Thin position for 3-manifolds

Semester Report - MATH4349

Benjamin Morris - u6678371

June 11, 2021

Abstract

This report presents the notion of thin position for three manifolds as developed in [4]. We begin with a discussion of a more general notion of width and thin position in manifold topology based on [3] before studying the specific case of thin position for three manifolds in §2. §3 examines a simple corollary of this theory relating Heegaard genus and incompressible surfaces to the width of a 3-manifold.

1 Introduction

The notion of thin position for knots was pioneered by Gabai in [2]. This lead to Scharlemann and Thompsons work in developing a notion of thin position for 3-manifolds in their paper [4] which will be the main focus of this report. At first glance the analogy between these two constructions may seem obscure, however, [3] details how they may both be considered two examples of a more general notion of thin position. In this report we take the viewpoint that the added clarity from discussing this generalisation is both interesting and useful. To that end, the rest of this introduction will be dedicated to defining the general concept of thin position, while §2 deals with thin position for 3-manifolds and §3 discusses a simple consequences thereof.

To define the generalised notion of thin position we start with a pair of manifolds (N, M) with $N \subset M$. Denote by $\mathcal{M}$ the set of morse functions on the pair (N, M), that is, morse functions $h : M \to \mathbb{R}$ such that $h|_N$ is also a morse function. We may further require a constraint C to be imposed on $\mathcal{M}$ and if so take $\mathcal{C} = \{h \in \mathcal{M} : h \text{ satisfies } C\}$. Then for $h \in \mathcal{C}$ let $c_0, \dots c_n$ denote the critical points of either $h|_N$ or $h|_M$ arranged in ascending order, and choose some $r_i \in \mathbb{R}$ with $c_{i-1} < r_i < c_i$ for $i = 1, \dots, n$. We now require a function $g : \mathcal{L} \to \mathbb{R}$ where $\mathcal{L} = \{(h|_N^{-1}(r), h|_M^{-1}(r)) : h \in \mathcal{C}, r \text{ non-critical}\}$ is the set of ordered pairs of so called level sets of $h \in \mathcal{C}$. Then for a given $h \in \mathcal{C}$ with the $r_i \in \mathbb{R}$ as above we define the n-tuple

$$(g(h|_N^{-1}(r_1), h|_M^{-1}(r_1)), g(h|_N^{-1}(r_2), h|_M^{-1}(r_2)), \dots, g(h|_N^{-1}(r_n), h|_M^{-1}(r_n))) \in \mathbb{R}^n \subset \mathbb{R}^\infty, \tag{1.1}$$

where a real valued n-tuple is viewed as an element of $\mathbb{R}^\infty$ with all but the first n entries set to 0. To complete the notion of thin position we want to be able to compare the above n-tuples as we vary h. This is done via a function $f : \mathbb{R}^\infty \to \mathcal{O}$ where $\mathcal{O}$ is a well ordered set. Then the width of N with respect to h is defined as

$$w_h(N) = f(g(h|_N^{-1}(r_1), h|_M^{-1}(r_1)), g(h|_N^{-1}(r_2), h|_M^{-1}(r_2)), \dots, g(h|_N^{-1}(r_n), h|_M^{-1}(r_n))). \tag{1.2}$$

The width of N is $w(N) = \min_{h \in \mathcal{C}}\{w_h(N)\}$, and N is said to be in thin position if it is presented with a morse function h for which $w_h(N) = w(N)$.

The intuition behind this construction is that g measures the "complexity" of a pair of level sets $(h|_N^{-1}(r_i), h|_M^{-1}(r_i))$ in some sense, and that f measures the complexity of the collection of level sets $\{(h|_N^{-1}(r_i), h|_M^{-1}(r_i)) : i = 1, \dots n\}$. Thin positions give a way to present N which minimises this complexity. That we had a useful notion of complexity requires a posteriori justification but if this is the case then presenting N in thin position should be optimal in some sense. For example the notion of thin position for knots was developed and used by Gabai to prove property R for knots. This is beyond the scope of this report, however, we will briefly define thin position for knots.

To define thin position for knots we take $(N, M) = (K, S^3)$ for $K \subset S^3$ a knot. We restrict to morse functions for which $h|_{S^3}$ has exactly two critical points $c_0 < c_n$, which are necessarily the global minimum and maximum respectively. Then if $c_1, \ldots c_{n-1}$ are the critical points of $h|_K$ and $r_i \in \mathbb{R}$ are chosen so that $c_{i-1} < r_i < c_i$, the complexity of the pair $(h|_K^{-1}(r_i), h|_{S^3}^{-1}(r_i))$ is given by $g(h|_K^{-1}(r_i), h|_{S^3}^{-1}(r_i)) = \chi(h|_K^{-1}(r_i)) = |K \cap h|_{S^3}^{-1}(r_i)| \in \mathbb{N}$. The width of K, $w_h(K)$, is then given by applying $f : \mathbb{R}^\infty \to \mathbb{R}$, defined by summing all (non-zero) entries, as per (1.2).

2 Thin Position for 3-manifolds

To define thin position for three manifolds we take $(N, M) = (M, M)$ and just identify this with the three manifold M (for simplicity we will restrict to closed, connected, orientable three manifolds). We restrict to morse functions on M for which the index 0 (3) critical points are situated below (above) all index 1 and 2 critical points. For such a morse function h on M with critical points in ascending order $c_0, c_1, \ldots, c_n$ and values $r_i \in (c_{i-1}, c_i)$ the complexity of the level set $h^{-1}(r_i)$[1] is defined to be

$$g(h^{-1}(r_i)) = C_i + s_i - \chi(h^{-1}(r_i)) \in \mathbb{N}, \tag{2.1}$$

where C_i is the number of connected components of $h^{-1}(r_i)$ and s_i is the number of S^2 components. Since euler characteristic is additive and since $\chi(S^2) = 2$, the above definition for complexity of a surface agrees with that given in [4] which takes complexity of a connected surface S to be $1 - \chi(S)$ or $2g - 1$ for $S \neq S^2$ a genus g surface and 0 for $S = S^2$ and then extends this additively. Finally, the width of M with respect to h is $w_h(M) := f(g(h^{-1}(r_1)), \ldots, g(h^{-1}(r_N)))$, where $f : \mathbb{N}^\infty \to \mathbb{N}^\infty$ is the function which takes an m-tuple $(x_1, \ldots, x_m)$, deletes an entry x_i if either $x_{i-1} > x_i$ or $x_{i+1} > x_i$, and then arranges the remaining entries in non-increasing order. After this process the resulting k-tuples are then ordered lexicographically.

When dealing with specific examples of thin position, the above formalism leads to obfuscation. Let us now intepret thin position for three manifolds in a more natural way. We have seen that a morse function h on M details how we may build M by repeatedly gluing handles together. By grouping together handles of the same index a morse function of the type specified above decribes a manifold with a decomposition

$$M = b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_N \cup T_N \cup b_3, \tag{2.2}$$

where b_0 is a collection of 0-handles, N_i is a collection of 1-handles, T_i is a collection of 2-handles, b_3 is a collection of 3-handles, and only N_1 and T_N may be empty. The level set $h^{-1}(r_i)$ is the resultant boundary after adding the first i handles. Let us now make some observations about how the complexity (2.1) interacts with surgery on a surface.

Suppose that $S = \partial N$, is a closed orientable surface that is the boundary of some three manifold N. Now suppose that $S' = \partial(N \cup (i - \text{handle}))$. If i is 0 or 3 then S' differs from S only in the number of S^2 components, so $g(S) = g(S')$. If $i = 1$ then $g(S') \geq g(S)$ since adding a 1-handle lowers χ by 2. Equality is achieved precisely when a 1-handle joins two separate components of S one of which being an S^2 so that we lower the number of components and sphere components by 1. Anything short of this results in the strict inequality $g(S') > g(S)$. Dually, we see that if $i = 2$ then $g(S') \leq g(S)$ since adding a 2 handle raises χ by 2. Equality is achieved precisely when the 2-handle is glued onto an inessential curve raising the number of components and sphere components by 1. Gluing a 2-handle onto an essential curve will give the strict inequality $g(S') < g(S)$.

Using this let us now analyse the sequence

$$g(h^{-1}(r_1)), g(h^{-1}(r_2)), \ldots, g(h^{-1}(r_n)). \tag{2.3}$$

The first few level sets are the surfaces after adding only one handles, which are simply a disjoint union of spheres with complexity 0. The sequence of complexity then increases (non-strictly) as we add the 1-handles in N_1, and decreases (again non-strictly) as we add the 2-handles in T_1 and so on until we have added all the two handles in T_N. After adding all 2-handles in T_N the manifold is completed by capping off the remaining S^2 boundary components with 3-handles, so what remains of the sequence of complexities is identically 0. Since entries of 0 are irrelevant in the lexicographical ordering on N^∞, we need only concern ourselves with what occurs when adding 1 and 2 handles matching our intuition that this is the only non-trivial part of the decomposition (2.2).

[1] Since $N = M$ we need not consider a pair of level sets now.

Now let us consider f applied to (2.3). We note that f selects the local maxima of this sequence, i.e. $g(h^{-1}(r_i))$ satisfying $g(h^{-1}(r_{i-1})) \leq g(h^{-1}(r_i)) \geq g(h^{-1}(r_{i+1}))$. Local maxima must occur between (non-strict) increasing and decreasing sections of the sequence of complexities, which as we saw *almost* correspond to collections N_i of 1-handles and T_i of 2-handles. Thus the level sets selected by f are the surfaces separating the collection N_i of 1-handles from the collection T_i of 2-handles as well as possibly some other surfaces resulting from 1-handles or 2-handles attached which fixed the complexity.

2.1 Scharlemann & Thompson's description

In this subsection we present the description of thin position for 3-manifolds as in [4] and verify that it is consistent with the previous description. We start with a three manifold M with a decomposition as per (2.2) resultant from some morse function h. Now construct the two following families of surfaces

$$S_i := \partial(b_0 \cup N_1 \cup T_1 \cup \cdots \cup T_{i-1} \cup N_i) \setminus \{S^2 \text{ component bounding a 0 or 3 handle}\}, \tag{2.4}$$

$$F_i := \partial(b_0 \cup N_1 \cup T_1 \cup \cdots \cup N_i \cup T_i) \setminus \{S^2 \text{ component bounding a 0 or 3 handle}\}. \tag{2.5}$$

Morse theoretically, we can view the surface S_i (F_i) as the level set at some value t_i (q_i) chosen so that it is situated between the collections N_i and T_i (T_i and N_{i+1}). The condition that we remove S^2 components bounding 0 or 3 handles is simply saying that viewing the morse function as a height function, we squish minima (maxima) as far upward (donward) as they can go. This is depicted schematically below.

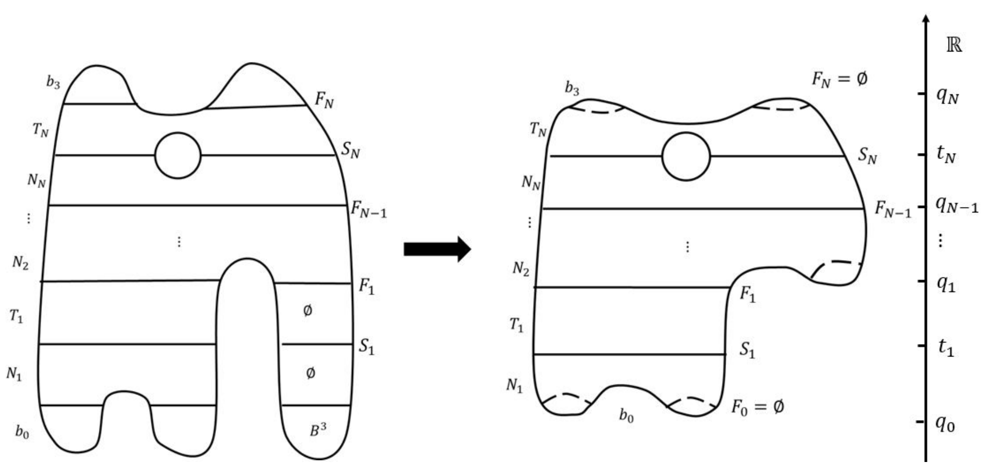

Figure 1: Schematic diagram for the decomposition (2.2), showing the squishing of a minima and maxima. Dashed lines represent the collection of 0 and 3 handles.

Our manifold M can then be written as the union of layers $M = \cup_{i=1}^{N} W_i$ where $W_i = h^{-1}[q_{i-1}, q_i]$ where $q_0, q_N \in \mathbb{R}$ are such that $h(M) \subset (q_0, q_N)$. Morse theory now informs us that each layer is of the form

$$W_i = (F_{i-1} \times I) \cup N_i \cup T_i \cup \{0 \text{ or } 3 \text{ handles incident to } N_i \text{ or } T_i\}. \tag{2.6}$$

The surface $S_i = h^{-1}(t_i) \subset W_i$ separates the layer W_i as $W_i = h^{-1}[q_{i-1}, t_i] \cup_{S_i} h^{-1}[t_i, q_i] := \overline{N}_i \cup \overline{T}_i$. Again we can use morse theory to deduce that $\overline{N}_i$ is of the form $\overline{N}_i = (F_{i-1} \times I) \cup N_i \cup \{0\text{-handles incident to } N_i\}$. Such an $\overline{N}_i$ is a very specific type of manifold; it is a cobordism between F_{i-1} and S_i obtained from $(F_{i-1} \times I)$ by gluing 1 and 0-handles to $F_{i-1} \times \{1\}$ where $F_{i-1} \simeq F_{i-1} \times \{0\}$ and $S_i \simeq \partial\overline{N}_i \setminus F_{i-1} \times \{0\}$. This is type of generalised handlebody known as a compression body. Dually, $\overline{T}_i = (S_i \times I) \cup T_i \cup \{3\text{-handles incident to } T_i\}$ is also a compression body, so in fact we may say that $S_i \subset W_i$ determines a Heegaard splitting of the layer W_i into compression bodies.

The width of a decomposition is then defined in [4] to be the N-tuple $W_h(M) := (g(S_1), g(S_2), \ldots, g(S_N)) \in \mathbb{N}^N \subset \mathbb{N}^\infty$ where the complexity g is as per 2.1, and tuples are ordered by arranging in non-increasing order and comparing lexicographically. Notice that for a given h, $W_h(M)$ does not necessarily agree with the width $w_h(M)$ from the previous section since the latter may contain (the complexity of) strictly more surfaces. However, this is remedied when we optimise over all morse functions h. The extra surfaces included in $w_h(M)$ arise from 1 or 2 handles attached at some step that fix the complexity of the surface they are attached to. We have already described precisely when this occurs. Suppose that h is a morse function with a 2-handle attached to an essential curve at any step. If it is not already, then we may promote this attachment to the "front of the queue" since we can always shrink the essential curve to be an ϵ neighbourhood of some point and move this point as necessary to realise this 2-handle attachment disjointly from all other previously attached 1 and 2-handles. Dually, any 1-handle attachment which joins an S^2 component to any other component S can be demoted to the back of the queue by simply handlesliding any subsequent 1 or 2 handle attachments off of the $S^2 \cup$ (1-handle), realising this attachment disjointly from the remaining attachments.

Repeating this process results in a Morse function h' where all the 1-handle (2-handle) attachments that fix complexity occur after (before) all other 1 and 2 handle attachments such that the intermediate surfaces, S_i and F_i, are the same as in h[2]. The N-tuple of complexities for h' differs from that of h by an additional collection of 0s at the start and end (due to the moved 2 and 1 handles respectively) and that we now have strictly increasing/decreasing sections corresponing to the collection of 1 and 2-handles. Therefore, the only non-zero elements selected by f are exactly the complexities of the surfaces S_i separating collections N_i and T_i, in other words $w'_h(M) = W_h(M)$. This combined with the inequality $w_h(M) \geq W_h(M)$ for all Morse functions h implies that minimising over both results in the same width $w(M)$ for M. In light of this we restrict to morse functions of the same type as h' since this subset of functions always achieves the minimal width. Let us call a decomposition as per figure 1 with $W_h(M) = w(M)$ a thin decomposition.

2.2 Properties of thin decompositions

In this subsection we will prove 6 propositions for a three manifold M presented in thin position, following [4] closely. First we briefly set up some terminology.

Let S be a (closed, orientable) surface and M be a (closed, connected, orientable) 3-manifold. A sphere $S^2 \subset M$ is essential if it does not bounds a 3-ball in M. M is irreducible if it contains no essential S^2. Suppose $S \subset M$. An essential disc in (M, S) is a disc $D^2 \hookrightarrow M$ with $D \cap S = \partial D$ such that ∂D is essential in S. We say S is incompressible in M if no such disk exists and otherwise we say S is compressible in M. M is Haken if it is irreducible and contains an incompressible surface.

Let N be an orientable 3-manifold. A Heegaard splitting of $N = N_1 \cup_S N_2$ into compression bodies N_i is weakly reducible if there exist essential discs ∂D_i for (N_i, S) such that $\partial D_1 \cap \partial D_2 = \emptyset \subset S$. A Heegaard splitting which is not weakly reducible is strongly irreducible.

Before beginning the proofs we signpost a common technique; we "thin" a given decomposition by altering it, giving a new decomposition with lesser width.

Proposition 1: In a thin decomposition for M, any S^2 component of any F_i is essential in M.

Proof: Suppose that $S \subset F_i$ is an inessential sphere. Then either $h^{-1}[q_i, \infty)$, $h^{-1}(-\infty, q_i]$ contains a 3-ball component B with $\partial B = S$. Replace B with a 3-handle in the first case and a 0-handle in the second. B must have consisted of a non-empty collection of both 1 and 2 handles so doing this lowers the complexity of at least one intermediate surface S_j, decreasing $w_h(M)$. □

Proposition 2: In a thin decomposition for M, each component from F_{i-1} either persists into F_i or has handles from both N_i and T_i attached to it in W_i.

Proof: First suppose that some component $F \subset F_{i-1}$ has only 2-handles in T_i attached to it. Since these attachments could not have depended on any 1-handle attachments in N_i we can simply regard these 2-handles as being part of T_{i-1}. This only changes the surface S_i, and it changes it precisely by 2-surgery on essential curves, thus decreasing the width $w_h(M)$. Dually, if we had only attached 1-handles to F we could regard these as being part of the subsequent collection of 1-handles N_{i+1} changing only S_i by removing 1-handles thus decreasing the width. □

Proposition 3: In a thin decomposition for M, each W_i is strongly irreducible.

[2]This process may require adding an additional empty collection of 1-handles at the start or 2-handles at the end

Proof: Suppose for a contradiction that the Heegaard splitting $W_i = \overline{N}_i \cup \overline{T}_i$ is weakly reducible and let D_N (D_T) be an essential disc for $\overline{N}_i$ ($\overline{T}_i$). We can then replace the layer W_i with two layers $W_i^{\pm} = \overline{N}_i^{\pm} \cup \overline{T}_i^{\pm}$, where $\overline{N}_i^{-} = \overline{N}_i \setminus (D_N \times I)$, and $\overline{T}_i^{-}$ is obtained by gluing a 2-handle with core D_T onto $\overline{N}_i^{-}$. Then $\overline{N}_i^{+}$ obtained from $\overline{T}_i^{-}$ by gluing a 1-handle with co-core D_N and $\overline{T}_i^{+}$ is obtained from $\overline{N}_i^{+}$ by adding the remainder of the two handles in T_i. This replaces the layer W_i with the composite layer $W_i^{-} \cup W_i^{+}$ each with Heegaard surfaces $S_i^{\pm}$ obtained from S_i by either compression along or two surgery on an essential disc giving $g(S_i^{\pm}) < g(S_i)$, thus lowering the width $w_h(M)$. □

An immediate consequence of proposition 3 is the following.

Proposition 4: In a thin decomposition for M, all 1-handles in N_i and 2-handles in T_i are incident to the same component S_i. We call such a component the active component of S_i.

Proof: If this fails then the Heegaard splitting for W_i is weakly irreducible. □

The next proposition relies on Theorem 2.1 from [1], the converse of which says that if $\partial W_i = F_{i-1} \cup F_i$ is compressible in W_i then any Heegard splitting for W_i is weakly reducible.

Proposition 5: In a thin decomposition for M, every component of F_i is incompressible in M.

Proof: Suppose that $S \subset F_i$ is a compressible (in M) component and let $D \subset M$ be an essential disc for S. Now let $F = \cup_{i=1}^{N} F_i$, and by restricting to an innermost disc of $D \cap F$ if necessary, we may assume that D is contained entirely within either W_i or W_{i+1}. In either case we obtain that ∂W_j is compressible in W_j for some j which by the aforementioned theorem implies that W_j must have a weakly reducible Heegaard splitting contradicting Proposition 3. □

Our final proposition is the following.

Proposition 6: Suppose M is irreducible and not a Lens space, then in a thin decomposition no component of any S_i is a torus.

Proof: Pick a thin decomposition of M and suppose that T is a torus component of some S_i. Without loss of generality we may assume that T is the active component of S_i. Now let W be the component of W_i containing T and by assumption on the form of the morse function h we have that $g(\partial W) < g(T) = 1$ giving $g(\partial W) = 0$ so $\partial W \subset F_i \cup F_{i-1}$ is a disjoint union of 2-spheres. Since these spheres would be essential in M by proposition 1, we must have $\partial W = \emptyset$ as not to contradict irreducibility of M. Therefore, $W = M$ as connected, orientable, closed submanifold of M. The surface $T \subset W$ determines a Heegaard splitting of M into solid torii which must be strongly irreducible by proposition 3, so M was a Lens space. □

3 A simple corollary

In this section we prove a simple corollary which relates Heegaard genus and incompressible surfaces to the width of a 3-manifold M.

Corollary: Suppose that M is an irreducible manifold of Heegaard genus g. Furthermore, suppose M contains no incompressible surfaces of genus $< g$. Then a minimal genus Heegaard splitting for M is a thin decomposition giving $w(M) = \{2g - 1\}$.

Proof: Suppose for a contradiction that $w(M) < \{2g - 1\}$. Then in a thin decomposition for M, every surface S_i has $g(S_i) < 2g-1$. If all F_i were empty then this thin decomposition would be a Heegaard splitting of genus $< g$ which is impossible. Combining non-emptiness of F_i with irreducibility of M we obtain that $0 < g(F_i) \leq S_i < 2g - 1$ as F_i is obtained from S_i by 2-surgery. Thus each F_i has a component of genus $< g$ which must be incompressible in M contradicting our assumption on M. □

In the following example this corollary is applied in contrapositive form.

Example: Let $M = S^1 \times S^1 \times S^1$ be the 3-torus. Then M is irreducible and has Heegaard genus 3. We saw in class that M has a weakly reducible minimal genus Heegard splitting, which combined with proposition 3 implies that $w(M) < \{2g - 1 = 5\}$. Thus we conclude that M contains an incompressible surface of genus < 3. In fact this result generalises; if $M = S^1 \times S$ is irreducible for $S \neq S^2$ a closed, connected orientable surface then from proposition 3.1 of [5] we obtain that a minimal genus Heegard splitting (say genus g) for M is weakly reducible. Thus we have $w(M) < \{2g - 1\}$ and the corollary tells us that M contains an incompressible surface of genus $< g$.

References

[1] A.J. Casson and C.McA. Gordon. "Reducing heegaard splittings". In: *Topology and its Applications* 27.3 (1987), pp. 275–283. ISSN: 0166-8641. DOI: `https://doi.org/10.1016/0166-8641(87)90092-7`.

[2] David Gabai. "Foliations and the topology of 3-manifolds. III". In: *Journal of Differential Geometry* 26.3 (1987), pp. 479–536. DOI: `10.4310/jdg/1214441488`.

[3] Hugh Howards, Yo'av Rieck, and Jennifer Schultens. "Thin position for knots and 3–manifolds: a unified approach". In: *Heegaard splittings of 3–manifolds (Haifa 2005)* (Dec. 2007). ISSN: 1464-8997. DOI: `10.2140/gtm.2007.12.89`.

[4] Thompson A. Scharlemann M. "Thin position for 3-manifolds". In: *Contemp. Math.* 164 (1994), pp. 231–238.

[5] Jennifer Schultens. "The Classification of Heegaard Splittings for (Compact Orient Able Surface) × S1". In: *Proceedings of the London Mathematical Society* s3-67.2 (Sept. 1993), pp. 425–448. ISSN: 0024-6115. DOI: `10.1112/plms/s3-67.2.425`.