-Baxter equation, with $\mathfrak{sl}_n$ and $U_q(\mathfrak{sl}_n)$ symmetry, and which act in products of spaces of power series in $n(n-1)/2$ variables. Since the undeformed $\mathfrak{sl}_n$ case has been well studied the focus of this thesis was the deformed $U_q(\mathfrak{sl}_n)$ case. The main contribution made here was to study the $U_q(\mathfrak{sl}_4)$ case where we obtained a novel factorisated L-operator ((3.41) to (3.44)), which generalises the known factorisation for the $\mathfrak{sl}_4$ case [9, 7]. This result poses some questions which deserve further study.

Firstly, whilst the $U_q(\mathfrak{sl}_4)$ intertwiners were found in § 4.2.2, a complete solution to the YBE also requires exchange operators (4.3c) which allow two copies of the $U_q(\mathfrak{sl}_4)$ L-operator $L(\boldsymbol{u})$ to exchange their defining parameters. The factorisation obtained for $L(\boldsymbol{u})$ ((3.41) to (3.44)) suggests that it is possible to construct such an operator via a similar method as in the $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ cases [26, 8]. Furthermore, if verified, the conjectured representation of $U_q(\mathfrak{sl}_n)$ on $\mathcal{V}^{(n)}$ (Conjecture A) may provide a pathway to extending this construction to the general $U_q(\mathfrak{sl}_n)$ case.

Secondly, the representation $\rho : U_q(\mathfrak{sl}_4) \to \mathcal{V}^{(4)}$ (defined by (3.35a) to (3.35d) and (3.38a) to (3.38e)) which was used to build the $U_q(\mathfrak{sl}_4)$ L-operator exhibits a phenomena not seen in the $U_q(\mathfrak{sl}_n)$ cases for $n < 4$. That is, the q-deformed Cartan-Weyl elements $E_{42}, E_{24} \in \mathrm{End}(\mathcal{V}^{(4)}_{\boldsymbol{\rho}})$ contain terms with a prefactor of $(q - q^{-1})$ and which are second order in the q-derivative operators D_{ij}. As a result these terms disappear in the rational limit $q \to 1$ and are not detected in the $\mathfrak{sl}_4$ representation on $\mathcal{V}^{(4)}$. It is currently unknown whether these terms are an artefact of the specific representation used here or whether they are an inherent part of representations of $U_q(\mathfrak{sl}_4)$ (or more generally $U_q(\mathfrak{sl}_n)$ for $n > 3$) on the space $\mathcal{V}^{(4)}$. It is also unknown if these terms will affect the construction of a $U_q(\mathfrak{sl}_4)$ exchange operator discussed in the previous paragraph.

Finally, Derkachov, Chicherin and Isaev made use of the Lie algebra isomorphism $\mathfrak{sl}_4 \simeq (\mathfrak{so}(6))$ to obtain an R-matrix acting in the product of spaces of functions in 6 variables with 4-dimensional conformal symmetry [7]. In this setting, the Coxeter relations between elementary transposition operators used to build the R-matrix are realised as integral operators (see Remark 4.4). This allows them to take on a remarkable intertepretation as a star-triangle relation for propagators in a conformal field theory [30]. It is hoped that a q-deformation of this analysis using the Hopf-algebra isomorphism $U_q(\mathfrak{sl}_4) \simeq U_q(\mathfrak{so}(6))$ will provide an appropriate q-analog of this star-triangle relation.

Appendix A

q-analog Results

The goal of this appendix is to present some useful results from q-analog theory which are used in the main text.

A.1 Elementary q-arithmetic

In this section we collect some helpful elementary properties of q-numbers $[\xi]_q = \frac{q^{\xi}-q^{-\xi}}{q-q^{-1}}$. Frequently used in this thesis are the results

$$[A]_q[B+C]_q - [A+C]_q[B]_q = [C]_q[A-B]_q, \tag{A.1}$$
$$[A]_q q^{\pm B} + [B]_q q^{\mp A} = [A+B]_q, \tag{A.2}$$

and the following generalisation of (A.2)

$$\sum_{k=1}^{n} [A_k]_q q^{\alpha_k} = \left[\textstyle\sum_{k=1}^{n} A_k\right]_q, \quad \text{where} \quad \alpha_k = -\left(\sum_{l=1}^{k-1} A_l\right) + \sum_{l=k+1}^{n} A_l, \tag{A.3}$$

which is proved by induction on n. Note that the exponents α_k are dependent on the ordering of terms in the leftmost sum in (A.3) and as such are not uniquely determined. Indeed, for distinct A_k this gives $n!$ n-tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ which solve the leftmost relation in (A.3). The $C = 1$ case of (A.1) is particularly useful

$$[A]_q[B+1]_q - [A+1]_q[B]_q = [A-B]_q. \tag{A.4}$$

Another useful result is the following q-analog of an arithmetic sequence:

$$\sum_{k=1}^{n} [A + 2(k-1)]_q = [n]_q[A + (n-1)]_q. \tag{A.5}$$

This is proved by induction, where the inductive step makes use of (A.4) in the form

$$[A+2n]_q = [n+1]_q[A+n]_q - [n]_q[A+(n-1)]. \tag{A.6}$$

Whilst expressions $[\xi]_q$ are designed to generalise some properties of numbers, note that the only assumption needed on arguments (A, A_i, B, C) in deriving (A.1) to (A.6) is pairwise commutativity, so that indices behave additively under multiplication. Hence these results are appropriate for a family of mutually commuting operators.

A.2 q-special Functions

In this section we define and examine properties of some q-deformed versions of special functions, which are used throughout this text. The main building block from which these functions are built is the infinite q-shifted factorial or the q-Pochhammer symbol

$$(a;q) = \prod_{k=0}^{\infty}(1-aq^k). \tag{A.7}$$

This function is an analytic function of q for $|q| < 1$ [14]. We can recover a finite product by taking an appropriate ratio of such functions

$$(a;q)_n = \frac{(a;q)}{(aq^n;q)} = \begin{cases} \prod_{k=0}^{n-1}(1-aq^k), & n \in \mathbb{Z}_{>0} \\ \left(\prod_{k=0}^{n-1}(1-aq^{-k})\right)^{-1}, & n \in \mathbb{Z}_{<0} \\ 1, & n = 0. \end{cases} \tag{A.8}$$

Note in particular the limiting behaviour $(a;q)_n \to (1-a)^n$ as $q \to 1$. This provides us with a natural q-generalisation of the power $(1-a)^\alpha$ as the ratio

$$(a;q)_\alpha = \frac{(a;q)}{(aq^\alpha;q)}. \tag{A.9}$$

One has the following relation between the finite product (A.8) and the two q-factorials (1.17) and (1.21)

$$(q^2;q^2)_n = (1-q^2)^n [\![n]\!]_{q^2}! = (q-q^{-1})^n q^{n(n-1)/2}[n]_q!, \tag{A.10}$$

which we make use of in equation (4.80).

An important result involving (A.7) is the q-binomial theorem:

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \tag{A.11}$$

for $|z| < 1, |q| < 1$. This series form is implied in expressions such as (4.42), (4.51), (4.58), (4.64) and (4.68), making it clear that they describe operators on the space $\hat{\mathcal{V}}$.

The identity (A.11) gives two candidates for a q-deformed exponential function

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \tag{A.12}$$

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-z;q)_\infty, \tag{A.13}$$

$$\lim_{q\to 1^-} e_q(z(1-q)) = \lim_{q\to 1^-} E_q(z(1-q)) = e^z. \tag{A.14}$$

Using the rightmost form for $e_q(z)$ one can verify

$$e_q(qz) = e_q(z) \cdot (1-z). \tag{A.15}$$

Using the series forms one can obtain $e_{q^{-1}}(z) = E_q(-qz)$, which allows for $e_q(z)$ to be analytically continued to $|q| \neq 1$ in such a way that (A.15) continues to be true. Also note that the identity $(e_q(z))^{-1} = E_q(-z)$, is an identity of power series so expressions such as $(e_q(z))^{-1}$ make arguments for all arguments.

As with much of q-anologue theory, we obtain nice results when considering variables with q-deformed commutation relations. For indeterminates x and y such that $xy = qyx$ we have the following identities [21] (as formal power series)

$$e_q(x+y) = e_q(y)e_q(x), \quad E_q(x+y) = E_q(x)E_q(y), \tag{A.16a}$$

$$\begin{aligned} e_q(x)e_q(y) &= e_q(y-yx)e_q(x) = e_q(x+y-yx) = \\ &= e_q(y)e_q(-yx)e_q(x) = e_q(y)e_q(x-yx), \end{aligned} \tag{A.16b}$$

$$E_q(y)E_q(x) = E_q(x+y+yx) = E_q(x)E_q(yx)E_q(y). \tag{A.16c}$$

The q-Gamma function is defined as

$$\Gamma_q(z) = \begin{cases} \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{(1-z)}, & 0 < |q| < 1, \\ \frac{(q^{-1};q^{-1})_\infty}{(q^{-z};q^{-1})_\infty}(1-q)^{(1-z)}, & |q| > 1. \end{cases} \tag{A.17}$$

It enjoys many properties analagous to the standard Gamma function [14] chief among them are the following

$$\lim_{q\to 1} \Gamma_q(z) = \Gamma(z), \quad \Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z). \tag{A.18}$$

The rest of this section will be dedicated to proving the equality of the coefficients $\Theta_{i,j,k}$ (4.82), and $\Omega_{i,j,k}$ (4.84), which appear in the proof of Proposition 4.9.

To do this we will rewrite both coefficients in terms of the basic hypergeometric functions ${}_r\phi_s$ defined by

$$ {}_r\phi_s\begin{bmatrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{bmatrix}; q, z\Bigg] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \dots (a_r;q)_n}{(q;q)_n(b_1;q)_n \dots (b_s;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} z^n. \tag{A.19} $$

Then we can use the transformation formulae from appendices I and III in [14] to obtain equality.

Firstly, using (I.11) [14] we can rewrite $\Theta_{i,j,k}$ as

$$ \begin{aligned} \Theta_{i,j,k} &= \frac{(q^{-2(\alpha+\beta)};q^2)_{j+k}(q^{-2\beta};q^2)_i}{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_k} q^{k(k-1+2(\alpha+i))-2i\alpha} \sum_{n=0}^{i} \frac{(q^{-2i};q^2)_n(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n(q^{2(1-i+\beta)};q^2)_n} q^{2n(1+\beta+\alpha-k)} \\ &= \frac{(q^{-2(\alpha+\beta)};q^2)_{j+k}(q^{-2\beta};q^2)_i}{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_k} q^{k(k-1+2(\alpha+i))-2i\alpha}\, {}_2\phi_1\left[\begin{matrix} q^{-2i}, q^{-2\alpha} \\ q^{2(1-i+\beta)} \end{matrix}; q^2, q^{2(1+\alpha+\beta-k)}\right], \end{aligned} \tag{A.20} $$

where we note that the argument q^{-2i} truncates the infinite sum in the definition (A.19) as $(q^{-2i};q^2)_n = 0$ for $n > i$. Combining results (III.6) and (I.9) [14] we obtain

$$ {}_2\phi_1\left[\begin{matrix} q^{-2i}, q^{-2\alpha} \\ q^{2(1-i+\beta)} \end{matrix}; q^2, q^{2(1+\alpha+\beta-k)}\right] = \frac{(q^{-2(\alpha+\beta)};q^2)_i}{(q^{-2\beta};q^2)_i} q^{2i(\alpha+\beta-k)}\, {}_3\phi_2\left[\begin{matrix} q^{-2i}, q^{-2(\alpha+\beta-k)}, q^{-2\beta} \\ q^{-2(\alpha+\beta)}, 0 \end{matrix}; q^2, q^2\right]. \tag{A.21} $$

Combining (A.20) and (A.21) we obtain

$$ \Theta_{i,j,k} = \frac{(q^{-2(\alpha+\beta)};q^2)_{j+k}(q^{-2(\alpha+\beta)};q^2)_i}{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_k} q^{k(k-1+2\alpha)+2i\beta}\, {}_3\phi_2\left[\begin{matrix} q^{-2i}, q^{-2(\alpha+\beta-k)}, q^{-2\beta} \\ q^{-2(\alpha+\beta)}, 0 \end{matrix}; q^2, q^2\right]. \tag{A.22} $$

Now with three applications of (I.10) [14] we obtain the following form for $\Omega_{i,j,k}$

$$ \begin{aligned} \Omega_{i,j,k} &= \frac{(q^{-2(\alpha+\beta)};q^2)_i(q^{-2\alpha};q^2)_{k+j}}{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_k} q^{k(k-1+2\alpha)-2\beta j} \\ &\times \left(\sum_{m=0}^{j}\sum_{l=0}^{k} \frac{(q^{-2k};q^2)_l(q^{-2j};q^2)_m(q^{-2\beta};q^2)_{m+l}}{(q^{2(1-k-j+\alpha)};q^2)_{m+l}(q^2;q^2)_l(q^2;q^2)_m} q^{2(m(1-k+\alpha+\beta)+l(1+i))}\right). \end{aligned} \tag{A.23} $$

The double sum in brackets in the second line can be identified with the (terminating) first q-Appell function $\Phi^{(1)}(q^{-2\beta}; q^{-2k}, q^{-2j}; q^{2(1+\alpha-(k+j))}; q^2; q^{2(i+1)}, q^{2(1-k+\alpha+\beta)})$

(10.2.5) [14]. In this case, the arguments of $\Phi^{(1)}$ are such that we may apply (10.3.7) [14] to rewrite this function as

$$\begin{aligned}
&\Phi^{(1)}(q^{-2\beta}; q^{-2k}, q^{-2j}; q^{2(1+\alpha-(k+j))}; q^2; q^{2(i+1)}, q^{2(1-k+\alpha+\beta)}) \\
&= \frac{(q^{-2(\alpha+\beta)}; q^2)_{j+k}(q^{-2\alpha}; q^2)_k}{(q^{-2(\alpha+\beta)}; q^2)_k(q^{-2\alpha}; q^2)_{j+k}} q^{2j\beta}\, {}_2\phi_1 \left[\begin{matrix} q^{-2\beta}, q^{-2k} \\ q^{2(1+\alpha-k)} \end{matrix} ; q^2, q^{2(i+1)} \right], \qquad \text{(A.24)}
\end{aligned}$$

after two applications of (I.14). The terminating ${}_2\phi_1$ series is rewritten with (III.6) and (I.9) [14] as

$$\begin{aligned}
{}_2\phi_1 \left[\begin{matrix} q^{-2\beta}, q^{-2k} \\ q^{2(1+\alpha-k)} \end{matrix} ; q^2, q^{2(i+1)} \right] &= \frac{(q^{-2(\alpha+\beta)}; q^2)_k}{(q^{-2\alpha}; q^2)_k} q^{2ki}\, {}_3\phi_2 \left[\begin{matrix} q^{-2k}, q^{-2i}, q^{-2\alpha} \\ q^{-2(\alpha+\beta)}, 0 \end{matrix} ; q^2, q^2 \right] \\
&= \frac{(q^{-2(\alpha+\beta)}; q^2)_k}{(q^{-2\alpha}; q^2)_k} q^{2\beta i}\, {}_3\phi_2 \left[\begin{matrix} q^{-2(\alpha+\beta-k)}, q^{-2i}, q^{-2\beta} \\ q^{-2(\alpha+\beta)}, 0 \end{matrix} ; q^2, q^2 \right], \qquad \text{(A.25)}
\end{aligned}$$

where in the last line we applied the transformation formula (III.11). We now combine (A.23) to (A.25) to obtain the desired equality

$$\begin{aligned}
\Omega_{i,j,k} &= \frac{(q^{-2(\alpha+\beta)}; q^2)_i (q^{-2(\alpha+\beta)}; q^2)_{k+j}}{(q^2; q^2)_i (q^2; q^2)_j (q^2; q^2)_k} q^{k(k-1+2\alpha)+2i\beta}\, {}_3\phi_2 \left[\begin{matrix} q^{-2(\alpha+\beta-k)}, q^{-2i}, q^{-2\beta} \\ q^{-2(\alpha+\beta)}, 0 \end{matrix} ; q^2, q^2 \right] \\
&= \Theta_{i,j,k}. \qquad \text{(A.26)}
\end{aligned}$$

Appendix B

Some Complex Analysis

The goal of this appendix is to provide justification for Remark 4.1 in which we claimed that the space $\mathcal{V}^{(n)} \subset \hat{\mathcal{V}}^{(n)}$, of power series in the x_{ij} centered at $x_{ij} = 1$ $(1 \leq j < i \leq n)$, provides a sufficient realisation of the principal valued complex power function x^α (for $\alpha \notin \mathbb{Z}_{<-1}$). This will require some complex analysis. For $x, \alpha \in \mathbb{C}$ the principal valued complex power function x^α is given by

$$x^\alpha = e^{\alpha \log(x)}, \tag{B.1}$$

where $\log(x)$ denotes the principal valued logarithm. Since $\log(x)$ is holomorphic on $\mathbb{C} \setminus \{\mathbb{R}_{\leq 0}\}$ it follows that the power function x^α given by (B.1) is also. The function x^α has the following Taylor series expansion about $x = 1$

$$\begin{aligned} x^\alpha &= 1 + \alpha(x-1) + \frac{\alpha(\alpha-1)}{2}(x-1)^2 + \dots \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-m)} \frac{(x-1)^m}{m!} := p_\alpha(x), \end{aligned} \tag{B.2}$$

which has radius of convergence $R = 1$. The ratio of Gamma functions in the second line is valid provided $\alpha + 1 \notin \mathbb{Z}_{\leq 0}$ which are poles of the numerator.

The series $p_\alpha(x_{ij})$ (B.2) now provides us a with a candidate representative for the power function $x_{ij}^\alpha \in \hat{\mathcal{V}}$ for $\alpha \notin \mathbb{Z}_{<0}$ where convergence no longer needs to be considered. Suitability of this candidate is based on three properties. Firstly, if $\alpha = k \in \mathbb{Z}_{\geq 0}$ then the sum in (B.2) is truncated after $m = k$ by poles of the denominator and we obtain

$$
\begin{aligned}
p_k(x) &= \sum_{m=0}^{k} \frac{k!}{(k-m)!}\frac{(x-1)^m}{m!} = \sum_{m=0}^{k}\sum_{l=0}^{m} \frac{m!}{(m-l)!\,l!}\frac{k!}{(k-m)!\,m!}x^l(-1)^{m-l}, \\
&= \sum_{l=0}^{k} \frac{k!}{l!}x^l \left(\sum_{m=l}^{k} \frac{1}{(m-l)!(k-m)!}(-1)^{m-l}\right) = \sum_{l=0}^{k} \binom{k}{l} x^l (1-1)^{k-l} = x^k.
\end{aligned}
\tag{B.3}
$$

In other words $p_k(x)$ reproduces the integer power $x^k \in \mathcal{V}$.

Secondly, since one has $x^\alpha x^\beta = x^{\alpha+\beta}$ from (B.1) this ensures the equality of power series $p_\alpha(x_{ij})p_\beta(x_{ij}) = p_{\alpha+\beta}(x_{ij})$ in $\hat{\mathcal{V}}$ (note a similar argument could have replaced the calculation (B.3)).

Finally, analyticity of (B.1) ensures that termwise algebraic differentation behaves as expected so we have

$$
\partial_{ij} p_\alpha(x_{ij}) = \alpha p_{\alpha-1}(x_{ij}), \tag{B.4}
$$

as an equality in $\hat{\mathcal{V}}$. Combining this with the first two properties we obtain $N_{ij}p_\alpha(x_{ij}) = \alpha p_\alpha(x_{ij})$ and hence $P(N_{ij})p_\alpha(x_{ij}) = P(\alpha)p_\alpha(x_{ij})$, where $P(N_{ij})$ is an operator constructed as a power series in the N_{ij}. In particular note $q^{\beta N_{ij}} p_\alpha(x_{ij}) = q^{\alpha\beta} p_\alpha(x_{ij})$. Thus for all intents and purposes we may regard the series $p_\alpha(x_{ij}) \in \hat{\mathcal{V}}$ (B.2) as the power function x_{ij}^α and our notation will therefore reflect this.

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